Physics 506 **Winter 2008**

Homework Assignment #10 — Due Thursday, March 27

Textbook problems: Ch. 12: 12.10, 12.13, 12.16, 12.19

12.10 A charged particle finds itself instantaneously in the equatorial plane of the earth's magnetic field (assumed to be a dipole field) at a distance R from the center of the earth. Its velocity vector at that instant makes an angle α with the equatorial plane ($v_\parallel/v_\perp = \tan\alpha$). Assuming that the particle spirals along the lines of force with a gyration radius $a \ll R$, and that the flux linked by the orbit is a constant of the motion, find an equation for the maximum magnetic latitude λ reached by the particle as a function of the angle α. Plot a graph (*not a sketch*) of λ versus α. Mark parametrically along the curve the values of α for which a particle at radius R in the equatorial plane will hit the earth (radius R_0) for $R/R_0 = 1.2, 1.5, 2.0, 2.5, 3, 4, 5$.

12.13 *a*) Specialize the Darwin Lagrangian (12.82) to the interaction of two charged particles (m_1, q_1) and (m_2, q_2). Introduce reduced particle coordinates, $\vec{r} = \vec{x}_1 - \vec{x}_2$, $\vec{v} = \vec{v}_1 - \vec{v}_2$ and also center of mass coordinates. Write out the Lagrangian in the reference frame in which the velocity of the center of mass vanishes and evaluate the canonical momentum components, $p_x = \partial L/\partial v_x$, etc.

b) Calculate the Hamiltonian to first order in $1/c^2$ and show that it is

$$H = \frac{p^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) + \frac{q_1 q_2}{r} - \frac{p^4}{8c^2}\left(\frac{1}{m_1^3} + \frac{1}{m_2^3}\right) + \frac{q_1 q_2}{2m_1 m_2 c^2}\left(\frac{p^2 + (\vec{p}\cdot\hat{r})^2}{r}\right)$$

[You may disregard the comparison with Bethe and Salpeter.]

12.16 *a*) Starting with the Proca Lagrangian density (12.91) and following the same procedure as for the electromagnetic fields, show that the symmetric stress-energy-momentum tensor for the Proca fields is

$$\Theta^{\alpha\beta} = \frac{1}{4\pi}\left[g^{\alpha\gamma}F_{\gamma\lambda}F^{\lambda\beta} + \frac{1}{4}g^{\alpha\beta}F_{\lambda\nu}F^{\lambda\nu} + \mu^2\left(A^\alpha A^\beta - \frac{1}{2}g^{\alpha\beta}A_\lambda A^\lambda\right)\right]$$

b) For these fields in interaction with the external source J^β, as in (12.91), show that the differential conservation laws take the same form as for the electromagnetic fields, namely

$$\partial_\alpha \Theta^{\alpha\beta} = \frac{J_\lambda F^{\lambda\beta}}{c}$$

c) Show explicitly that the time-time and space-time components of $\Theta^{\alpha\beta}$ are

$$\Theta^{00} = \frac{1}{8\pi}[E^2 + B^2 + \mu^2(A^0A^0 + \vec{A}\cdot\vec{A})]$$
$$\Theta^{i0} = \frac{1}{4\pi}[(\vec{E}\times\vec{B})_i + \mu^2 A^i A^0]$$

12.19 Source-free electromagnetic fields exist in a localized region of space. Consider the various conservation laws that are contained in the integral of $\partial_\alpha M^{\alpha\beta\gamma} = 0$ over all space, where $M^{\alpha\beta\gamma}$ is defined by (12.117).

a) Show that when β and γ are both space indices conservation of the total field angular momentum follows.

b) Show that when $\beta = 0$ the conservation law is

$$\frac{d\vec{X}}{dt} = \frac{c^2 \vec{P}_{\text{em}}}{E_{\text{em}}}$$

where $\vec{X}$ is the coordinate of the center of mass of the electromagnetic fields, defined by

$$\vec{X} \int u \, d^3x = \int \vec{x} u \, d^3x$$

where u is the electromagnetic energy density and E_{em} and $\vec{P}_{\text{em}}$ are the total energy and momentum of the fields.