Physics 506 **Winter 2008**

Homework Assignment #8 — Solutions

Textbook problems: Ch. 11: 11.5, 11.13, 11.14, 11.18

11.5 A coordinate system K' moves with a velocity $\vec{v}$ relative to another system K. In K' a particle has a velocity $\vec{u}\,'$ and an acceleration $\vec{a}\,'$. Find the Lorentz transformation law for accelerations, and show that in the system K the components of acceleration parallel and perpendicular to $\vec{v}$ are

$$\vec{a}_\parallel = \frac{(1-v^2/c^2)^{3/2}}{(1+\vec{v}\cdot\vec{u}\,'/c^2)^3}\vec{a}_\parallel{}'$$
$$\vec{a}_\perp = \frac{(1-v^2/c^2)}{(1+\vec{v}\cdot\vec{u}\,'/c^2)^3}\left(\vec{a}_\perp{}' + \frac{\vec{v}}{c^2}\times(\vec{a}\,'\times\vec{u}\,')\right)$$

Instead of working directly with perpendicular and parallel components, we may start with a particular boost in the x-t direction, and then generalize our results. We thus take a boost of the form

$$x^0 = \gamma(x^{0\prime} + \beta x'), \qquad x = \gamma(x' + \beta x^{0\prime}), \qquad y = y', \qquad z = z' \tag{1}$$

Note that $\gamma = 1/\sqrt{1-\beta^2}$ and $\beta = v/c$ are *constants* specifying the Lorentz boost. In frame K, the path of a particle is specified by the vector function $\vec{x}(x^0)$, while in frame K' this is instead $\vec{x}'(x^{0\prime})$. Three-velocities and 3-accelerations are then defined in a frame dependent manner

$$\text{frame } K: \qquad \vec{u} = c\frac{\partial\vec{x}}{\partial x^0}, \qquad \vec{a} = c\frac{\partial\vec{u}}{\partial x^0}$$
$$\text{frame } K': \qquad \vec{u}\,' = c\frac{\partial\vec{x}\,'}{\partial x^{0\prime}}, \qquad \vec{a}\,' = c\frac{\partial\vec{u}}{\partial x^{0\prime}}$$

To transform between the two frames, we need not just the transformation of the 3-vectors, but also the transformation relating times x^0 and $x^{0\prime}$. Noting from (1) that a particle following a path $\vec{x}\,'(x^{0\prime})$ yields a time relation

$$x^0 = \gamma(x^{0\prime} + \beta x'(x^{0\prime}))$$

we may write

$$\frac{dx^0}{dx^{0\prime}} = \gamma(1+\beta u'_x/c)$$

The inverse relation is simply

$$\frac{dx^{0\prime}}{dx^0} = \frac{1}{\gamma(1+\beta u'_x/c)}$$

This useful expression is basically all we need. We start with velocities

$$u_x = c\frac{dx}{dx^0} = c\frac{dx^{0\prime}}{dx^0}\frac{dx}{dx^{0\prime}} = \frac{c}{\gamma(1+\beta u'_x/c)}\frac{d}{dx^{0\prime}}\gamma(x'+\beta x^{0\prime}) = \frac{u'_x+c\beta}{1+\beta u'_x/c} \tag{2}$$

and

$$u_y = c\frac{dy}{dx^0} = c\frac{dx^{0\prime}}{dx^0}\frac{dy}{dx^{0\prime}} = \frac{c}{\gamma(1+\beta u'_x/c)}(u'_y/c) = \frac{u'_y}{\gamma(1+\beta u'_x/c)} \tag{3}$$

Writing $\beta u'_x = \vec{\beta}\cdot\vec{u}'$, and noting that the x direction is the parallel direction while the y direction is the perpendicular direction, it is easy to see that these velocity transformations may be written as

$$\vec{u}_\parallel = \frac{\vec{u}'_\parallel + c\vec{\beta}}{1+\vec{\beta}\cdot\vec{u}\,'/c}, \qquad \vec{u}_\perp = \frac{\vec{u}'_\perp}{\gamma(1+\vec{\beta}\cdot\vec{u}\,'/c)}$$

