Physics 506 **Winter 2008**

Homework Assignment #11 — Solutions

Textbook problems: Ch. 13: 13.1, 13.2, 13.3, 13.5

13.1 If the light particle (electron) in the Coulomb scattering of Section 13.1 is treated classically, scattering through an angle θ is correlated uniquely to an incident trajectory of impact parameter b according to

$$b = \frac{ze^2}{pv} \cot \frac{\theta}{2}$$

where $p = \gamma m v$ and the differential scattering cross section is $\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta} \left| \frac{db}{d\theta} \right|$.

a) Express the invariant momentum transfer squared in terms of impact parameter and show that the energy transfer $T(b)$ is

$$T(b) = \frac{2z^2 e^4}{mv^2} \frac{1}{b^2 + b_{\text{min}}^{(c)\,2}}$$

where $b_{\text{min}}^{(c)} = ze^2/pv$ and $T(0) = T_{\text{max}} = 2\gamma^2\beta^2 mc^2$.

The invariant momentum transfer squared is defined as $Q^2 = -(p - p')^2$ where p^μ and $p^{\mu\,\prime}$ are the initial and final 4-momenta of the electron. Expanding this out, and using $p^2 = p'^2 = m^2c^2$ gives

$$Q^2 = 2(p^\mu p'_\mu - m^2c^2) = 2(EE'/c^2 - m^2c^2 - \vec{p} \cdot \vec{p}\,') \qquad (1)$$

Now consider the center of mass frame, where the heavy particle is essentially stationary and the electron undergoes scattering by an angle θ. Since this is an elastic scattering, we use conservation of energy to write $E = E' = \sqrt{m^2c^4 + |\vec{p}|^2c^2}$. In addition the scattering angle is related to 3-momentum transfer according to $\vec{p} \cdot \vec{p}\,' = |\vec{p}|^2 \cos\theta$. Inserting this into (1) for the Q^2 invariant gives

$$Q^2 = 2|\vec{p}|^2(1 - \cos\theta) = 4|\vec{p}|^2 \sin^2 \frac{\theta}{2} = 4p^2 \sin^2 \frac{\theta}{2}$$

where in the final expression we simply use p to denote the magnitude of the 3-momentum $\vec{p}$. Rewriting $\sin^2(\theta/2)$ in terms of $\cot^2(\theta/2)$ according to

$$\sin^2 \frac{\theta}{2} = \frac{1}{1 + \cot^2 \frac{\theta}{2}}$$

and inserting the relation between b and θ given above results in

$$Q^2 = \left(\frac{2ze^2}{v} \right)^2 \frac{1}{b^2 + b_{\text{min}}^{(c)\,2}} \qquad (2)$$

with $b_{\rm min}^{(c)} = ze^2/pv$.

We now examine the kinetic energy transfer in the lab frame. In this frame, the electron is initially at rest. Hence $E = mc^2$ and $\vec{p} = 0$. Inserting this into (1) gives

$$Q^2 = 2m(E' - mc^2) = 2mT$$

where $T \equiv E' - mc^2$ is the kinetic energy transfer. Finally, using this relation $Q^2 = 2mT$ in (2) gives

$$T = \frac{2z^2e^4}{mv^2}\frac{1}{b^2 + b_{\rm min}^{(c)\,2}} \tag{3}$$

b) Calculate the small transverse impulse Δp given to the (nearly stationary) light particle by the transverse electric field (11.152) of the heavy particle $q = ze$ as it passes by at large impact parameter b in a (nearly) straight line path at speed v. Find the energy transfer $T \approx (\Delta p)^2/2m$ in terms of b. Compare with the exact classical result of part a. Comment.

