Physics 506 **Winter 2008**

Homework Assignment #9 — Solutions

Textbook problems: Ch. 11: 11.27, 11.30
Ch. 12: 12.2, 12.3

11.27 *a*) A charge density ρ' of zero total charge, but with a dipole moment $\vec{p}$, exists in reference frame K'. There is no current density in K'. The frame K' moves with a velocity $\vec{v} = \vec{\beta}c$ in the frame K. Find the charge and current densities ρ and $\vec{J}$ in the frame K and show that there is a magnetic dipole moment, $\vec{m} = (\vec{p}\times\vec{\beta})/2$, correct to first order in β. What is the electric dipole moment in K to the same order in β?

We assume the charge density ρ' is static (independent of time t') in the rest frame. Thus $\rho' = \rho'(\vec{x}\,')$ is only a function of $\vec{x}\,'$. Furthermore, we define

$$0 = q' \equiv \int \rho' d^3x', \qquad \vec{p}\,' \equiv \int \vec{x}\,'\rho' d^3x'$$

We use the prime notation (ie $\vec{p}\,'$) to denote the electric dipole in the rest frame. To boost to the lab frame K, we first construct the 4-vector current $J'^{\mu} = (c\rho', 0)$. The boosted current is then

$$J^{\mu} = (\gamma c\rho', \gamma\vec{\beta}c\rho')$$

while the coordinates are related by

$$\begin{aligned} x^0 &= \gamma(x'^0 + \vec{\beta}\cdot\vec{x}\,') \\ \vec{x} &= \vec{x}\,' + \frac{\gamma-1}{\beta^2}\vec{\beta}(\vec{\beta}\cdot\vec{x}\,') + \gamma\vec{\beta}x'^0 \end{aligned} \qquad (1)$$

As a result, the lab frame charge and current densities are

$$\rho(x^0,\vec{x}) = \gamma\rho'(\vec{x}\,'), \qquad \vec{J}(x^0,\vec{x}) = \gamma\vec{v}\rho'(\vec{x}\,')$$

where $\vec{x}\,'$ may be given by the boost $\vec{x}\,' = \vec{x} + [(\gamma-1)/\beta^2]\vec{\beta}(\vec{\beta}\cdot\vec{x}) - \gamma\vec{\beta}x^0$, and we recall that ρ' is independent of x'^0.

In the lab frame, we define the electric and magnetic dipole moments as integrations over the source distrbutions at a fixed lab time x^0. For simplicity, we take $x^0 = 0$. In this case, the boost relations (1) may be solved to give $x'^0 = -\vec{\beta}\cdot\vec{x}\,'$ as well as

$$\vec{x} = \vec{x}\,' - \frac{\gamma}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{x}\,')$$

At fixed x^0 time, the 3-volume integration in frames K and K' are related by

$$\left[\int d^3x = \frac{1}{\gamma}\int d^3x'\right]_{x^0=\text{const}}$$

We now have enough information to relate moments computed in frames K and K'. In particular, for the magnetic dipole moment, we have

$$\begin{aligned}\vec{m} = \frac{1}{2c}\int \vec{x}\times\vec{J}d^3x &= \frac{1}{2c}\int\left(\vec{x}\,' - \frac{\gamma}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{x}\,')\right)\times(\gamma\vec{\beta}c\rho')\frac{d^3x'}{\gamma}\\ &= \frac{1}{2}\int\vec{x}\,'\times\vec{\beta}\rho'd^3x' = \left(\frac{1}{2}\int\vec{x}'\rho'd^3x'\right)\times\vec{\beta}\\ &= \tfrac{1}{2}\vec{p}\,'\times\vec{\beta}\end{aligned}$$