We now go on to accelerations. From (2), we have

$$\begin{aligned} a_x = c\frac{du_x}{dx^0} &= \frac{c}{\gamma(1+\beta u'_x/c)}\frac{d}{dx^{0\prime}}\frac{u'_x+c\beta}{1+\beta u'_x/c} \\ &= \frac{c}{\gamma(1+\beta u'_x/c)}\frac{(1+\beta u'_x/c)(a'_x/c)-(u'_x+c\beta)(\beta a'_x/c^2)}{(1+\beta u'_x/c)^2} \\ &= \frac{(1-\beta^2)a'_x}{\gamma(1+\beta u'_x/c)^3} = \frac{a'_x}{\gamma^3(1+\beta u'_x/c)^3} \end{aligned}$$

And from (3) we have

$$\begin{aligned} a_y = c\frac{du_y}{dx^0} &= \frac{c}{\gamma(1+\beta u'_x/c)}\frac{d}{dx^{0\prime}}\frac{u'_y}{\gamma(1+\beta u'_x/c)} \\ &= \frac{c}{\gamma^2(1+\beta u'_x/c)}\frac{(1+\beta u'_x/c)(a'_y/c)-u'_y(\beta a'_x/c^2)}{(1+\beta u'_x/c)^2} \\ &= \frac{a'_y+\beta(u'_x a'_y - u'_y a'_x)/c}{\gamma^2(1+\beta u'_x/c)^3} \end{aligned} \tag{4}$$

It is straightforward to convert the expression for a_x into one for $\vec{a}_\parallel$. The result is

$$\vec{a}_\parallel = \frac{\vec{a}'_\parallel}{\gamma^3(1+\vec{\beta}\cdot\vec{u}\,'/c)^3}$$

For the perpendicular direction, we have to be a bit more clever. Noting that x components in (4) are related to $\vec{\beta}\cdot(\ )$, while y components are directly related to the $\perp$ direction, we have

$$\vec{a}_\perp = \frac{\vec{a}'_\perp + \vec{a}\,'(\vec{\beta}\cdot\vec{u}\,') - \vec{u}'(\vec{\beta}\cdot\vec{a}\,')/c}{\gamma^2(1+\vec{\beta}\cdot\vec{u}\,'/c)^3}$$

Use of the BAC–CAB rule finally gives

$$\vec{a}_\perp = \frac{\vec{a}\,'_\perp + \vec{\beta} \times (\vec{a}\,' \times \vec{u}\,')/c}{\gamma^2(1 + \vec{\beta} \cdot \vec{u}\,'/c)^3}$$

11.13 An infinitely long straight wire of negligible cross-sectional area is at rest and has a uniform linear charge density q_0 in the inertial frame K'. The frame K' (and the wire) move with a velocity $\vec{v}$ parallel to the direction of the wire with respect to the laboratory frame K.

a) Write down the electric and magnetic fields in cylindrical coordinates in the rest frame of the wire. Using the Lorentz transformation properties of the fields, find the components of the electric and magnetic fields in the laboratory.

The K' frame is the rest frame of the wire, while the K frame may be considered as the lab frame. Starting in the K' rest frame, we take the wire to be oriented along the $\hat{z}$ direction. Since the charges are at rest in K', there is no magnetic field. The electric field is given by a simple application of Gauss' law. Thus (in cylindrical coordinates, and with Gaussian units)

$$\vec{E}' = \frac{2q_0}{\rho'}\hat{\rho}, \qquad \vec{B}' = 0$$

We now transform to the lab frame K using a boost along the $\hat{z}$ axis $\vec{\beta} = (v/c)\hat{z}$. In general, the electric and magnetic fields are related by

$$\vec{E} = \gamma(\vec{E}' - \vec{\beta} \times \vec{B}') - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta} \cdot \vec{E}')$$
$$\vec{B} = \gamma(\vec{B}' + \vec{\beta} \times \vec{E}') - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta} \cdot \vec{B}')$$

In this case, since $\vec{B}' = 0$ and $\vec{\beta} \cdot \vec{E}' = 0$ we have simply

$$\vec{E} = \gamma\vec{E}' = \frac{2\gamma q_0}{\rho'}\hat{\rho}$$
$$\vec{B} = \gamma\vec{\beta} \times \vec{E}' = \frac{2v\gamma q_0}{c\rho'}\hat{\phi}$$

Finally, noting that the $\hat{\rho}$ direction is transverse to the boost, we have the simple relation $\rho' = \rho$ (ie there is no length contraction in the transverse direction). Thus we find

$$\vec{E} = \frac{2\gamma q_0}{\rho}\hat{\rho}, \qquad \vec{B} = \frac{2v\gamma q_0}{c\rho}\hat{\phi} \tag{5}$$

b) What are the charge and current densities associated with the wire in its rest frame? In the laboratory?

