Physics 506 **Winter 2008**

Homework Assignment #10 — Solutions

Textbook problems: Ch. 12: 12.10, 12.13, 12.16, 12.19

12.10 A charged particle finds itself instantaneously in the equatorial plane of the earth's magnetic field (assumed to be a dipole field) at a distance R from the center of the earth. Its velocity vector at that instant makes an angle α with the equatorial plane ($v_\parallel/v_\perp = \tan\alpha$). Assuming that the particle spirals along the lines of force with a gyration radius $a \ll R$, and that the flux linked by the orbit is a constant of the motion, find an equation for the maximum magnetic latitude λ reached by the particle as a function of the angle α. Plot a graph (*not a sketch*) of λ versus α. Mark parametrically along the curve the values of α for which a particle at radius R in the equatorial plane will hit the earth (radius R_0) for $R/R_0 = 1.2, 1.5, 2.0, 2.5, 3, 4, 5$.

Since the particle spirals along the lines of force (ie magnetic field lines), we first set out to calculate what these lines are. For a dipole field with a magnetic dipole moment $\vec{m} = -M\hat{z}$, the magnetic field is

$$\vec{B} = \frac{3\hat{r}(\hat{r}\cdot\vec{m}) - \vec{m}}{r^3} = \frac{M}{r^3}(\hat{z} - 3\cos\theta\hat{r})$$

where θ is the standard polar angle in spherical coordinates. This expression may be transformed entirely into spherical coordinates by writing $\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$. The result is

$$\vec{B} = -\frac{M}{r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta}) \tag{1}$$

Because of azimuthal symmetry, we can think of this as a vector field in the r and θ directions. What we want to do now is to come up with a parametric equation $r = r(\lambda)$, $\theta = \theta(\lambda)$ describing the field lines. Here λ is a parameter along the curve. The key to relating this parametric equation to the magnetic field is to realize that the tangent to the curve should be identified with the magnetic field vector $\vec{B}$. Since the tangent to the curve is given by

$$\frac{\partial}{\partial\lambda} = \frac{dr}{d\lambda}\hat{r} + r\frac{d\theta}{d\lambda}\hat{\theta} \tag{2}$$

we may take ratios of $\hat{r}$ and $\hat{\theta}$ components of (1) and (2) to obtain

$$\frac{2\cos\theta}{\sin\theta} = \frac{dr/d\lambda}{r d\theta/d\lambda} = \frac{1}{r}\frac{dr}{d\theta}$$

This gives rise to the separable equation $dr/r = 2\cot\theta\, d\theta$ which may be integrated to yield

$$r(\theta) = R\sin^2\theta \tag{3}$$

Note that we have chosen the initial condition that $r(\pi/2) = R$, since $\theta = \pi/2$ corresponds to the equatorial plane.

In addition to the equation for a magnetic field line, we also need the magnitude of the magnetic field. This may be computed from (1)

$$B = \frac{M\sqrt{1+3\cos^2\theta}}{r^3}$$

Along the line $r = R\sin^2\theta$, this becomes

$$B(\theta) = \frac{M}{R^3}\frac{\sqrt{1+3\cos^2\theta}}{\sin^6\theta} \tag{4}$$

Since the flux linked by the orbit is a constant of motion (an adiabatic invariant), we end up with the velocity relation

$$v_\parallel(\theta)^2 = v_0^2 - v_{\perp,0}^2\frac{B(\theta)}{B_0} = v_{\parallel,0}^2 + v_{\perp,0}^2\left(1 - \frac{B(\theta)}{B_0}\right)$$

where we have used $v_0^2 = v_{\parallel,0}^2 + v_{\perp,0}^2$. The particle starts at an angle $\theta_0 = \pi/2$. From (4), the initial magnetic field is $B_0 = M/R^3$. hence

$$v_\parallel(\theta)^2 = v_{\parallel,0}^2 + v_{\perp,0}^2\left(1 - \frac{\sqrt{1+3\cos^2\theta}}{\sin^6\theta}\right)$$

The minimum value of θ is reached at the turning point when $v_\parallel(\theta) = 0$. This corresponds to

$$v_{\parallel,0}^2 + v_{\perp,0}^2\left(1 - \frac{\sqrt{1+3\cos^2\theta}}{\sin^6\theta}\right) = 0 \qquad \Rightarrow \qquad \frac{\sqrt{1+3\cos^2\theta}}{\sin^6\theta} = 1 + \frac{v_{\parallel,0}^2}{v_{\perp,0}^2}$$

