Physics 506 **Winter 2008**

Homework Assignment #7 — Solutions

Textbook problems: Ch. 10: 10.12, 10.14, 10.18
Ch. 11: 11.3

10.12 A linearly polarized plane wave of amplitude E_0 and wave number k is incident on a circular opening of radius a in an otherwise perfectly conducting flat screen. The incident wave vector makes an angle α with the normal to the screen. The polarization vector is perpendicular to the plane of incidence.

a) Calculate the diffracted fields and the power per unit solid angle transmitted through the opening, using the vector Smythe-Kirchhoff formula (10.101) with the assumption that the tangential electric field in the opening is the unperturbed incident field.

We use the vector Smythe-Kirchhoff formula

$$\vec{E} = \frac{ie^{ikr}}{2\pi r}\vec{k} \times \int_{S_1} \hat{n}' \times \vec{E}(\vec{x}\,')e^{-i\vec{k}\cdot\vec{x}\,'}da' \tag{1}$$

To set up the problem, we assume the screen lies in the x-y plane (at $z = 0$), with the circular hole centered at the origin. We then take the incident wave vector $\vec{k}_0$ to lie in the x-z plane and at an angle α with the z axis

$$\vec{k}_0 = k(\hat{x}\sin\alpha + \hat{z}\cos\alpha)$$

This defines the plane of incidence to be the x-z plane. Since the polarization vector is perpendicular to the plane of incidence, we take

$$\hat{\epsilon}_0 = \hat{y}$$

As a result, the incident plane wave is described as

$$\vec{E}(\vec{x}\,') = E_0\hat{\epsilon}_0 e^{i\vec{k}_0\cdot\vec{x}\,'} = E_0\hat{y}e^{ik(x'\sin\alpha + z'\cos\alpha)}$$

Since the normal to the screen is $\hat{n}' = \hat{z}$, we see that

$$\hat{n}' \times \vec{E}(\vec{x}\,') = \hat{z} \times \vec{E}(\vec{x}\,') = -E_0\hat{x}e^{ik(x'\sin\alpha + z'\cos\alpha)}$$

Inserting this into (1) gives

$$\begin{aligned}\vec{E} &= -\frac{iE_0e^{ikr}}{2\pi r}(\vec{k}\times\hat{x})\int e^{ik(x'\sin\alpha + z'\cos\alpha)}e^{-i\vec{k}\cdot\vec{x}\,'}\Big|_{z'=0}da' \\ &= -\frac{iE_0e^{ikr}}{2\pi r}(\vec{k}\times\hat{x})\int e^{ikx'\sin\alpha}e^{-i(k_xx' + k_yy')}da'\end{aligned}$$

The integral is over the area of the circular aperture of radius a. This is best done in polar coordinates. Taking

$$x' = \rho\cos\beta, \qquad y' = \rho\sin\beta$$

along with the the spherical coordinates decomposition of the scattered wavevector

$$k_x = k\sin\theta\cos\phi, \qquad k_y = k\sin\theta\sin\phi, \qquad k_z = k\cos\theta$$

gives

$$\vec{E} = -\frac{iE_0 e^{ikr}}{2\pi r}(\vec{k}\times\hat{x})\int_0^a \rho\, d\rho \int_0^{2\pi} d\beta e^{ik\rho[\sin\alpha\cos\beta - \sin\theta\cos(\phi-\beta)]}$$

This integral is identical to that encountered in Section 10.9. The result is then straightforward

$$\vec{E} = -\frac{iE_0 e^{ikr}}{r}a^2(\vec{k}\times\hat{x})\frac{J_1(ka\xi)}{ka\xi}$$

where

$$\xi = \sqrt{\sin^2\theta + \sin^2\alpha - 2\sin\theta\sin\alpha\cos\phi}$$

Writing out explicitly

$$\vec{k}\times\hat{x} = \hat{y}k_z - \hat{z}k_y = k(\hat{y}\cos\theta - \hat{z}\sin\theta\sin\phi)$$

gives

$$\vec{E} = -\frac{iE_0 e^{ikr}}{r}ka^2(\hat{y}\cos\theta - \hat{z}\sin\theta\sin\phi)\frac{J_1(ka\xi)}{ka\xi}$$