In the rest frame, and in cylindrical coordinates, the charge density may be expressed as

$$\rho' = q_0 \frac{\delta(\rho')}{2\pi\rho'}$$

Since the 3-current vanishes, the 4-current density may be written as

$$J'^{\mu} = \left(c q_0 \frac{\delta(\rho')}{2\pi\rho'}, 0 \right)$$

Boosting back to the lab frame K gives

$$J^{\mu} = (\gamma J'^0, \vec{\beta}\gamma J'^0) = \left(c\gamma q_0 \frac{\delta(\rho)}{2\pi\rho}, v\gamma q_0 \hat{z} \frac{\delta(\rho)}{2\pi\rho} \right)$$

where we once again used the fact that $\rho = \rho'$. The explicit charge and 3-current densities are

$$\rho = \gamma q_0 \frac{\delta(\rho)}{2\pi\rho}, \qquad \vec{J} = v\gamma q_0 \hat{z} \frac{\delta(\rho)}{2\pi\rho} \tag{6}$$

c) From the laboratory charge and current densities, calculate directly the electric and magnetic fields in the laboratory. Compare with the results of part a.

From (6), the wire in the lab frame has linear charge density γq_0 and carries a current $I = v\gamma q_0$ in the $\hat{z}$ direction. Gauss' law and Ampère's law then gives

$$\vec{E} = \frac{2\gamma q_0}{\rho}\hat{\rho}, \qquad \vec{B} = \frac{2v\gamma q_0}{c\rho}\hat{\phi}$$

in agreement with the result (5) of part a.

11.14 *a*) Express the Lorentz scalars $F^{\alpha\beta}F_{\alpha\beta}$, $\mathcal{F}^{\alpha\beta}F_{\alpha\beta}$ and $\mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta}$ in terms of $\vec{E}$ and $\vec{B}$. Are there any other invariants quadratic in the field strengths $\vec{E}$ and $\vec{B}$?

Recall that the field-strength tensors are given by

$$F_{\alpha\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}, \qquad \mathcal{F}_{\alpha\beta} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix}$$

while their raised index counterparts are

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \qquad \mathcal{F}^{\alpha\beta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

As a result, we find the scalar contractions to be

$$F^{\alpha\beta}F_{\alpha\beta} = -\mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta} = -2(\vec{E}^2 - \vec{B}^2)$$
$$\mathcal{F}^{\alpha\beta}F_{\alpha\beta} = -4\vec{E}\cdot\vec{B}$$

No other independent scalar invariants can be formed that are quadratic in the field strengths. One way to motivate this is to note that a scalar contraction may be written as

$$s = t^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

where $t^{\mu\nu\rho\sigma}$ is a rank-4 tensor that can only be created out of Lorentz invariants such as the metric $\eta_{\mu\nu}$ and the antisymmetric tensor $\epsilon^{\mu\nu\rho\sigma}$. The only two independent possibilities that respect the antisymmetry of the field strengths are

$$\tfrac{1}{2}[\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}], \qquad \epsilon^{\mu\nu\rho\sigma}$$

which give rise to the $F^{\alpha\beta}F_{\alpha\beta}$ and $\mathcal{F}^{\alpha\beta}F_{\alpha\beta}$ invariants. Note, furthermore, that

$$\tfrac{1}{2}\epsilon^{\mu\nu\kappa\lambda}\epsilon^{\rho\sigma}{}_{\kappa\lambda} = -\tfrac{1}{2}[\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}]$$

which explains why $\mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta} = -F^{\alpha\beta}F_{\alpha\beta}$.

b) Is it possible to have an electromagnetic field that appears as a purely electric field in one inertial frame and as a purely magnetic field in some other inertial frame? What are the criteria imposed on $\vec{E}$ and $\vec{B}$ such that there is an inertial frame in which there is no electric field?

It is impossible to have an electromagnetic field that is pure electric in one frame and pure magnetic in another. To see this, we note that the scalar invariant $F^{\alpha\beta}F_{\alpha\beta} = -2(\vec{E}^2 - \vec{B}^2)$ must be the same in all frames. In particular, a purely electric field would have $F^{\alpha\beta}F_{\alpha\beta} < 0$, while a purely magnetic field would have $F^{\alpha\beta}F_{\alpha\beta} > 0$. This cannot happen, since the invariant cannot have different signs in different frames.