In fact, this is exact, not just to first order, but to all orders in β. The electric dipole moment calculation is similar

$$\begin{aligned}\vec{p} = \int\vec{x}\rho\,d^3x &= \int\left(\vec{x}\,' - \frac{\gamma}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{x}\,')\right)\gamma\rho'\frac{d^3x'}{\gamma}\\ &= \int\left(\vec{x}\,' - \frac{\gamma}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{x}\,')\right)\rho'd^3x'\\ &= p\,' - \frac{\gamma}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{p}\,')\end{aligned}$$

(If we had non-zero total charge, this expression would be corrected by the addition of a $q'\vec{v}t$ term; this was not apparent in the above, since we had only worked at fixed time $t = 0$.) This indicates that the electric dipole moment remains uncorrected to first order in β, ie $\vec{p} = \vec{p}\,' + \mathcal{O}(\beta^2)$.

b) Instead of the charge density, but no current density, in K', consider no charge density, but a current density $\vec{J}'$ that has a magnetic dipole moment $\vec{m}$. Find the charge and current densities in K and show that to first order in β there is an electric dipole moment $\vec{p} = \vec{\beta}\times\vec{m}$ in addition to the magnetic dipole moment.

In this case, the 4-current densities are

$$J'^{\mu} = (0, \vec{J}'), \qquad J^{\mu} = \left(\gamma\vec{\beta}\cdot\vec{J}', \vec{J}' + \frac{\gamma-1}{\beta^2}\vec{\beta}(\vec{\beta}\cdot\vec{J}')\right)$$

For the lab electric dipole moment, we have

$$\begin{aligned}\vec{p} = \int\vec{x}\rho\,d^3x &= \int\left(\vec{x}\,' - \frac{\gamma}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{x}\,')\right)\frac{\gamma}{c}\vec{\beta}\cdot\vec{J}'\frac{d^3x'}{\gamma}\\ &= \frac{1}{c}\int\left(\vec{x}\,' - \frac{\gamma}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{x}\,')\right)(\vec{\beta}\cdot\vec{J}')\,d^3x'\end{aligned} \tag{2}$$

We now recall the identity (written in 3-space, and without primes for simplicity, but valid in either K or K')

$$\begin{aligned}\int x_i J_j d^3x &= \tfrac{1}{2}\int (x_i J_j + x_j J_i) d^3x + \tfrac{1}{2}\int (x_i J_j - x_j J_i) d^3x \\ &= \tfrac{1}{2}\int J_k \partial_k (x_i x_j)\, d^3x + \tfrac{1}{2}\epsilon_{ijk}\int [\vec{x}\times\vec{J}]_k\, d^3x \\ &= -\tfrac{1}{2}\int x_i x_j (\vec{\nabla}\cdot\vec{J})\, d^3x + \tfrac{1}{2}\epsilon_{ijk}\int [\vec{x}\times\vec{J}]_k\, d^3x\end{aligned}$$

(where we have used integration by parts). In the K' frame, and with a static current density, current conservation gives $\vec{\nabla}'\cdot\vec{J}'=0$. Moreover, the antisymmetric term corresponds to the magnetic dipole moment $\vec{m}'$. This indicates that

$$\int x'_i J'_j d^3x' = c\epsilon_{ijk} m'_k$$

Inserting this into (2) gives

$$p_i = \epsilon_{ijk}\beta_j m'_k - \frac{\gamma}{\gamma+1}\beta_i\beta_j\epsilon_{jln}\beta_l m'_n = \epsilon_{ijk}\beta_j m'_k$$