The transverse electric field of (11.152) is given by

$$E_\perp = \frac{q\gamma b}{(b^2 + \gamma^2v^2t^2)^{3/2}}$$

We now calculate the impulse according to

$$\Delta p_\perp = \int F_\perp \, dt = e\int E_\perp \, dt = ze^2\gamma b\int \frac{dt}{(b^2 + \gamma^2v^2t^2)^{3/2}}$$

where we used $q = ze$ for the charge of the heavy particle. This integral can be performed by trig substitution $t = (b/\gamma v)\tan\theta$ with the result

$$\Delta p_\perp = \frac{ze^2}{bv}\int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta = \frac{2ze^2}{bv}$$

As a result, the energy transfer is approximately

$$T \approx \frac{(\Delta p_\perp)^2}{2m} = \frac{2z^2e^4}{mv^2}\frac{1}{b^2} \tag{4}$$

This is similar to the exact classical result (3), with the exception that the $b_{\rm min}^{(c)\,2}$ term is missing. That this term is missing is actually not surprising, because we have assumed the particle passes by at large impact parameter. This is essentially the limit $b \gg b_{\rm min}^{(c)}$, and it corresponds to having almost no deflection from the straight line path. When the impact parameter gets too small, the electron suffers a large deflection, and the straight line approximation breaks down. Thus instead of going to infinity as this approximate result does, the exact result (3) remains finite as $b \to 0$.

13.2 Time-varying electromagnetic fields $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$ of finite duration act on a charged particle of charge e and mass m bound harmonically to the origin with natural frequency ω_0 and small damping constant Γ. The fields may be caused by a passing charged particle or some other external source. The charge's motion in response to the fields is nonrelativistic and small in amplitude compared to the scale of spatial variation of the fields (dipole approximation). Show that the energy transferred to the oscillator in the limit of very small damping is

$$\Delta E = \frac{\pi e^2}{m}|\vec{E}(\omega_0)|^2$$

where $\vec{E}(\omega)$ is the symmetric Fourier transform of $\vec{E}(0,t)$:

$$\vec{E}(0,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\vec{E}(\omega)e^{-i\omega t}d\omega, \qquad \vec{E}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\vec{E}(0,t)e^{i\omega t}dt$$

The classical dynamics of the charged particle is given by $\vec{F} = m\vec{a}$

$$m\ddot{\vec{x}} = -m\omega_0^2\vec{x} - m\Gamma\dot{\vec{x}} + e\vec{E}(\vec{x},t) + \frac{e}{c}\dot{\vec{x}}\times\vec{B}(\vec{x},t)$$

In general, the Lorentz force terms are non-linear in displacement $\vec{x}(t)$. However for small amplitudes we may replace $\vec{E}(\vec{x},t) \approx \vec{E}(0,t)$ and $\vec{B}(\vec{x},t) \approx \vec{B}(0,t)$ on the right hand side. This gives the equation

$$\ddot{\vec{x}} + \Gamma\dot{\vec{x}} + \omega_0^2\vec{x} = \frac{e}{m}\vec{E}(t) + \frac{e}{mc}\dot{\vec{x}}\times\vec{B}(t)$$

Note that this equation is still rather awkward to solve because of the magnetic field coupling. Fortunately, this $\dot{\vec{x}}\times\vec{B}$ term can also be dropped at the same linearized level of approximation. This is because it can be treated as a perturbation: if $\vec{x}$ is first order in the external fields, then $\dot{\vec{x}}\times\vec{B}$ will be second order. As a result, we have the familiar damped driven harmonic oscillator

$$\ddot{\vec{x}} + \Gamma\dot{\vec{x}} + \omega_0^2\vec{x} = \frac{e}{m}\vec{E}(t)$$

with frequency domain solution

$$\vec{x}(\omega) = \frac{e/m}{\omega_0^2 - \omega^2 - i\omega\Gamma}\vec{E}(\omega) \tag{5}$$

The energy transfer is then obtained by integrating the power

$$\Delta E = \int_{-\infty}^{\infty}\vec{F}(t)\cdot\dot{\vec{x}}(t)\,dt = e\int_{-\infty}^{\infty}\vec{E}(t)\cdot\dot{\vec{x}}(t)\,dt$$