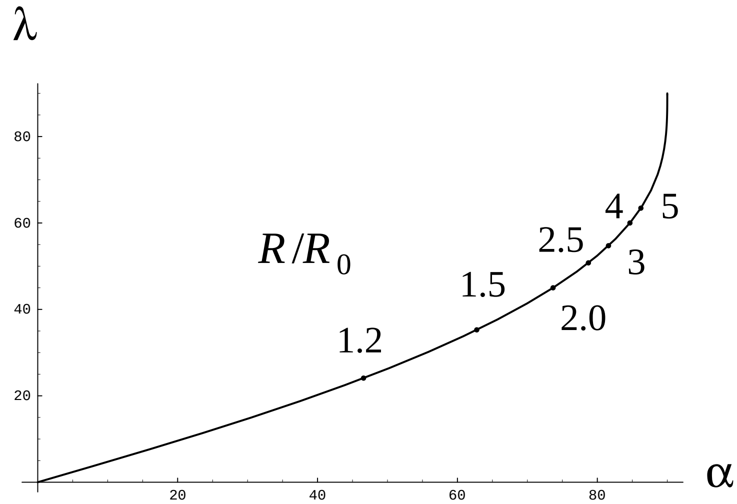
Setting $\theta = \pi/2 - \lambda$ where λ is the magnetic latitude, and using $v_{\parallel,0}/v_{\perp,0} = \tan\alpha$ then gives

$$1 + \tan^2\alpha = \frac{\sqrt{1+3\sin^2\lambda}}{\cos^6\lambda}$$

or

$$\alpha = \tan^{-1}\left[\left(\frac{\sqrt{1+3\sin^2\lambda}}{\cos^6\lambda} - 1\right)^{1/2}\right]$$

We may plot λ versus α as



Since the magnetic field line is given by (3), the particle will hit the earth when $R_0 = R\sin^2\theta = R\cos^2\lambda$, or $\lambda = \cos^{-1}\sqrt{R_0/R}$. These values are indicated on the plot.

12.13 *a*) Specialize the Darwin Lagrangian (12.82) to the interaction of two charged particles (m_1, q_1) and (m_2, q_2). Introduce reduced particle coordinates, $\vec{r} = \vec{x}_1 - \vec{x}_2$, $\vec{v} = \vec{v}_1 - \vec{v}_2$ and also center of mass coordinates. Write out the Lagrangian in the reference frame in which the velocity of the center of mass vanishes and evaluate the canonical momentum components, $p_x = \partial L/\partial v_x$, etc.

The two particle Darwin Lagrangian reads

$$L = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{8c^2}(m_1v_1^4 + m_2v_2^4) - \frac{q_1}{q_2}r_{12} + \frac{q_1q_2}{2r_{12}c^2}[\vec{v}_1\cdot\vec{v}_2 + (\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r})] \quad (5)$$

We take a standard (non-relativistic) transformation to center of mass coordinates

$$\vec{r} = \vec{x}_1 - \vec{x}_2, \qquad \vec{R} = \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{M}$$

where $M = m_1 + m_2$. Inverting this gives

$$\vec{x}_1 = \vec{R} + \frac{m_2}{M}\vec{r}, \qquad \vec{x}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$$

As a result, the individual terms in the Lagrangian (5) become

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}MV^2 + \frac{1}{2}\mu v^2$$

$$\frac{(m_1v_1^4 + m_2v_2^4)}{8c^2} = \frac{1}{8c^2}\left(MV^4 + 6\mu V^2v^2 + 4\mu\frac{m_2 - m_1}{M}(\vec{V}\cdot\vec{v})v^2 + \mu\frac{m_1^3 + m_2^3}{M^3}v^4\right)$$

$$\vec{v}_1\cdot\vec{v}_2 = V^2 + \frac{m_2 - m_1}{M}\vec{V}\cdot\vec{v} - \frac{\mu}{M}v^2$$

$$(\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r}) = (\vec{V}\cdot\hat{r})^2 + \frac{m_2 - m_1}{M}(\vec{V}\cdot\hat{r})(\vec{v}\cdot\hat{r}) - \frac{\mu}{M}(\vec{v}\cdot\hat{r})^2$$

where $\mu = m_1 m_2/M$ is the reduced mass. For vanishing center of mass velocity ($\vec{V} = 0$) the Lagrangian becomes

$$L = \frac{1}{2}\mu v^2 + \frac{1}{8c^2}\mu \frac{m_1^3 + m_2^3}{M^3} v^4 - \frac{q_1 q_2}{r} - \frac{\mu q_1 q_2}{2Mrc^2}[v^2 + (\vec{v}\cdot\hat{r})^2] \tag{6}$$