The scattered power is

$$\frac{dP}{d\Omega} = \frac{r^2}{2Z_0}|\vec{E}|^2 = \frac{|E_0|^2a^2}{8Z_0}(ka)^2(\cos^2\theta + \sin^2\theta\sin^2\phi)\left|\frac{2J_1(ka\xi)}{ka\xi}\right|^2$$

This may be rewritten in terms of the normally incident power on the aperture

$$P_i = \frac{|E_0|^2}{2Z_0}(\pi a^2)\cos\alpha$$

The result is

$$\epsilon_\perp: \qquad \frac{dP}{d\Omega} = P_i\cos\alpha\frac{(ka)^2}{4\pi}\left(\frac{\cos^2\theta + \sin^2\theta\sin^2\phi}{\cos^2\alpha}\right)\left|\frac{2J_1(ka\xi)}{ka\xi}\right|^2 \qquad (2)$$

b) Compare your result in part a with the standard scalar Kirchhoff approximation and with the result in Section 10.9 for the polarization vector in the plane of incidence.

According to Section 10.9, if the polarization vector is in the plane of incidence, we would have

$$\epsilon_\parallel : \qquad \frac{dP}{d\Omega} = P_i \cos\alpha \frac{(ka)^2}{4\pi}\left(\cos^2\theta + \sin^2\theta\cos^2\phi\right)\left|\frac{2J_1(ka\xi)}{ka\xi}\right|^2$$

while for diffraction, we would have

$$\text{scalar}: \qquad \frac{dP}{d\Omega} = P_i \cos\alpha \frac{(ka)^2}{4\pi}\left(\frac{\cos\alpha + \cos\theta}{2\cos\alpha}\right)^2\left|\frac{2J_1(ka\xi)}{ka\xi}\right|^2$$

These two expressions, along with the perpendicular polarization one of (2), differ by the slowly varying angular factors in the parentheses. However, the main diffraction feature arising from $J_1(\zeta)/\zeta$ is the same in all three cases. Thus the diffraction patterns are essentially the same regardless of the scalar or vector nature of the radiation. Note also that, for normal incidence (ie $\alpha = 0$), the perpendicular and parallel polarization expressions are identical, up to a 90° rotation of the polarization vector.

10.14 A rectangular opening with sides of length a and $b \geq a$ defined by $x = \pm(a/2)$, $y = \pm(b/2)$ exists in a flat, perfectly conducting plane sheet filling the x-y plane. A plane wave is normally incident with its polarization vector making an angle β with the long edges of the opening.

a) Calculate the diffracted fields and power per unit solid angle with the vector Smythe-Kirchhoff relation (10.109), assuming that the tangential electric field in the opening is the incident unperturbed field.

The vector Smythe-Kirchhoff relation states

$$\vec{E} = \frac{ie^{ikr}}{2\pi r}\vec{k} \times \int_{S_1} \hat{n}' \times \vec{E}(\vec{x}\,')e^{-i\vec{k}\cdot\vec{x}\,'}da'$$

where for a normally incident plane wave, the incident unperturbed field may be taken as

$$\vec{E}(\vec{x}\,') = E_0\hat{\epsilon}_0 e^{ikz'} = E_0(\hat{x}\sin\beta + \hat{y}\cos\beta)e^{ikz'}$$

For the rectangular screen, the surface S_1 is the rectangle at $z' = 0$ with $|x| \leq a/2$ and $|y| \leq b/2$ and surface normal $\hat{n}' = \hat{z}$. The resulting integral is then

$$\begin{aligned}
\vec{E} &= \frac{iE_0 e^{ikr}}{2\pi r}\vec{k} \times \int \hat{z} \times (\hat{x}\sin\beta + \hat{y}\cos\beta)e^{-i\vec{k}\cdot\vec{x}\,'}da' \\
&= \frac{iE_0 e^{ikr}}{2\pi r}\vec{k} \times \int_{-a/2}^{a/2} dx' \int_{-b/2}^{b/2} dy'\,(\hat{y}\sin\beta - \hat{x}\cos\beta)e^{-i\vec{k}\cdot\vec{x}\,'} \\
&= \frac{iE_0 e^{ikr}}{2\pi r}\vec{k} \times (\hat{y}\sin\beta - \hat{x}\cos\beta)\int_{-a/2}^{a/2} dx'\,e^{-ik_x x}\int_{-b/2}^{b/2} dy'\,e^{-ik_y y}
\end{aligned}$$