or, in vector notation

$$\vec{p} = \vec{\beta}\times\vec{m}\,'$$

The magnetic dipole moment picks up a correction

$$\begin{aligned}\vec{m} &= \frac{1}{2c}\int \vec{x}\times\vec{J}\, d^3x \\ &= \frac{1}{2c}\int \left(\vec{x}\,' - \frac{\gamma}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{x}\,')\right)\times\left(\vec{J}' + \frac{\gamma-1}{\beta^2}\vec{\beta}(\vec{\beta}\cdot\vec{J}')\right)\frac{d^3x'}{\gamma} \\ &= \frac{1}{2c\gamma}\int \left(\vec{x}\times\vec{J}' - \frac{\gamma}{\gamma+1}(\vec{\beta}\times\vec{J}')(\vec{\beta}\cdot\vec{x}\,') - \frac{\gamma-1}{\beta^2}(\vec{\beta}\times\vec{x}\,')(\vec{\beta}\cdot\vec{J}')\right) d^3x' \\ &= \frac{1}{\gamma}\vec{m}\,' + \frac{1}{2\gamma}\left(\frac{\gamma-1}{\beta^2} - \frac{\gamma}{\gamma+1}\right)(\beta^2\vec{m}\,' - \vec{\beta}(\vec{\beta}\cdot\vec{m}\,')) \\ &= (1-\tfrac{1}{2}\beta^2)\vec{m}\,' - \frac{\gamma-1}{2(\gamma+1)}\vec{\beta}(\vec{\beta}\cdot\vec{m}\,')\end{aligned}$$

To first order in β, this is simply $\vec{m} = \vec{m}\,' + \mathcal{O}(\beta^2)$. Finally, we note that the results of a and b are not directly related by electric/magnetic duality. This is because physically there is actually a subtle difference between magnetic dipole moments generated by current loops versus ones generated by magnetic monopoles.

11.30 An isotropic linear material medium, characterized by the constitutive relations (in its rest frame K'), $\vec{D}' = \epsilon\vec{E}'$ and $\mu\vec{H}' = \vec{B}'$, is in uniform translation with velocity $\vec{v}$ in the inertial frame K. By exploiting the fact that $F_{\mu\nu} = (\vec{E}, \vec{B})$ and $G_{\mu\nu} = (\vec{D}, \vec{H})$ transform as second rank 4-tensors under Lorentz transformations, show that the macroscopic fields $\vec{D}$ and $\vec{H}$ are given in terms of $\vec{E}$ and $\vec{B}$ by

$$\vec{D} = \epsilon\vec{E} + \gamma^2\left(\epsilon - \frac{1}{\mu}\right)[\beta^2\vec{E}_\perp + \vec{\beta}\times\vec{B}]$$
$$\vec{H} = \frac{1}{\mu}\vec{B} + \gamma^2\left(\epsilon - \frac{1}{\mu}\right)[-\beta^2\vec{B}_\perp + \vec{\beta}\times\vec{E}]$$

where $\vec{E}_\perp$ and $\vec{B}_\perp$ are components perpendicular to $\vec{v}$.

Since $F_{\mu\nu}$ transforms as a rank-2 tensor, we have seen that the components $\vec{E}$ and $\vec{B}$ transform according to

$$\vec{E}' = \gamma(\vec{E} + \vec{\beta}\times\vec{B}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{E})$$
$$\vec{B}' = \gamma(\vec{B} - \vec{\beta}\times\vec{E}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{B})$$

For this problem, it is actually convenient to rewrite these expressions in terms of the perpendicular and parallel field components

$$\begin{aligned}\vec{E}' &= \gamma(\vec{E}_\perp + \vec{\beta}\times\vec{B}) + \hat{\beta}(\hat{\beta}\cdot\vec{E})\\ \vec{B}' &= \gamma(\vec{B}_\perp - \vec{\beta}\times\vec{E}) + \hat{\beta}(\hat{\beta}\cdot\vec{B})\end{aligned} \tag{3}$$

where

$$\vec{E}_\perp = \vec{E} - \hat{\beta}(\hat{\beta}\cdot\vec{E}) = -\hat{\beta}\times(\hat{\beta}\times\vec{E})$$

(and similarly for $\vec{B}_\perp$). Since these are the relativistic transformations of the components of a rank-2 tensor, any other rank-2 tensor must transform similarly. In particular, since $\vec{D}$ and $\vec{H}$ are components of the $G_{\mu\nu}$ tensor, they also transform as

$$\vec{D}' = \gamma(\vec{D}_\perp + \vec{\beta}\times\vec{H}) + \hat{\beta}(\hat{\beta}\cdot\vec{D})$$
$$\vec{H}' = \gamma(\vec{H}_\perp - \vec{\beta}\times\vec{D}) + \hat{\beta}(\hat{\beta}\cdot\vec{H})$$