The canonical momentum is $p_i = \partial L/\partial v_i$, which gives

$$\vec{p} = \mu\vec{v} + \frac{1}{2c^2}\mu \frac{m_1^3 + m_2^3}{M^3} v^2 \vec{v} - \frac{\mu q_1 q_2}{2Mrc^2}[\vec{v} + (\vec{v}\cdot\hat{r})\hat{r}] \tag{7}$$

b) Calculate the Hamiltonian to first order in $1/c^2$ and show that it is

$$H = \frac{p^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) + \frac{q_1 q_2}{r} - \frac{p^4}{8c^2}\left(\frac{1}{m_1^3} + \frac{1}{m_2^3}\right) + \frac{q_1 q_2}{2m_1 m_2 c^2}\left(\frac{p^2 + (\vec{p}\cdot\hat{r})^2}{r}\right)$$

[You may disregard the comparison with Bethe and Salpeter.]

The Hamiltonian is obtained from the Lagrangian (6) by the transformation $H = \vec{p}\cdot\vec{v} - L$. Note, however, that we must invert the relation (7) to write the resulting H as a function of $\vec{p}$ and $\vec{r}$. We start with

$$\begin{aligned} H &= \vec{p}\cdot\vec{v} - \frac{1}{2}\mu v^2 - \frac{1}{8c^2}\mu \frac{m_1^3 + m_2^3}{M^3} v^4 + \frac{q_1 q_2}{r} + \frac{\mu q_1 q_2}{2Mrc^2}[v^2 + (\vec{v}\cdot\hat{r})^2] \\ &= \frac{p^2}{2\mu} - \frac{1}{2\mu}(\vec{p} - \mu\vec{v})^2 - \frac{1}{8c^2}\mu \frac{m_1^3 + m_2^3}{M^3} v^4 + \frac{q_1 q_2}{r} + \frac{\mu q_1 q_2}{2Mrc^2}[v^2 + (\vec{v}\cdot\hat{r})^2] \end{aligned} \tag{8}$$

Since we only work to first order in $1/c^2$, we do not need to completely solve (7) for $\vec{v}$ in terms of $\vec{p}$. Instead, it is sufficient to note that

$$\vec{v} = \frac{1}{\mu}\vec{p} + \mathcal{O}\left(\frac{1}{c^2}\right)$$

Inserting this into (8) gives (to order $1/c^2$)

$$\begin{aligned} H &= \frac{p^2}{2\mu} - \frac{1}{8c^2}\frac{m_1^3 + m_2^3}{M^3\mu^3} p^4 + \frac{q_1 q_2}{r} + \frac{q_1 q_2}{2M\mu rc^2}[p^2 + (\vec{p}\cdot\hat{r})^2] \\ &= \frac{p^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) - \frac{p^4}{8c^2}\left(\frac{1}{m_1^3} + \frac{1}{m_2^3}\right) + \frac{q_1 q_2}{r} + \frac{q_1 q_2}{2m_1 m_2 rc^2}[p^2 + (\vec{p}\cdot\hat{r})^2] \end{aligned}$$

12.16 *a*) Starting with the Proca Lagrangian density (12.91) and following the same procedure as for the electromagnetic fields, show that the symmetric stress-energy-momentum tensor for the Proca fields is

$$\Theta^{\alpha\beta} = \frac{1}{4\pi}\left[g^{\alpha\gamma}F_{\gamma\lambda}F^{\lambda\beta} + \frac{1}{4}g^{\alpha\beta}F_{\lambda\nu}F^{\lambda\nu} + \mu^2\left(A^\alpha A^\beta - \frac{1}{2}g^{\alpha\beta}A_\lambda A^\lambda\right)\right]$$

The Proca Lagrangian density is

$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} + \frac{1}{8\pi} \mu^2 A_\mu A^\mu$$

Since

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \mathcal{A}_\lambda} \partial^\nu A_\lambda - \eta^{\mu\nu} \mathcal{L}$$

we find

$$T^{\mu\nu} = -\frac{1}{4\pi} F^{\mu\lambda} \partial^\nu A_\lambda + \frac{1}{16\pi} \eta^{\mu\nu} F^2 - \frac{1}{8\pi} \mu^2 \eta^{\mu\nu} A^2$$

where we have used a shorthand notation $F^2 \equiv F_{\mu\nu} F^{\mu\nu}$ and $A^2 \equiv A_\mu A^\mu$. In order to convert this canonical stress tensor to the symmetric stress tensor, we write $\partial^\nu A_\lambda = F^\nu{}_\lambda + \partial_\lambda A^\nu$. Then