The integrals are simple to perform, and yield

$$\vec{E} = \frac{2iE_0 e^{ikr}}{\pi r}[-\hat{x}k_z \sin\beta - \hat{y}k_z\cos\beta + \hat{z}(k_x\sin\beta + k_y\cos\beta)]\frac{\sin(k_x a/2)\sin(k_y b/2)}{k_x k_y}$$

Because of the rectangular geometry, this expression is simplest when expressed in cartesian components. However, if we choose to write $\vec{k}$ in terms of spherical components, we may substitute in

$$k_x = k\sin\theta\cos\phi, \qquad k_y = k\sin\theta\sin\phi, \qquad k_z = k\cos\theta$$

to obtain

$$\vec{E} = \frac{2iE_0 e^{ikr}}{\pi k r}[-\hat{x}\cos\theta\sin\beta - \hat{y}\cos\theta\cos\beta + \hat{z}\sin\theta\sin(\phi+\beta)] \\ \times \frac{\sin\left(\frac{ka}{2}\sin\theta\cos\phi\right)}{\sin\theta\cos\phi}\frac{\sin\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\sin\theta\sin\phi}$$

Note the standard $\sin\zeta/\zeta$ diffraction patterns for the x and y directions.

The scattered power may be expressed as

$$\frac{dP}{d\Omega} = \frac{r^2}{2Z_0}|\vec{E}|^2 = \frac{1}{2Z_0}\frac{4|E_0|^2}{\pi^2 k^2}[\cos^2\theta + \sin^2\theta\sin^2(\phi+\beta)] \\ \times \frac{\sin^2\left(\frac{ka}{2}\sin\theta\cos\phi\right)}{(\sin\theta\cos\phi)^2}\frac{\sin^2\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{(\sin\theta\sin\phi)^2}$$

In terms of the normally incident power on the aperture

$$P_i = \frac{|E_0|^2}{2Z_0}ab$$

the above becomes

$$\frac{dP}{d\Omega} = \frac{P_i}{\pi^2}[\cos^2\theta + \sin^2\theta\sin^2(\phi+\beta)] \\ \times \frac{\sin^2\left(\frac{ka}{2}\sin\theta\cos\phi\right)}{\frac{ka}{2}(\sin\theta\cos\phi)^2}\frac{\sin^2\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\frac{kb}{2}(\sin\theta\sin\phi)^2} \tag{3}$$

Note that, for small openings, this reduces to

$$\frac{dP}{d\Omega} = \frac{P_i}{\pi^2}\frac{ka}{2}\frac{kb}{2}[\cos^2\theta + \sin^2\theta\sin^2(\phi+\beta)]$$

$b)$ Calculate the corresponding result of the sclar Kirchhoff approximation.

For the scalar Kirchhoff approximation, we have

$$\psi = -\frac{e^{ikr}}{4\pi r}\int_{S_1}[\hat{n}'\cdot\vec{\nabla}'\psi + i\vec{k}\cdot\hat{n}'\psi]e^{-i\vec{k}\cdot\vec{x}\,'}\,da'$$

Here we take

$$\psi(\vec{x}\,') = \psi_0 e^{ikz'}, \qquad \hat{n}'\cdot\vec{\nabla}'\psi = \hat{z}\cdot\vec{\nabla}'\psi = \frac{\partial}{\partial z'}\psi = ik\psi_0 e^{ikz'}$$