This may be converted into the frequency domain using Parseval's relation (or the convolution theorem)

$$\Delta E = e\int_{-\infty}^{\infty} \dot{\vec{x}}(\omega)\cdot\vec{E}^*(\omega)\,d\omega$$

Substituting in (5) and expressing d/dt in frequency space then gives

$$\begin{aligned}\Delta E &= \frac{e^2}{m}\int_{-\infty}^{\infty}\frac{-i\omega}{\omega_0^2-\omega^2-i\omega\Gamma}|\vec{E}(\omega)|^2 d\omega\\ &= \frac{e^2}{m}2\Re\int_0^{\infty}\frac{-i\omega}{\omega_0^2-\omega^2-i\omega\Gamma}|\vec{E}(\omega)|^2 d\omega\\ &= \frac{2e^2}{m}\int_0^{\infty}\frac{\omega^2\Gamma}{(\omega^2-\omega_0^2)^2+\omega^2\Gamma^2}|\vec{E}(\omega)|^2 d\omega\end{aligned}$$

This general expression simplifies in the limit $\Gamma\to 0$ where the fraction in the integrand becomes a delta function

$$\lim_{\Gamma\to 0}\frac{\omega^2\Gamma}{(\omega^2-\omega_0^2)^2+\omega^2\Gamma^2} = \frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$$

This gives

$$\lim_{\Gamma\to 0}\Delta E = \frac{\pi e^2}{m}|\vec{E}(\omega_0)|^2 \tag{6}$$

13.3 The external fields of Problem 13.2 are caused by a charge ze passing the origin in a straight-line path at speed v and impact parameter b. The fields are given by (11.152).

a) Evaluate the Fourier transforms for the perpendicular and parallel components of the electric field at the origin and show that

$$E_\perp(\omega) = \frac{ze}{bv}\left(\frac{2}{\pi}\right)^{1/2}\xi K_1(\xi),\qquad E_\parallel(\omega) = -i\frac{ze}{\gamma bv}\left(\frac{2}{\pi}\right)^{1/2}\xi K_0(\xi)$$

where $\xi = \omega b/\gamma v$, and $K_\nu(\xi)$ is the modified Bessel function of the second kind and order ν. [See references to tables of Fourier transforms in Section 13.3]

The external fields for a charge ze are given by

$$E_\parallel = \frac{-ze\gamma vt}{(b^2+\gamma^2v^2t^2)^{3/2}},\qquad E_\perp = \frac{ze\gamma b}{(b^2+\gamma^2v^2t^2)^{3/2}} \tag{7}$$

Before evaluating the Fourier transforms, we recall that the modified Bessel functions K_0 and K_1 may be defined by

$$K_0(x) = \int_0^{\infty}\frac{\cos(xt)}{(t^2+1)^{1/2}}dt,\qquad K_1(x) = \int_0^{\infty}\frac{t\sin(xt)}{(t^2+1)^{1/2}}dt$$

Based on symmetry/antisymmetry, these may be extended to the entire real line

$$K_0(x) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{ixt}}{(t^2+1)^{1/2}}dt, \qquad K_1(x) = \frac{1}{2i}\int_{-\infty}^{\infty} \frac{te^{ixt}}{(t^2+1)^{1/2}}dt$$

Comparing these expressions to (7), we see some similarities. However, the denominators in (7) are raised to the 3/2 power. This suggests that we integrate by parts to obtain the transform expressions

$$\int_{-\infty}^{\infty} \frac{te^{ixt}}{(t^2+1)^{3/2}}dt = 2ixK_0(x), \qquad \int_{-\infty}^{\infty} \frac{e^{ixt}}{(t^2+1)^{3/2}}dt = 2xK_1(x)$$