If there is an inertial frame in which there is no electric field, we would have

$$F^{\alpha\beta}F_{\alpha\beta} = -2(\vec{E}^2 - \vec{B}^2) \geq 0 \qquad \text{and} \qquad \mathcal{F}^{\alpha\beta}F_{\alpha\beta} = -4\vec{E}\cdot\vec{B} = 0$$

where saturation of the inequality corresponds to both $\vec{E}$ and $\vec{B}$ vanishing. Hence $\vec{E}$ and $\vec{B}$ must satisfy the requirements

$$\vec{E}^2 < \vec{B}^2, \qquad \vec{E}\cdot\vec{B} = 0$$

in any inertial frame. Along with the trivial case ($\vec{E} = \vec{B} = 0$), these are the criteria on the fields such that there is an inertial frame in which the electric field vanishes.

c) For macroscopic media, $\vec{E}$, $\vec{B}$ form the field tensor $F^{\alpha\beta}$ and $\vec{D}$, $\vec{H}$ the tensor $G^{\alpha\beta}$. What further invariants can be formed? What are their explicit expressions in terms of the 3-vector fields?

The macroscopic field-strength tensor $G_{\alpha\beta}$ is given by

$$G_{\alpha\beta} = \begin{pmatrix} 0 & D_x & D_y & D_z \\ -D_x & 0 & -H_z & H_y \\ -D_y & H_z & 0 & -H_x \\ -D_z & -H_y & H_x & 0 \end{pmatrix}$$

This allows us to form the quadratic in $G_{\alpha\beta}$ invariants

$$G^{\alpha\beta}G_{\alpha\beta} = -\mathcal{G}^{\alpha\beta}\mathcal{G}_{\alpha\beta} = -2(\vec{D}^2 - \vec{H}^2)$$
$$\mathcal{G}^{\alpha\beta}G_{\alpha\beta} = -4\vec{D}\cdot\vec{H}$$

along with the mixed invariants

$$F^{\alpha\beta}G_{\alpha\beta} = -\mathcal{F}^{\alpha\beta}\mathcal{G}_{\alpha\beta} = -2(\vec{E}\cdot\vec{D} - \vec{B}\cdot\vec{H})$$
$$F^{\alpha\beta}\mathcal{G}_{\alpha\beta} = \mathcal{F}^{\alpha\beta}G_{\alpha\beta} = -2(\vec{E}\cdot\vec{H} + \vec{B}\cdot\vec{D})$$

11.18 The electric and magnetic fields of a particle of charge q moving in a straight line with speed $v = \beta c$, given by (11.152), become more and more concentrated as $\beta \to 1$, as is indicated in Fig. 11.9. Choose axes so that the charge moves along the z axis in the positive direction, passing the origin at $t = 0$. Let the spatial coordinates of the observation point be (x, y, z) and define the transverse vector $\vec{r}_\perp$, with components x and y. Consider the fields and the source in the limit of $\beta = 1$.

a) Show that the fields can be written as

$$\vec{E} = 2q\frac{\vec{r}_\perp}{r_\perp^2}\delta(ct - z); \qquad \vec{B} = 2q\frac{\hat{v}\times\vec{r}_\perp}{r_\perp^2}\delta(ct - z)$$

where $\hat{v}$ is a unit vector in the direction of the particle's velocity.

We take the particle motion to be along the $\hat{z}$ direction, so that $\vec{\beta} = (v/c)\hat{z}$. In this case, the fields in the rest frame are given by

$$\vec{E}' = \frac{q\vec{r}\,'}{r'^3}, \qquad \vec{B}' = 0$$

Boosting this back to the lab frame gives

$$\vec{E} = \gamma\vec{E}' - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{E}'), \qquad \vec{B} = \vec{\beta}\times\vec{E} \tag{7}$$

In particular, the latter expression indicates that once we compute the electric field $\vec{E}$, the magnetic field follows from a simple cross product with the velocity. To proceed, we rewrite $\vec{r}\,'$ in terms of unprimed (lab) quantities. This may be done by considering the explicit boost transformation

$$x' = x, \qquad y' = y, \qquad z' = \gamma(z - vt)$$

This gives

$$\vec{r}\,' = \vec{r}_\perp + \gamma(z - vt)\hat{z}$$

so that

$$\vec{E}' = \frac{q(\vec{r}_\perp + \gamma(z - vt)\hat{z})}{(r_\perp^2 + \gamma^2(z - vt)^2)^{3/2}}$$