Hence

$$\begin{aligned}\psi &= -\frac{e^{ikr}}{4\pi r}\int(ik\psi_0 + ik_z\psi_0)e^{-i\vec{k}\cdot\vec{x}\,'}\,da' \\ &= -\frac{i\psi_0 e^{ikr}}{4\pi r}(k+k_z)\int_{-a/2}^{a/2}dx'\,e^{-ik_x x}\int_{-b/2}^{b/2}dy'\,e^{-ik_y y}\end{aligned}$$

The integrals are identical to the ones performed above. The result (using spherical components of $\vec{k}$) is

$$\psi = -\frac{i\psi_0 e^{ikr}}{\pi kr}(1+\cos\theta)\frac{\sin\left(\frac{ka}{2}\sin\theta\cos\phi\right)}{\sin\theta\cos\phi}\,\frac{\sin\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\sin\theta\sin\phi}$$

Using $dP/d\Omega = r^2|\psi|^2$ and $P_i = |\psi|^2 ab$, the scalar expression for scattered power becomes

$$\frac{dP}{d\Omega} = \frac{P_i}{\pi^2}\left[\cos^4\frac{\theta}{2}\right]\frac{\sin^2\left(\frac{ka}{2}\sin\theta\cos\phi\right)}{\frac{ka}{2}(\sin\theta\cos\phi)^2}\,\frac{\sin^2\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\frac{kb}{2}(\sin\theta\sin\phi)^2}$$

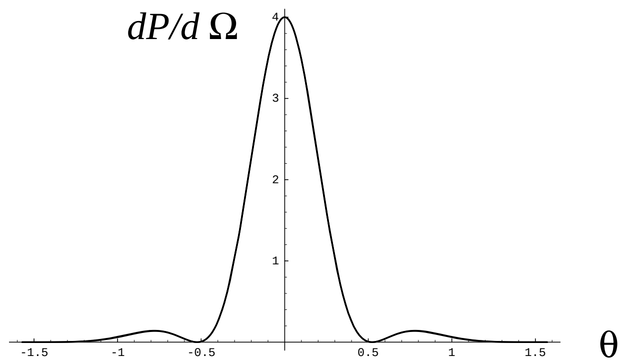
Comparing this scalar expression to the vector expression (3), we see that the only difference lies in the additional polarization factors enclosed in the square brackets.

c) For $b=a$, $\beta = 45^\circ$, $ka = 4\pi$, compute the vector and scalar approximations to the diffracted power per unit solid angle as a function of the angle θ for $\phi = 0$. Plot a graph showing a comparison between the two results.

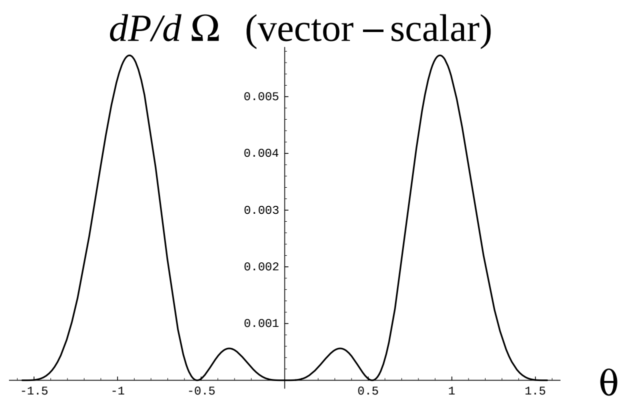
For the above parameters, the vector and scalar expressions reduce to

$$\begin{aligned}\left.\frac{dP}{d\Omega}\right|_{\text{vector}} &= \frac{P_i}{\pi^2}[\tfrac{1}{2}(1+\cos^2\theta)]\frac{\sin^2(2\pi\sin\theta)}{\sin^2\theta} \\ \left.\frac{dP}{d\Omega}\right|_{\text{scalar}} &= \frac{P_i}{\pi^2}[\cos^4(\theta/2)]\frac{\sin^2(2\pi\sin\theta)}{\sin^2\theta}\end{aligned}$$