The inverse transformation may be obtained by taking $\vec{\beta}\to-\vec{\beta}$

$$\vec{D} = \gamma(\vec{D}'_\perp - \vec{\beta}\times\vec{H}') + \hat{\beta}(\hat{\beta}\cdot\vec{D}')$$
$$\vec{H} = \gamma(\vec{H}'_\perp + \vec{\beta}\times\vec{D}') + \hat{\beta}(\hat{\beta}\cdot\vec{H}')$$

Using the constitutive relations $\vec{D}' = \epsilon\vec{E}'$ and $\mu\vec{H}' = \vec{B}'$ gives

$$\begin{aligned}\vec{D} &= \gamma\left(\epsilon\vec{E}'_\perp - \frac{1}{\mu}\vec{\beta}\times\vec{B}'\right) + \epsilon\hat{\beta}(\hat{\beta}\cdot\vec{E}')\\ \vec{H} &= \gamma\left(\frac{1}{\mu}\vec{B}'_\perp + \epsilon\vec{\beta}\times\vec{E}'\right) + \frac{1}{\mu}\hat{\beta}(\hat{\beta}\cdot\vec{B}')\end{aligned} \tag{4}$$

It is now a simple matter of substituting (3) into the right hand side of the above and simplifying the resulting expressions to obtain the desired relations between $(\vec{D}, \vec{H})$ and $(\vec{E}, \vec{B})$.

To proceed, we split (3) into parallel and perpendicular field components

$$\vec{\beta} \cdot \vec{E}' = \vec{\beta} \cdot \vec{E}, \qquad \vec{\beta} \cdot \vec{B}' = \vec{\beta} \cdot \vec{B}$$

and

$$\vec{E}'_\perp = \gamma(\vec{E}_\perp + \vec{\beta} \times \vec{B}), \qquad \vec{B}'_\perp = \gamma(\vec{B}_\perp - \vec{\beta} \times \vec{E})$$

As a result, we easily see that

$$\vec{\beta} \times \vec{E}' = \gamma(\vec{\beta} \times \vec{E} - \beta^2 \vec{B}_\perp), \qquad \vec{\beta} \times \vec{B}' = \gamma(\vec{\beta} \times \vec{B} + \beta^2 \vec{E}_\perp)$$

Substituting these expressions into (4) gives

$$\begin{aligned}
\vec{D} &= \gamma^2 \left(\epsilon(\vec{E}_\perp + \vec{\beta} \times \vec{B}) - \frac{1}{\mu}(\vec{\beta} \times \vec{B} + \beta^2 \vec{E}_\perp) \right) + \epsilon \hat{\beta}(\hat{\beta} \cdot \vec{E}) \\
\vec{H} &= \gamma^2 \left(\frac{1}{\mu}(\vec{B}_\perp - \vec{\beta} \times \vec{E}) + \epsilon(\vec{\beta} \times \vec{E} - \beta^2 \vec{B}_\perp) \right) + \frac{1}{\mu} \hat{\beta}(\hat{\beta} \cdot \vec{B})
\end{aligned}$$

which simplifies to

$$\begin{aligned}
\vec{D} &= \epsilon[\vec{E}_\perp + \hat{\beta}(\hat{\beta} \cdot \vec{E})] + \gamma^2 \left(\epsilon - \frac{1}{\mu} \right) (\beta^2 \vec{E}_\perp + \vec{\beta} \times \vec{B}) \\
\vec{H} &= \frac{1}{\mu}[\vec{B}_\perp + \hat{\beta}(\hat{\beta} \cdot \vec{B})] - \gamma^2 \left(\epsilon - \frac{1}{\mu} \right) (\beta^2 \vec{B}_\perp - \vec{\beta} \times \vec{E})
\end{aligned}$$