Boosting back using (7) gives the lab frame electric field

$$\vec{E} = \frac{\gamma q(\vec{r}_\perp + (z - vt)\hat{z})}{(r_\perp^2 + \gamma^2(z - vt)^2)^{3/2}}$$

which is an exact expression, valid for any velocity β. We are interested, however, in the limit $\beta \to 1$. This is equivalent to taking the limit $\gamma \to \infty$. Here we may use the fact that the denominator gets highly peaked, and approaches a delta function in this limit. More precisely,

$$\lim_{\gamma\to\infty} \frac{\gamma}{(A^2 + B^2\gamma^2)^{3/2}} = \frac{1}{A^2} \lim_{\xi\to\infty} \frac{\xi}{(1 + \xi^2 B^2)^{3/2}} = \frac{2}{A^2}\delta(B)$$

for any B and non-zero A. Taking $A = r_\perp$ and $B = z - vt$ then gives the result

$$\vec{E} = \frac{2q(\vec{r}_\perp + (z - vt)\hat{z})}{r_\perp^2}\delta(z - vt) = \frac{2q\vec{r}_\perp}{r_\perp^2}\delta(z - vt)$$

Since we have taken the limit $v \to c$, we finally arrive at

$$\vec{E} = 2q\frac{\vec{r}_\perp}{r_\perp^2}\delta(z - ct), \qquad \vec{B} = \vec{\beta} \times \vec{E} = 2q\frac{\hat{v} \times \vec{r}_\perp}{r_\perp^2}\delta(z - ct)$$

b) Show by substitution into the Maxwell equations that these fields are consistent with a 4-vector source density

$$J^\alpha = qcv^\alpha\delta^{(2)}(\vec{r}_\perp)\delta(ct - z)$$

where the 4-vector $v^\alpha = (1, \hat{v})$.

Maxwell's equations are $\partial_\mu F^{\mu\nu} = (4\pi/c)J^\nu$. In particular, the time component gives

$$\frac{4\pi}{c}J^0 = \vec{\nabla} \cdot \vec{E} = 2q\vec{\nabla}_\perp \cdot \left(\frac{\vec{r}_\perp}{r_\perp^2}\right)\delta(z - ct)$$

Using Gauss' theorem in two dimensions, it is easy to see that

$$\vec{\nabla}_\perp \cdot \left(\frac{\vec{r}_\perp}{r_\perp^2}\right) = 2\pi\delta^{(2)}(\vec{r}_\perp)$$

Hence we obtain

$$J^0 = cq\delta^{(2)}(\vec{r}_\perp)\delta(z - ct)$$

In the space directions, we have

$$\begin{aligned}
\frac{4\pi}{c}\vec{J} = -\frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \vec{\nabla}\times\vec{B} &= 2q\frac{\vec{r}_\perp}{r_\perp^2}\delta'(z-ct) + 2q\vec{\nabla}\times\left[\hat{z}\times\left(\frac{\vec{r}_\perp}{r_\perp^2}\right)\delta(z-ct)\right] \\
&= 2q\frac{\vec{r}_\perp}{r_\perp^2}\delta'(z-ct) + 2q\hat{z}\times\left[\hat{z}\times\left(\frac{\vec{r}_\perp}{r_\perp^2}\right)\right]\delta'(z-ct) \\
&\qquad + 2q\left[\hat{z}\vec{\nabla}\cdot\left(\frac{\vec{r}_\perp}{r_\perp^2}\right) - (\hat{z}\cdot\vec{\nabla})\left(\frac{\vec{r}_\perp}{r_\perp^2}\right)\right]\delta(z-ct) \\
&= 4\pi q\hat{z}\delta^{(2)}(\vec{r}_\perp)\delta(z-ct)
\end{aligned}$$

(Note that $\hat{z}\cdot\vec{\nabla} = \partial_z$.) Noting that $\hat{v} = \hat{z}$, this is equivalent to

$$\vec{J} = cq\hat{v}\delta^{(2)}(\vec{r}_\perp)\delta(z-ct)$$

Putting together the time and space components gives

$$J^\mu = cq(1, \hat{v})\delta^{(2)}(\vec{r}_\perp)\delta(z-ct)$$

c) Show that the fields of part a are derivable from either of the following 4-vector potentials

$$A^0 = A^z = -2q\delta(ct-z)\ln(\lambda r_\perp); \qquad \vec{A}_\perp = 0$$

or

$$A^0 = 0 = A^z; \qquad \vec{A}_\perp = -2q\Theta(ct-z)\vec{\nabla}_\perp\ln(\lambda r_\perp)$$

where λ is an irrelevant parameter setting the scale of the logarithm. Show that the two potentials differ by a gauge transformation and find the gauge function, χ.