$$\begin{aligned} T^{\mu\nu} &= -\frac{1}{4\pi} [F^{\mu\lambda} F^\nu{}_\lambda - \tfrac{1}{4} \eta^{\mu\nu} F^2 + \tfrac{1}{2} \mu^2 \eta^{\mu\nu} A^2] - \frac{1}{4\pi} F^{\mu\lambda} \partial_\lambda A^\nu \\ &= -\frac{1}{4\pi} [F^{\mu\lambda} F^\nu{}_\lambda - \tfrac{1}{4} \eta^{\mu\nu} F^2 + \tfrac{1}{2} \mu^2 \eta^{\mu\nu} A^2 - (\partial_\lambda F^{\mu\lambda}) A^\nu] - \frac{1}{4\pi} \partial_\lambda (F^{\mu\lambda} A^\nu) \end{aligned}$$

Using the Proca equation of motion $\partial_\lambda F^{\lambda\mu} + \mu^2 A^\mu = 0$ then gives

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\lambda S^{\lambda\mu\nu}$$

where

$$\Theta^{\mu\nu} = -\frac{1}{4\pi} \left[F^{\mu\lambda} F^\nu{}_\lambda - \tfrac{1}{4} \eta^{\mu\nu} F^2 - \mu^2 (A^\mu A^\nu - \tfrac{1}{2} \eta^{\mu\nu} A^2) \right] \tag{9}$$

is the symmetric stress tensor and $S^{\lambda\mu\nu} = (1/4\pi) F^{\lambda\mu} A^\nu$ is antisymmetric on the first two indices.

b) For these fields in interaction with the external source J^β, as in (12.91), show that the differential conservation laws take the same form as for the electromagnetic fields, namely

$$\partial_\alpha \Theta^{\alpha\beta} = \frac{J_\lambda F^{\lambda\beta}}{c}$$

Taking a 4-divergence of the symmetric stress tensor (9) gives

$$\begin{aligned} \partial_\mu \Theta^{\mu\nu} &= -\frac{1}{4\pi} \big[\partial_\mu F^{\mu\lambda} F^\nu{}_\lambda + F^{\mu\lambda} \partial_\mu F^\nu{}_\lambda - \tfrac{1}{2} F_{\rho\lambda} \partial^\nu F^{\rho\lambda} \\ &\qquad - \mu^2 (\partial_\mu A^\mu A^\nu + A^\mu \partial_\mu A^\nu - A^\lambda \partial^\nu A_\lambda) \big] \\ &= -\frac{1}{4\pi} \left[\partial_\mu F^{\mu\lambda} F^\nu{}_\lambda + \tfrac{1}{2} F_{\rho\lambda} (2\partial^\rho F^{\nu\lambda} - \partial^\nu F^{\rho\lambda}) + \mu^2 A^\lambda (\partial^\nu A_\lambda - \partial_\lambda A^\nu) \right] \\ &= -\frac{1}{4\pi} \left[(\partial_\mu F^{\mu\lambda} + \mu^2 A^\lambda) F^\nu{}_\lambda + \tfrac{1}{2} F_{\rho\lambda} (\partial^\rho F^{\nu\lambda} + \partial^\lambda F^{\rho\nu} + \partial^\nu F^{\lambda\rho}) \right] \\ &= -\frac{1}{c} J^\lambda F^\nu{}_\lambda = \frac{1}{c} J_\lambda F^{\lambda\nu} \end{aligned}$$

Note that in the second line we have used the fact that $\partial_\mu A^\mu = 0$, which is automatic for the Proca equation. To obtain the last line, we used the Bianchi identity $3\partial^{[\rho}F^{\nu\lambda]} = 0$ as well as the Proca equation of motion.

c) Show explicitly that the time-time and space-time components of $\Theta^{\alpha\beta}$ are

$$\Theta^{00} = \frac{1}{8\pi}[E^2 + B^2 + \mu^2(A^0A^0 + \vec{A}\cdot\vec{A})]$$
$$\Theta^{i0} = \frac{1}{4\pi}[(\vec{E}\times\vec{B})_i + \mu^2 A^i A^0]$$

Given the explicit form of the Maxwell tensor, it is straightforward to show that

$$F^2 \equiv F_{\mu\nu}F^{\mu\nu} = -2(E^2 - B^2), \qquad A^2 \equiv A_\mu A^\mu = (A^0)^2 - \vec{A}^{\,2}$$