We are now ready to evaluate the Fourier transforms. For $E_\perp$, we have

$$\begin{aligned} E_\perp(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{ze\gamma b}{(b^2+\gamma^2 v^2 t^2)^{3/2}} e^{i\omega t}dt \\ &= \frac{ze\gamma}{b^2\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(1+(\gamma v t/b)^2)^{3/2}}dt \\ &= \frac{ze}{bv\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{e^{i\xi t'}}{(1+t'^2)^{3/2}}dt' = \frac{ze}{bv}\sqrt{\frac{2}{\pi}}\xi K_1(\xi) \end{aligned} \tag{8}$$

where we made the change of variables $t' = \gamma v t/b$ and introduced the parameter $\xi = \omega b/\gamma v$. The transform for $E_\parallel$ is similar

$$\begin{aligned} E_\parallel(\omega) &= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{ze\gamma v t}{(b^2+\gamma^2 v^2 t^2)^{3/2}} e^{i\omega t}dt \\ &= -\frac{ze\gamma v}{b^3\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{te^{i\omega t}}{(1+(\gamma v t/b)^2)^{3/2}}dt \\ &= -\frac{ze}{\gamma bv\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{t'e^{i\xi t'}}{(1+t'^2)^{3/2}}dt' = -i\frac{ze}{\gamma bv}\sqrt{\frac{2}{\pi}}\xi K_0(\xi) \end{aligned} \tag{9}$$

b) Using the result of Problem 13.2, write down the energy transfer ΔE to a harmonically bound charged particle. From the limiting forms of the modified Bessel functions for small and large argument, show that your result agrees with the appropriate limit of $T(b)$ in Problem 13.1 on the one hand and the arguments at the end of Section 13.1 on the adiabatic behavior for $b \gg \gamma v/\omega_0$ on the other.

The energy transfer ΔE is approximately given by (6)

$$\Delta E = \frac{\pi e^2}{m}|\vec{E}(\omega_0)|^2$$

Substituting in $E_\perp$ and $E_\parallel$ from (8) and (9) gives

$$\Delta E = \frac{2z^2e^4}{mv^2}\frac{\xi_0^2[K_1(\xi_0)^2 + \gamma^{-2}K_0(\xi_0)^2]}{b^2}$$

where $\xi_0 = \omega_0 b/\gamma v$.

Note that the adiabatic regime is governed by the scale of b compared to $b^{(c)}_{\text{max}} \equiv \gamma v/\omega_0$. In particular, since $\xi_0 = b/b^{(c)}_{\text{max}}$, the two regimes of interest (small and large impact parameter) correspond directly to small and large argument of the modified Bessel functions. In the small impact parameter regime $b \ll b^{(c)}_{\text{max}}$ we expand

$$K_0(\xi) = -\ln\left(\frac{\xi e^{\gamma}}{2}\right) + \cdots, \qquad K_1(\xi) = \frac{1}{\xi} + \cdots$$

Thus

$$\Delta E \approx \frac{2z^2e^4}{mv^2}\frac{1 + (\gamma^{-1}\xi\ln(\xi e^{\gamma}/2))^2}{b^2} \approx \frac{2z^2e^4}{mv^2}\frac{1}{b^2} \qquad (\xi \to 0)$$

This agrees with the large (but not so large as to be in the adiabatic regime) impact parameter limit expression (4) of the previously computed energy transfer. (Note that Problem 13.1 concerned a free electron, namely $\omega_0 \to 0$ or $b^{(c)}_{\text{max}} \to \infty$.) Of course this expression breaks down for zero impact parameter for the same reason that (4) breaks down. Finally, for large impact parameters $b \gg b^{(c)}_{\text{max}}$, we use the asymptotic expansion

$$K_\nu(\xi) \sim \sqrt{\frac{\pi}{2\xi}}e^{-\xi}$$

In this case, we obtain

$$\Delta E \sim \frac{\pi z^2 e^4}{mv^2}\frac{(1+\gamma^{-2})e^{-2b/b^{(c)}_{\text{max}}}}{b\,b^{(c)}_{\text{max}}}$$

This vanishes exponentially as $e^{-2b/b^{(c)}_{\text{max}}}$, which agrees with the notion that there is no significant energy transfer in the adiabatic limit (corresponding to $b > b^{(c)}_{\text{max}}$).