For the first case ($A^0 = A^z$ non-vanishing) we have

$$\begin{aligned}
\vec{E} = -\vec{\nabla}A^0 - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} &= 2q\vec{\nabla}[\delta(z-ct)\ln(\lambda r_\perp)] + \frac{2q}{c}\hat{z}\frac{\partial}{\partial t}[\delta(z-ct)\ln(\lambda r_\perp)] \\
&= 2q\delta(z-ct)\vec{\nabla}_\perp\ln(\lambda r_\perp) + 2q\hat{z}\delta'(z-ct)\ln(\lambda r_\perp) \\
&\qquad - 2q\hat{z}\delta'(z-ct)\ln(\lambda r_\perp) \\
&= 2q\delta(z-ct)\frac{\hat{r}_\perp}{r_\perp^2}
\end{aligned}$$

and

$$\vec{B} = \vec{\nabla} \times \vec{A} = -2q\vec{\nabla} \times [\hat{z}\delta(z-ct)\ln(\lambda r_\perp)]$$
$$= -2q\vec{\nabla}_\perp \times [\delta(z-ct)\ln(\lambda r_\perp)\hat{z}]$$
$$= -2q\frac{\vec{r}_\perp}{r_\perp^2} \times \hat{z}\delta(z-ct) = 2q\frac{\hat{v}\times\vec{r}_\perp}{r_\perp^2}\delta(z-ct)$$

For the second case ($\vec{A}_\perp$ non-vanishing) we have

$$\vec{A}_\perp = -2q\Theta(ct-z)\vec{\nabla}_\perp \ln(\lambda r_\perp) = -2q\Theta(ct-z)\frac{\vec{r}_\perp}{r_\perp^2}$$

This gives

$$\vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} = 2q\delta(ct-z)\frac{\vec{r}_\perp}{r_\perp^2}$$

and

$$\vec{B} = \vec{\nabla} \times \vec{A} = -2q\vec{\nabla} \times \left[\Theta(ct-z)\frac{\vec{r}_\perp}{r_\perp^2}\right]$$
$$= 2q\delta(ct-z)\hat{z} \times \frac{\vec{r}_\perp}{r_\perp^2} - 2q\Theta(ct-z)\vec{\nabla} \times \left(\frac{\vec{r}_\perp}{r_\perp^2}\right)$$
$$= 2q\frac{\hat{v}\times\vec{r}_\perp}{r_\perp^2}\delta(ct-z)$$

(since the function $\vec{r}_\perp/r_\perp^2 = \vec{\nabla}_\perp \ln(\lambda r_\perp)$ is curl-free).

The above two vector potentials are necessarily related by a gauge transformation χ through

$$A^\mu_{(1)} - A^\mu_{(2)} = \partial^\mu\chi$$

To find the explicit gauge transformation, we write

$$\begin{aligned}
\partial^\mu\chi &= A^\mu_{(1)} - A^\mu_{(2)} \\
&= 2q\big(-\delta(ct-z)\ln(\lambda r_\perp), \Theta(ct-z)\vec{\nabla}_\perp \ln(\lambda r_\perp) - \delta(ct-z)\ln(\lambda r_\perp)\hat{z}\big) \\
&= 2q\bigg(-\frac{\partial}{\partial ct}\Theta(ct-z)\ln(\lambda r_\perp), \Theta(ct-z)\vec{\nabla}_\perp \ln(\lambda r_\perp) + \hat{z}\frac{\partial}{\partial z}\Theta(ct-z)\ln(\lambda r_\perp)\bigg) \\
&= -2q\left(\frac{\partial}{\partial x^0}, -\vec{\nabla}\right)\Theta(ct-z)\ln(\lambda r_\perp)
\end{aligned}$$

We thus conclude that the gauge function is

$$\chi = -2q\Theta(ct-z)\ln(\lambda r_\perp)$$