Thus

$$\Theta\mu\nu = -\frac{1}{4\pi}\left[F^{\mu\lambda}F^\nu{}_\lambda + \tfrac{1}{2}\eta^{\mu\nu}(E^2 - B^2) - \mu^2(A^\mu A^\nu - \tfrac{1}{2}\eta^{\mu\nu}((A^0)^2 - \vec{A}^{\,2}))\right]$$

The time-time component of this is

$$\begin{aligned}\Theta^{00} &= -\frac{1}{4\pi}\left[F^{0i}F^0{}_i + \tfrac{1}{2}(E^2 - B^2) - \mu^2((A^0)^2 - \tfrac{1}{2}((A^0)^2 - \vec{A}^{\,2}))\right]\\ &= -\frac{1}{4\pi}\left[-\tfrac{1}{2}(E^2 + B^2) - \tfrac{1}{2}\mu^2((A^0)^2 + \vec{A}^{\,2})\right]\\ &= \frac{1}{8\pi}\left[E^2 + B^2 + \mu^2((A^0)^2 + \vec{A}^{\,2})\right]\end{aligned}$$

Similarly, the time-space components are

$$\begin{aligned}\Theta^{0i} &= -\frac{1}{4\pi}\left[F^0{}_jF^{ij} - \mu^2A^0A^i\right] = -\frac{1}{4\pi}\left[E^j(-\epsilon_{ijk}B^k) - \mu^2A^0A^i\right]\\ &= -\frac{1}{4\pi}\left[-\epsilon_{ijk}E^jB^k - \mu^2A^0A^i\right] = \frac{1}{4\pi}\left[(\vec{E}\times\vec{B})^i + \mu^2A^0A^i\right]\end{aligned}$$

12.19 Source-free electromagnetic fields exist in a localized region of space. Consider the various conservation laws that are contained in the integral of $\partial_\alpha M^{\alpha\beta\gamma} = 0$ over all space, where $M^{\alpha\beta\gamma}$ is defined by (12.117).

a) Show that when β and γ are both space indices conservation of the total field angular momentum follows.

Note that

$$M^{\alpha\beta\gamma} = \Theta^{\alpha\beta}x^\gamma - \Theta^{\alpha\gamma}x^\beta$$

Hence

$$M^{0ij} = \Theta^{0i}x^j - \Theta^{0j}x^i = c(g^ix^j - g^jx^i) = c\epsilon^{ijk}(\vec{g}\times\vec{x})^k = -c\epsilon^{ijk}(\vec{x}\times\vec{g})^k$$

where $\vec{g}$ is the linear momentum density of the electromagnetic field. Since $\vec{x} \times \vec{g}$ is the angular momentum density, integrating M^{0ij} over 3-space gives the field angular momentum

$$M^{ij} \equiv \int M^{0ij} d^3x = -c\epsilon^{ijk} \int (\vec{x} \times \vec{g}\,)^k \, d^3x = -c\epsilon^{ijk} L^k$$

The conservation law $\partial_\mu M^{\mu ij} = 0$ then corresponds to the conservation of angular momentum in the electromagnetic field.

b) Show that when $\beta = 0$ the conservation law is

$$\frac{d\vec{X}}{dt} = \frac{c^2 \vec{P}_{\text{em}}}{E_{\text{em}}}$$

where $\vec{X}$ is the coordinate of the center of mass of the electromagnetic fields, defined by

$$\vec{X} \int u \, d^3x = \int \vec{x} u \, d^3x$$

where u is the electromagnetic energy density and E_{em} and $\vec{P}_{\text{em}}$ are the total energy and momentum of the fields.

In this case, we have

$$\begin{aligned} M^{0i} \equiv \int M^{00i} d^3x &= \int (\Theta^{00} x^i - \Theta^{0i} x^0) \, d^3x \\ &= \int (ux^i - cg^i x^0) \, d^3x = \int (ux^i - c^2 t g^i) \, d^3x \end{aligned}$$

Making use of the definition $\int ux^i d^3x = EX^i$ where $E = \int u \, d^3x$ is the total field energy, we have simply

$$M^{0i} = EX^i - c^2 t P^i$$

where $\vec{P} = \int \vec{g} \, d^3x$ is the (linear) field momentum. Since M^{0i} is a conserved charge, its time derivative must vanish. This gives

$$0 = \frac{d}{dt}(E\vec{X}) - c^2 \frac{d}{dt}(t\vec{P}) = E\frac{d\vec{X}}{dt} - c^2 \vec{P}$$

(where we used the fact that energy and momentum are conserved, namely $dE/dt = 0$ and $d\vec{P}/dt = 0$). The result $d\vec{X}/dt = c^2 \vec{P}/E$ then follows.