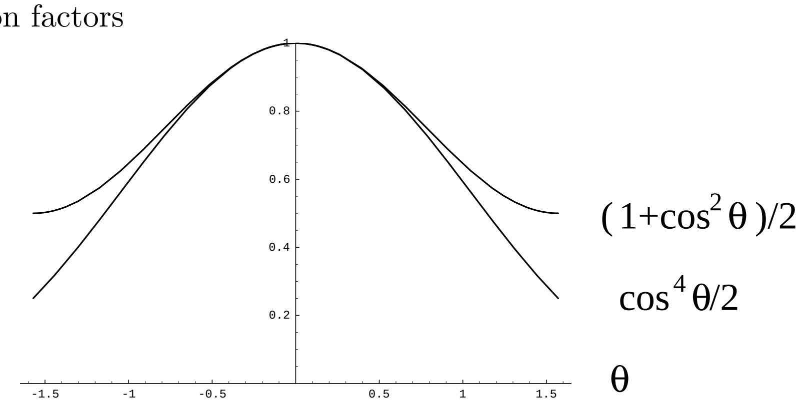
These two expressions (normalized to unit power) may be plotted on the same graph



In fact, they are virtually indistinguishable. To show that the vector and scalar expressions are actually not identical, we may plot the difference $dP/d\Omega|_{\text{vector}} - dP/d\Omega|_{\text{scalar}}$



(note the different scale on the vectical axis). This difference is entirely dependent on the polarization factors



These factors are nearly identical in the forward direction (at the diffraction peak). Although the difference gets large off axis, there is so little power there that this difference is essentially unimportant.

10.18 Discuss the diffraction due to a small, circular hole of radius a in a flat, perfectly conducting sheet, assuming that $ka \ll 1$.

a) If the fields near the screen on the incident side are normal $\vec{E}_0 e^{-i\omega t}$ and tangential

$\vec{B}_0 e^{-i\omega t}$, show that the diffracted electric field in the Fraunhofer zone is

$$\vec{E} = \frac{e^{ikr-i\omega t}}{3\pi r} k^2 a^3 \left[2c\frac{\vec{k}}{k} \times \vec{B}_0 + \frac{\vec{k}}{k} \times \left(\vec{E}_0 \times \frac{\vec{k}}{k} \right) \right]$$

where $\vec{k}$ is the wave vector in the direction of observation.

A small hole in a perfectly conducting sheet can be treated as if it were a small aperture in the side of a waveguide. This was discussed in Section 9.5. In general, the fields in the hole can be expanded in a multipole expansion, with the electric and magnetic dipoles being the dominant terms (for $ka \ll 1$). In terms of the normal electric field $\vec{E}_0$ and tangential magnetic field $\vec{H}_0$, the effective dipole moments for a small circular aperture of radius a turns out to be

$$\vec{p}_{\text{eff}} = \frac{4\epsilon_0 a^3}{3} \vec{E}_0, \qquad \vec{m}_{\text{eff}} = -\frac{8a^3}{3} \vec{H}_0 \tag{4}$$

These results were derived in Sections 3.13 and 5.13. Note that the signs are chosen for the diffraction region. While this is a diffraction problem, once we have determined the effective electric and dipole moments due to the aperture, we may treat this as a radiation problem, with the electric and magnetic fields given by

$$\begin{aligned} \vec{E} &= -\frac{k^2}{4\pi\epsilon_0} \frac{e^{ikr}}{r} [\hat{n} \times (\hat{n} \times \vec{p}_{\text{eff}}) + \hat{n} \times \vec{m}_{\text{eff}}/c] \\ \vec{H} &= \frac{1}{Z_0} \hat{n} \times \vec{E} \end{aligned}$$

Substituting in (4) gives, for the electric field

$$\vec{E} = -\frac{k^2 a^3}{3\pi} \frac{e^{ikr}}{r} [\hat{n} \times (\vec{E}_0 \times \hat{n}) + 2c\, \hat{n} \times (\mu_0 \vec{H}_0)] \tag{5}$$