13.5 Consider the energy loss by close collisions of a fast, but nonrelativistic, heavy particle of charge ze passing through an electronic plasma. Assume that the screened Coulomb interaction $V(r) = ze^2\exp(-k_D r)/r$, where k_D is the Debye screening parameter, acts between the electrons and the incident particle.

a) Show that the energy transfer in a collision at impact parameter b is given approximately by

$$\Delta E(b) \approx \frac{2(ze^2)^2}{mv^2}k_D^2 K_1^2(k_D b)$$

where m is the electron mass and v is the velocity of the incident particle.

As in Problem 13.1b, we may calculate the energy transfer from the impulse

$$\Delta E \approx \frac{(\Delta p_\perp)^2}{2m} \tag{10}$$

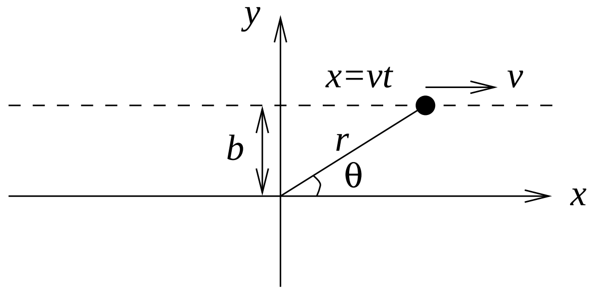
where

$$\Delta p_\perp = \int_{-\infty}^{\infty} F_\perp \, dt$$

For a central potential $V(r) = ze^2 \exp(-k_D r)/r$, the force is in the radial direction

$$\vec{F} = -\vec{\nabla} V = ze^2 \frac{e^{-k_D r}(1 + k_D r)}{r^2} \hat{r}$$

Setting up the collision as



we see that the perpendicular component of the force is given by

$$F_\perp = F_r \sin\theta = F_r \frac{b}{r} = ze^2 b \frac{e^{-k_D r}(1 + k_D r)}{r^3}$$

where $r = \sqrt{b^2 + x^2} = \sqrt{b^2 + v^2 t^2}$. The momentum-impulse theorem then gives

$$\Delta p_\perp = ze^2 b \int_{-\infty}^{\infty} \frac{e^{-k_D r}(1 + k_D r)}{r^3} dt$$

Instead of expressing r in terms of t, we may substitute in t as a function of r

$$t = \frac{1}{v}\sqrt{r^2 - b^2} \qquad \Rightarrow \qquad dt = \frac{r}{v} \frac{dr}{\sqrt{r^2 - b^2}}$$

so that

$$\Delta p_\perp = \frac{2ze^2 b}{v} \int_b^{\infty} \frac{e^{-k_D r}(1 + k_D r)}{r^2} \frac{dr}{\sqrt{r^2 - b^2}}$$

The particle's path from $t = -\infty$ to $t = \infty$ corresponds to taking $r = \infty$ from the left, to $r = b$ at minimum approach, and back out to $r = \infty$ on the right. Because of symmetry, we can simply double the integral for the particle to move from $r = b$ out to infinity. This integral can be simplified by hyperbolic trig substitution $r = b\cosh t$ to give

$$\Delta p_\perp = \frac{2ze^2}{vb} \int_0^{\infty} \frac{e^{-\xi \cosh t}(1 + \xi \cosh t)}{\cosh^2 t} dt \tag{11}$$

where $\xi = k_D b$. The integral

$$f(\xi) = \int_0^{\infty} \frac{e^{-\xi \cosh t}(1 + \xi \cosh t)}{\cosh^2 t} dt \tag{12}$$

is somewhat troublesome to evaluate. One approach is to note that this simplifies upon taking a derivative

$$f'(\xi) = -\xi \int_0^\infty e^{-\xi \cosh t} dt = -\xi K_0(\xi) = (\xi K_1(\xi))'$$