or equivalently

$$\begin{aligned}
\vec{D} &= \epsilon \vec{E} + \gamma^2 \left(\epsilon - \frac{1}{\mu} \right) (\beta^2 \vec{E}_\perp + \vec{\beta} \times \vec{B}) \\
\vec{H} &= \frac{1}{\mu} \vec{B} + \gamma^2 \left(\epsilon - \frac{1}{\mu} \right) (-\beta^2 \vec{B}_\perp + \vec{\beta} \times \vec{E})
\end{aligned} \tag{5}$$

Note that an alternate way of deducing these expressions is to start in the rest frame, where the four-velocity is

$$U^\mu = c(1, \vec{0}) \qquad \text{(rest frame)}$$

Since U^μ specifies the time direction, we may introduce a projection operator

$$\Pi^{\mu\nu} = \eta^{\mu\nu} - U^\mu U^\nu / c^2$$

This allows us to project onto the space components of a tensor. In particular, since the magnetic field is encoded in the space-space components of the Maxwell field strength $F^{\mu\nu}$, we may write

$$B^{\mu\nu} = \Pi^{\mu\alpha} F_{\alpha\beta} \Pi^{\beta\nu} = F^{\mu\nu} - \frac{1}{c^2}\left(F^{\mu\lambda} U_\lambda U^\nu + U^\mu U_\lambda F^{\lambda\nu}\right)$$

The electric field comes from the space-time and time-space components

$$E^{\mu\nu} = F^{\mu\nu} - B^{\mu\nu} = \frac{1}{c^2}\left(F^{\mu\lambda} U_\lambda U^\nu + U^\mu U_\lambda F^{\lambda\nu}\right)$$

In particular, the full Maxwell field strength is a sum of the electric and magnetic field components

$$F^{\mu\nu} = E^{\mu\nu} + B^{\mu\nu}$$

With this in mind, we may write the macroscopic field strength tensor $G^{\mu\nu}$ as a sum of ϵ times the electric field and $1/\mu$ times the magnetic field

$$\begin{aligned} G^{\mu\nu} &= \epsilon E^{\mu\nu} + \frac{1}{\mu} B^{\mu\nu} \\ &= \frac{1}{\mu} F^{\mu\nu} + \frac{1}{c^2}\left(\epsilon - \frac{1}{\mu}\right)\left(F^{\mu\lambda} U_\lambda U^\nu + U^\mu U_\lambda F^{\lambda\nu}\right) \end{aligned}$$

Although this expression was derived in the rest frame, since it is written in terms of 4-vectors and 4-tensors, it is valid in any frame. For a moving frame where

$$U^\mu = c\gamma(1, \vec{\beta})$$

we may work out the explicit form of $G^{\mu\nu}$ and show that it gives the same result as (5).

12.2 *a*) Show from Hamilton's principle that Lagrangians that differ only by a total time derivative of some function of the coordinates and time are equivalent in the sense that they yield the same Euler-Lagrange equations of motion.

Suppose Lagrangians L_1 and L_2 differ by a total time derivative of the form

$$L_2 = L_1 + \frac{d}{dt} f(q_i(t), t)$$

Writing out the time derivative explicitly gives

$$L_2 = L_1 + \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial t}$$

The Euler-Lagrange equations for L_2 are derived from

$$\begin{aligned} \frac{\partial L_2}{\partial q_i} &= \frac{\partial L_1}{\partial q_i} + \frac{\partial^2 f}{\partial q_i \partial q_j}\dot{q}_j + \frac{\partial^2 f}{\partial q_i \partial t} \\ \frac{\partial L_2}{\partial \dot{q}_i} &= \frac{\partial L_1}{\partial \dot{q}_i} + \frac{\partial f}{\partial q_i} \end{aligned}$$