Using $\vec{B}_0 = \mu_0 \vec{H}_0$ and $\hat{n} \equiv \hat{k} = \vec{k}/k$ then gives

$$\vec{E} = -\frac{k^2 a^3}{3\pi} \frac{e^{ikr-i\omega t}}{r} \left[\frac{\vec{k}}{k} \times \left(\vec{E}_0 \times \frac{\vec{k}}{k} \right) + 2c\frac{\vec{k}}{k} \times \vec{B}_0 \right]$$

where we have restored the explicit harmonic time dependence.

b) Determine the angular distribution of the diffracted radiation and show that the total power transmitted through the hole is

$$P = \frac{2}{27\pi Z_0} k^4 a^6 (4c^2 B_0^2 + E_0^2)$$

We may determine the diffracted power by calculating the Poynting vector

$$\frac{dP}{d\Omega} = r^2 \frac{dP}{da} = r^2 \hat{n} \cdot \vec{S} = \frac{r^2}{2} \hat{n} \cdot (\vec{E} \times \vec{H}^*) = \frac{r^2}{2Z_0} |\vec{E}|^2$$

Substituting in (5) gives

$$\begin{aligned}
\frac{dP}{d\Omega} &= \frac{k^4a^6}{18\pi^2 Z_0}|\hat{n}\times(\vec{E}_0\times\hat{n}+2c\vec{B}_0)|^2 \\
&= \frac{k^4a^6}{18\pi^2 Z_0}\left(|\vec{E}_0\times\hat{n}+2c\vec{B}_0|^2 - 4c^2|\hat{n}\cdot\vec{B}_0|^2\right) \\
&= \frac{k^4a^6}{18\pi^2 Z_0}\left(|\vec{E}_0|^2 - |\hat{n}\cdot\vec{E}_0|^2 + 4c^2(|\vec{B}_0|^2 - |\hat{n}\cdot\vec{B}_0|^2) + 4c\Re(\vec{B}_0^*\cdot(\vec{E}_0\times\hat{n}))\right)
\end{aligned}$$

To proceed, we may choose an explicit coordinate system. We let the screen with the small hole lie in the x-y plane (at $z=0$). Since $\vec{E}_0$ is normal and $\vec{B}_0$ is tangential to the screen, we let

$$\vec{E}_0 = E_0\hat{z}, \qquad \vec{B}_0 = B_0\hat{x}$$

Specifying the normal vector $\hat{n}$ in standard spherical coordinates

$$\hat{n} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$$

gives

$$\hat{n}\cdot\vec{E}_0 = E_0\cos\theta, \qquad \hat{n}\cdot\vec{B}_0 = B_0\sin\theta\cos\phi$$

and

$$\vec{B}_0^*\cdot(\vec{E}_0\times\hat{n}) = -B_0^*E_0\sin\theta\sin\phi$$

As a result, the angular power distribution becomes

$$\frac{dP}{d\Omega} = \frac{k^4a^6}{18\pi^2 Z_0}\left(|E_0|^2\sin^2\theta + 4c^2|B_0|^2(1-\sin^2\theta\cos^2\phi) - 4c\Re(E_0B_0^*)\sin\theta\sin\phi\right)$$

The power transmitted through the hole is obtained from the integral

$$\begin{aligned}
P &= \int_0^1 d(\cos\theta)\int_0^{2\pi} d\phi\frac{dP}{d\Omega} \\
&= \frac{k^4a^6}{18\pi^2 Z_0}\int_0^1 d(\cos\theta)\int_0^{2\pi} d\phi\,\big(|E_0|^2\sin^2\theta + 4c^2|B_0|^2(1-\sin^2\theta\cos^2\phi) \\
&\qquad\qquad - 4c\Re(E_0B_0^*)\sin\theta\sin\phi\big) \\
&= \frac{k^4a^6}{9\pi Z_0}\int_0^1 d(\cos\theta)\,\left(|E_0|^2\sin^2\theta + 2c^2|B_0|^2(2-\sin^2\theta)\right) \\
&= \frac{2k^4a^6}{27\pi Z_0}(|E_0|^2 + 4c^2|B_0|^2)
\end{aligned}$$

11.3 Show explicitly that two successive Lorentz transformations in the same direction are equivalent to a single Lorentz transformation with a velocity

$$v = \frac{v_1+v_2}{1+(v_1v_2/c^2)}$$

This is an alternative way to derive the parallel-velocity addition law.