As a result, we have simply $f(\xi) = \xi K_1(\xi)$ up to a possible constant of integration. Direct examination of (12) indicates that $f(\infty) = 0$, which fixes the constant to be zero. Substituting this integral into (11) then gives

$$\Delta p_\perp = \frac{2ze^2}{vb} \xi K_1(\xi)$$

This gives an energy transfer of

$$\Delta E \approx \frac{(\Delta p_\perp)^2}{2m} = \frac{2(ze^2)^2}{mv^2 b^2} \xi^2 K_1^2(\xi) = \frac{2(ze^2)^2}{mv^2} k_D^2 K_1^2(k_D b) \tag{13}$$

Note that an alternate starting point would be to take the zero frequency limit ($\omega_0 \to 0$) of the energy transfer expression (6) of Problem 13.2

$$\Delta E = \frac{\pi e^2}{m} |\vec{E}(\omega_0)|^2 = \frac{\pi}{m} |\vec{F}(\omega_0)|^2$$

where the force is given by $\vec{F} = e\vec{E}$. At zero frequency, the Fourier transform is

$$\vec{F}(\omega = 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \vec{F}(t) dt$$

so that

$$\Delta E = \frac{1}{2m} \left| \int_{-\infty}^\infty \vec{F}(t) dt \right|^2 = \frac{|\Delta \vec{p}|^2}{2m}$$

This is clearly equivalent to (10), up to dropping the parallel component of the momentum transfer.

b) Determine the energy loss per unit distance traveled for collisions with impact parameter greater than $b_{\rm min}$. Assuming $k_D b_{\rm min} \ll 1$, show that

$$\left(\frac{dE}{dx}\right)_{k_D b < 1} \approx \frac{(ze)^2}{v^2} \omega_P^2 \ln\left(\frac{1}{1.47 k_D b_{\rm min}}\right)$$

where $b_{\rm min}$ is given by the larger of the classical and quantum minimum impact parameters [(13.16) and above].

We may use the expression

$$\frac{dE}{dx} = 2\pi N \int_{b_{\rm min}}^\infty \Delta E \, b db$$

to compute the energy loss per unit distance. For ΔE given in (13), this becomes

$$\frac{dE}{dx} = \frac{4\pi N(ze^2)^2}{mv^2} \int_{\xi_{\min}}^{\infty} \xi K_1^2(\xi) d\xi = \frac{4\pi N(ze^2)^2}{mv^2} \left[\frac{\xi^2}{2} \left(K_1^2(\xi) - K_0(\xi) K_2(\xi) \right) \right]_{\xi_{\min}}^{\infty}$$

Since the modified Bessel functions are exponentially suppressed at infinity, the only contribution comes from the lower limit

$$\frac{dE}{dx} = \frac{(ze)^2}{v^2} \omega_P^2 \frac{\xi_{\min}^2 \left(K_0(\xi_{\min}) K_2(\xi_{\min}) - K_1^2(\xi_{\min}) \right)}{2}$$

where we the plasma frequency (in Gaussian units) is given by

$$\omega_P^2 = \frac{4\pi N e^2}{m}$$

Finally, using the small argument expansions of the modified Bessel functions

$$K_0(\xi) \approx \log \frac{2}{\xi} - \gamma, \qquad K_1(\xi) \approx \frac{1}{\xi}, \qquad K_2(\xi) \approx \frac{2}{\xi^2}$$

gives

$$\begin{aligned} \frac{\xi_{\min}^2 \left(K_0(\xi_{\min}) K_2(\xi_{\min}) - K_1^2(\xi_{\min}) \right)}{2} &\approx \log \frac{2}{\xi_{\min}} - \gamma - \frac{1}{2} \\ &= \log \frac{2e^{-\gamma - 1/2}}{\xi_{\min}} \approx \log \frac{1}{1.468 \xi_{\min}} \end{aligned}$$

Putting everything together then yields the energy loss per unit distance

$$\frac{dE}{dx} \approx \frac{(ze)^2}{v^2} \omega_P^2 \log \frac{1}{1.468 k_D b_{\min}}$$