Then

$$\begin{aligned}
\frac{\partial L_2}{\partial q_i} - \frac{d}{dt}\frac{\partial L_2}{\partial \dot{q}_i} &= \frac{\partial L_1}{\partial q_i} - \frac{d}{dt}\frac{\partial L_1}{\partial \dot{q}_i} + \frac{\partial^2 f}{\partial q_i \partial q_j}\dot{q}_j + \frac{\partial^2 f}{\partial q_i \partial t} - \frac{d}{dt}\frac{\partial f}{\partial q_i} \\
&= \frac{\partial L_1}{\partial q_i} - \frac{d}{dt}\frac{\partial L_1}{\partial \dot{q}_i} + \frac{\partial}{\partial q_i}\frac{df}{dt} - \frac{d}{dt}\frac{\partial f}{\partial q_i} \\
&= \frac{\partial L_1}{\partial q_i} - \frac{d}{dt}\frac{\partial L_1}{\partial \dot{q}_i}
\end{aligned}$$

As a result, both L_1 and L_2 yield the same Euler-Lagrange equations.

Note that it is perhaps more straightforward to consider the change in the action

$$S_2 = \int_{t_1}^{t_2} L_2\, dt = \int_{t_1}^{t_2} (L_1 + \frac{df}{dt})\, dt = \int_{t_1}^{t_2} L_1\, dt + f(q_i(t_2), t_2) - f(q_i(t_1), t_1)$$

In other words, the additional of a total time derivative only changes the action by a surface term. So long as we do not vary the path at its endpoints ($\delta q_i(t_1) = \delta q_i(t_2) = 0$) we end up with the same equations of motion.

b) Show explicitly that the gauge transformation $A^\alpha \to A^\alpha + \partial^\alpha \Lambda$ of the potentials in the charged-particle Lagrangian (12.12) merely generates another equivalent Lagrangian.

We start with the Lagrangian

$$L = -mc^2\sqrt{1 - u^2/c^2} + \frac{e}{c}\vec{u}\cdot\vec{A} - e\Phi$$

In components, the gauge transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$ reads

$$\Phi \to \Phi + \frac{1}{c}\frac{\partial}{\partial t}\Lambda, \qquad \vec{A} \to \vec{A} - \vec{\nabla}\Lambda$$

In this case, the Lagrangian changes by the term

$$\delta L = -\frac{e}{c}\left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)\Lambda$$

However, for $\Lambda = \Lambda(\vec{x}(t), t)$, the above is just the total time derivative

$$\delta L = -\frac{e}{c}\frac{d\Lambda}{dt}$$

As a result the Lagrangian changes by a total time derivative. Thus the gauge transformed Lagrangian is equivalent to the original one in the sense of part a.

12.3 A particle with mass m and charge e moves in a uniform, static, electric field $\vec{E}_0$.

a) Solve for the velocity and position of the particle as explicit functions of time, assuming that the initial velocity $\vec{v}_0$ was perpendicular to the electric field.

The charged particle dynamics is governed by the equation of motion

$$\frac{dU^\mu}{d\tau} = \frac{e}{mc} F^{\mu\nu} U_\nu$$

For a constant electric field only, this breaks up into time and space components

$$\frac{dU^0}{d\tau} = \frac{e}{mc}\vec{E}_0 \cdot \vec{u}, \qquad \frac{d\vec{u}}{d\tau} = \frac{e}{mc}\vec{E}_0 U^0 \tag{6}$$

where we have taken $U^\mu = (U^0, \vec{u})$. These set of coupled equations may be solved by taking a proper time derivative of the first equation and noting that

$$\frac{d^2 U^0}{d\tau^2} = \frac{e}{mc}\vec{E}_0 \cdot \frac{d\vec{u}}{d\tau} = \left(\frac{eE_0}{mc}\right)^2 U^0$$