To be explicit, consider a Lorentz boost in the x^0-x^1 plane with velocity v_1 from the unprimed to the primed frame followed by a boost with velocity v_2 from the primed to the double-primed frame

$$\begin{aligned} (x^0)' &= \gamma_1(x^0 - \beta_1 x^1) \\ (x^1)' &= \gamma_1(x^1 - \beta_1 x^0) \end{aligned} \qquad \text{then} \qquad \begin{aligned} (x^0)'' &= \gamma_2((x^0)' - \beta_2 (x^1)') \\ (x^1)'' &= \gamma_2((x^1)' - \beta_2 (x^0)') \end{aligned}$$

When combined, this gives

$$\begin{aligned} (x^0)'' &= \gamma_1\gamma_2((1+\beta_1\beta_2)x^0 - (\beta_1+\beta_2)x^1) \\ (x^1)'' &= \gamma_1\gamma_2((1+\beta_1\beta_2)x^1 - (\beta_1+\beta_2)x^0) \end{aligned}$$

If this were a single Lorentz boost, then we must be able to write it as

$$\begin{aligned} (x^0)'' &= \gamma_{12}(x^0 - \beta_{12} x^1) \\ (x^1)'' &= \gamma_{12}(x^1 - \beta_{12} x^0) \end{aligned}$$

Comparing the two expressions gives

$$\beta_{12} = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2} \qquad \gamma_{12} = \gamma_1\gamma_2(1+\beta_1\beta_2)$$

The first term gives the desired velocity addition relation

$$v_{12} = \frac{v_1+v_2}{1+(v_1v_2/c^2)}$$

However, for consistency, we also have to show that the second term is consistent with a Lorentz boost. To do this, we consider the square

$$\begin{aligned} \gamma_1^2\gamma_2^2(1+\beta_1\beta_2)^2 &= \frac{(1+\beta_1\beta_2)^2}{(1-\beta_1^2)(1-\beta_2^2)} \\ &= \frac{(1+\beta_1\beta_2)^2}{1-\beta_1^2-\beta_2^2+\beta_1^2\beta_2^2} \\ &= \frac{(1+\beta_1\beta_2)^2}{(1+\beta_1\beta_2)^2-(\beta_1+\beta_2)^2} \\ &= \frac{1}{1-((\beta_1+\beta_2)/(1+\beta_1\beta_2))^2} = \frac{1}{1-\beta_{12}^2} = \gamma_{12}^2 \end{aligned}$$

Hence this is indeed consistent with a Lorentz boost. Alternatively, the velocity addition relation can easily be seen in terms of rapidities. For boosts in the x^0-x^1 plane, we write down

$$\Lambda_1 = \begin{pmatrix} \cosh\zeta_1 & -\sinh\zeta_1 & & \\ -\sinh\zeta_1 & \cosh\zeta_1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

and

$$\Lambda_2 = \begin{pmatrix} \cosh\zeta_2 & -\sinh\zeta_2 & & \\ -\sinh\zeta_2 & \cosh\zeta_2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

By matrix multiplication, we see that the composition of two boosts is then

$$\Lambda_{12} = \Lambda_2\Lambda_1 = \begin{pmatrix} \cosh(\zeta_1+\zeta_2) & -\sinh(\zeta_1+\zeta_2) & & \\ -\sinh(\zeta_1+\zeta_2) & \cosh(\zeta_1+\zeta_2) & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Since rapidities ζ_i are related to velocities β_i according to

$$\beta_i = \tanh\zeta_i$$

we have simply $\zeta_{12} = \zeta_1 + \zeta_2$, or

$$\beta_{12} = \tanh(\tanh^{-1}\beta_1 + \tanh^{-1}\beta_2) = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}$$

which is identical to the result found above. Finally, note that two successive Lorentz boosts in *different* directions generically gives rise to a Lorentz boost along with a rotation. This is related to the non-Abelian nature of the Lorentz group.