Thus

$$U^0 = Ae^{(eE_0/mc)\tau} + Be^{-(eE_0/mc)\tau}$$

Substituting this into the second equation of (6) and integrating gives

$$\vec{u} = \vec{u}_0 + \hat{E}_0(Ae^{(eE_0/mc)\tau} - Be^{-(eE_0/mc)\tau})$$

Since we took a proper time derivative of the first equation of (6) and converted it into a second order equation, it is important to check that the original first order equation is satisfied. When we do this, we find the constraint

$$\vec{E}_0 \cdot \vec{u}_0 = 0$$

This indicates that the proper velocity $\vec{u}$ has two components, a τ-dependent parallel component in the direction $\hat{E}_0$ and a constant perpendicular component $\vec{u}_0$. At this stage, the solution for the four-velocity is given in terms of four independent parameters: A, B and two components of $\vec{u}_0$ perpendicular to the electric field. Note however, that the four-velocity satisfies the constraint $U^\mu U_\mu = c^2$. This gives rise to a relation among the parameters

$$4AB - \vec{u}_0^2 = c^2$$

Before proceeding, we simplify the notation by choosing proper time $\tau = 0$ to correspond to the actual time $t = 0$. In this case, the condition that the initial parallel velocity vanishes is equivalent to demanding that the component of $\vec{u}$

along the $\hat{E}_0$ direction vanishes at $\tau = 0$. This is satisfied by taking $A = B$, so that

$$U^0 = 2A\cosh\left(\frac{eE_0\tau}{mc}\right), \qquad \vec{u} = \vec{u}_0 + 2A\hat{E}_0 \sinh\left(\frac{eE_0\tau}{mc}\right)$$

where

$$2A = c\sqrt{1 + u_0^2/c^2}$$

The parameters of the solution are the two independent perpendicular components of the initial proper velocity $\vec{u}_0$. This may be converted to ordinary velocity by the usual relation

$$\vec{u}_0 = \gamma_0 \vec{v}_0 \qquad \text{where} \qquad \gamma_0^2 = \frac{1}{1 - v_0^2/c^2}$$

The result is

$$U^0 = c\gamma_0 \cosh\left(\frac{eE_0\tau}{mc}\right), \qquad \vec{u} = \gamma_0\vec{v}_0 + c\gamma_0\hat{E}_0 \sinh\left(\frac{eE_0\tau}{mc}\right)$$

We may now obtain the position four-vector by integrating

$$\begin{aligned} ct = x^0 &= \int_0^\tau U^0 d\tau' = \frac{mc^2\gamma_0}{eE_0} \sinh\left(\frac{eE_0\tau}{mc}\right) \\ \vec{x} &= \int_0^\tau \vec{u}\, d\tau' = \gamma_0\vec{v}_0\tau + \frac{mc^2\gamma_0}{eE_0}\hat{E}_0\left[\cosh\left(\frac{eE_0\tau}{mc}\right) - 1\right] \end{aligned} \tag{7}$$

Note that we have chosen the initial position to be at $x^\mu = 0$.

The above expressions have been given in terms of the proper time τ. To obtain $\vec{v}$ and $\vec{x}$ as a position of time, we note that

$$t = \frac{mc\gamma_0}{eE_0} \sinh\left(\frac{eE_0\tau}{mc}\right) \qquad \Rightarrow \qquad \tau = \frac{mc}{eE_0} \sinh^{-1}\left(\frac{eE_0 t}{mc\gamma_0}\right)$$

This gives

$$\gamma = \frac{U^0}{c} = \gamma_0\sqrt{1 + \left(\frac{eE_0 t}{mc\gamma_0}\right)^2}, \qquad \vec{u} = \gamma_0\left(\vec{v}_0 + \frac{e\vec{E}_0 t}{m\gamma_0}\right) \tag{8}$$

Hence

$$\vec{v}(t) = \frac{\vec{u}}{\gamma} = \left[1 + \left(\frac{eE_0 t}{mc\gamma_0}\right)^2\right]^{-1/2}\left(\vec{v}_0 + \frac{e\vec{E}_0 t}{m\gamma_0}\right) \tag{9}$$

For the position, we have

$$\vec{x}(t) = \frac{mc\gamma_0}{eE_0}\vec{v}_0 \sinh^{-1}\left(\frac{eE_0 t}{mc\gamma_0}\right) + \frac{mc^2\gamma_0}{eE_0}\hat{E}_0\left[\sqrt{1 + \left(\frac{eE_0 t}{mc\gamma_0}\right)^2} - 1\right]$$

Note that, for short times $t \ll mc\gamma_0/eE_0$, the velocity and position may be expanded as

$$\vec{v} \approx \vec{v}_0 + \frac{e\vec{E}_0 t}{m\gamma_0}, \qquad \vec{x} \approx \vec{v}_0 t + \frac{e\vec{E}_0}{2m\gamma_0} t^2$$

which corresponds to integrating

$$\frac{d\vec{p}}{dt} = e\vec{E}_0 \qquad \vec{p} = \gamma m\vec{v}$$

under the approximation $\gamma \approx \gamma_0$. In the non-relativistic case ($\gamma \approx 1$), this corresponds to the familiar uniform acceleration in a uniform electric field, $\vec{a} = (e/m)\vec{E}_0$.

Note that an alternate solution would be to start with the 3-vector equation

$$\frac{d\vec{p}}{dt} = e\vec{E}_0$$

which may immediately be integrated to yield

$$\vec{p} = \vec{p}_0 + e\vec{E}_0 t$$

Since we want to work relativistically, we must use $\vec{p} = \gamma m\vec{v}$. As a result, the velocity is

$$\gamma\vec{v} = \gamma_0\vec{v}_0 + \frac{e}{m}\vec{E}_0 t \tag{10}$$

This expression looks simple, except that the left-hand side is actually a non-linear function of the velocity (because $\gamma = 1/\sqrt{1 - v^2/c^2}$). To proceed, we may square this expression to obtain

$$\beta^2\gamma^2 = \beta_0^2\gamma_0^2 + \left(\frac{eE_0 t}{mc}\right)^2$$

where we have made use of the initial condition $\vec{E}_0 \cdot \vec{v}_0 = 0$. Noting that $\beta^2\gamma^2 = \gamma^2 - 1$ allows us to solve for γ

$$\gamma^2 = \gamma_0^2 + \left(\frac{eE_0 t}{mc}\right)^2$$

which is perhaps a quicker way to arrive at the γ factor of (8). Inserting this into (10) then gives

$$\vec{v} = \frac{\gamma_0}{\gamma}\left(\vec{v}_0 + \frac{e\vec{E}_0 t}{m\gamma_0}\right)$$

which agrees with (9). This can be integrated once more (using trig substitution) to get the position as a function of time.

b) Eliminate the time to obtain the trajectory of the particle in space. Discuss the shape of the path for short and long times (define "short" and "long" times).

It is best to work with (7) to eliminate time. Splitting the directions into perpendicular and parallel components, we write

$$x_{\|} = \frac{mc^2\gamma_0}{eE_0}\left[\cosh\left(\frac{eE_0\tau}{mc}\right) - 1\right], \qquad x_{\perp} = \gamma_0 v_0 \tau$$

Hence

$$x_{\|} = \frac{mc^2\gamma_0}{eE_0}\left[\cosh\left(\frac{eE_0}{mc\gamma_0}\frac{x_{\perp}}{v_0}\right) - 1\right]$$

Using the same criteria for short times ($t \ll mc\gamma_0/eE_0$), we may expand the cosh to obtain

$$x_{\|} \approx \frac{eE_0}{2m\gamma_0 v_0^2} x_{\perp}^2$$

demonstrating that the short time path is parabolic. For long times ($t \gg mc\gamma_0/eE_0$), we have instead an exponential path

$$x_{\|} \approx \frac{mc^2\gamma_0}{2eE_0}\exp\left(\frac{eE_0}{mc\gamma_0}\frac{x_{\perp}}{v_0}\right)$$