Physics 506 **Winter 2008**

Homework Assignment #6 — Solutions

Textbook problems: Ch. 10: 10.2, 10.3, 10.8, 10.9a

10.2 Electromagnetic radiation with elliptic polarization, described (in the notation of Section 7.2) by the polarization vector,

$$\vec{\epsilon} = \frac{1}{\sqrt{1+r^2}}(\vec{\epsilon}_+ + re^{i\alpha}\vec{\epsilon}_-)$$

is scattered by a perfectly conducting sphere of radius a. Generalize the amplitude in the scattering cross section (10.71), which applies for $r = 0$ or $r = \infty$, and calculate the cross section for scattering in the long-wavelength limit. Show that

$$\frac{d\sigma}{d\Omega} = k^4a^6\left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta - \frac{3}{4}\left(\frac{r}{1+r^2}\right)\sin^2\theta\cos(2\phi-\alpha)\right]$$

Compare with Problem 10.1.

The spherical wave scattering of Section 10.4 involves either a left or right circularly polarized incident wave. In order to treat elliptic polarization, we may use linear superposition of the incident and scattered waves. In particular, with the above polarization vector, we may write the incident plane wave as

$$\vec{E} = \sum_l i^l\sqrt{\frac{2\pi(2l+1)}{1+r^2}}\big[j_l(kr)\vec{X}_{l,1} + \frac{1}{k}\vec{\nabla}\times j_l(kr)\vec{X}_{l,1} + re^{i\alpha}(j_l(kr)\vec{X}_{l,-1} - \frac{1}{k}\vec{\nabla}\times j_l(kr)\vec{X}_{l,-1})\big]$$

(and a similar expression for $\vec{H}$). Note that we are using properly normalized circular polarizations

$$\vec{\epsilon}_\pm = \frac{1}{\sqrt{2}}(\vec{\epsilon}_1 \pm \vec{\epsilon}_2)$$

This accounts for the $\sqrt{2\pi(2l+1)}$ factor as opposed to the $\sqrt{4\pi(2l+1)}$ factor used in Sections 10.3 and 10.4. The scattered wave then takes the normalized form

$$\vec{E}_{\text{sc}} = \tfrac{1}{2}\sum_l i^l\sqrt{\frac{2\pi(2l+1)}{1+r^2}}\big[\alpha_+(l)h_l^{(1)}(kr)\vec{X}_{l,1} + \frac{\beta_+(l)}{k}\vec{\nabla}\times h_l^{(1)}(kr)\vec{X}_{l,1} + re^{i\alpha}(\alpha_-(l)h_l^{(1)}(kr)\vec{X}_{l,-1} - \frac{\beta_-(l)}{k}\vec{\nabla}\times h_l^{(1)}(kr)\vec{X}_{l,-1})\big]$$

Since the incident wave is a coherent combination of both $\vec{\epsilon}_+$ and $\vec{\epsilon}_-$ polarizations, the scattered electric field (essentially the scattering amplitude) is similarly a coherent sum of positive and negative helicities. The scattering cross section may then be written as

$$\frac{d\sigma_{\rm sc}}{d\Omega} = \frac{\pi}{2k^2(1+r^2)}\Big|\sum_l \sqrt{2l+1}\big[\alpha_+(l)\vec{X}_{l,1} + i\beta_+(l)\hat{n}\times\vec{X}_{l,1} + re^{i\alpha}(\alpha_-(l)\vec{X}_{l,-1} - i\beta_-(l)\hat{n}\times\vec{X}_{l,-1})\big]\Big|^2$$

Note that the factor of $1+r^2$ in the denominator arises from the scattered wave, $\vec{E}_{\rm sc}$. All factors of r drop out of the incident flux calculation as the incident wave is already properly normalized. This differential cross section is the generalization of (10.63), and leads to a total scattering cross section

$$\sigma_{\rm sc} = \frac{\pi}{2k^2(1+r^2)}\sum_l (2l+1)[|\alpha_+(l)|^2 + |\beta_+(l)|^2 + r^2(|\alpha_-(l)|^2 + |\beta_-(l)|^2)]$$

In the long wavelength limit, we only need to worry about the $l=1$ terms in the above. The partial wave coefficients $\alpha_\pm(l)$ and $\beta_\pm(l)$ are those for a perfectly conducting sphere, and are unchanged by the elliptical polarization. For $l=1$, we use the Jackson result

$$\alpha_\pm(1) = -\frac{1}{2}\beta_\pm(1) \approx -\frac{2i}{3}(ka)^3$$

As a result, we obtain

$$\frac{d\sigma_{\rm sc}}{d\Omega} \approx \frac{2\pi}{3k^2(1+r^2)}(ka)^6\big|\vec{X}_{1,1} - 2i\hat{n}\times\vec{X}_{1,1} + re^{i\alpha}(\vec{X}_{1,-1} + 2i\hat{n}\times\vec{X}_{1,-1})\big|^2 \quad (1)$$

We now work out the explicit functional forms of $\vec{X}_{1,1}$ and $\vec{X}_{1,-1}$. This is most straightforward in spherical coordinates where

$$\vec{X}_{1,\pm1} = \frac{1}{\sqrt{2}}\vec{L}Y_{1,\pm1}$$

Using

$$\vec{L} = \frac{r}{i}\hat{r}\times\vec{\nabla} = i\left(\hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} - \hat{\phi}\frac{\partial}{\partial\theta}\right)$$

as well as $Y_{1,\pm1} = \mp\sqrt{3/8\pi}\sin\theta e^{\pm i\phi}$, we find

$$\vec{X}_{1,\pm1} = \mp i\sqrt{\frac{3}{16\pi}}\left(\hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} - \hat{\phi}\frac{\partial}{\partial\theta}\right)\sin\theta e^{\pm i\phi} = \sqrt{\frac{3}{16\pi}}(\hat{\theta} \pm i\hat{\phi}\cos\theta)e^{\pm i\phi} \quad (2)$$

This also yields

$$\hat{n} \times \vec{X}_{1,\pm 1} = \hat{r} \times \vec{X}_{1,\pm 1} = \sqrt{\frac{3}{16\pi}}(\hat{\phi} \mp i\hat{\theta}\cos\theta)e^{\pm i\phi} \tag{3}$$

Inserting these expressions into (1) gives

$$\begin{aligned}\frac{d\sigma_{\text{sc}}}{d\Omega} &= \frac{(ka)^6}{8k^2(1+r^2)}\Big|\big[\hat{\theta}(1-2\cos\theta) + i\hat{\phi}(\cos\theta - 2)\big]e^{i\phi} \\ &\qquad + re^{i(\alpha-\phi)}\big[\hat{\theta}(1-2\cos\theta) - i\hat{\phi}(\cos\theta-2)\big]\Big|^2 \\ &= \frac{(ka)^6}{8k^2(1+r^2)}\big|\hat{\theta}(1-2\cos\theta)(1+re^{i(\alpha-2\phi)}) \\ &\qquad + i\hat{\phi}(\cos\theta-2)(1-re^{i(\alpha-2\phi)})\big|^2 \\ &= \frac{(ka)^6}{8k^2(1+r^2)}\big[(1-2\cos\theta)^2(1+r^2+2r\cos(\alpha-2\phi)) \\ &\qquad + (\cos\theta-2)^2(1+r^2-2r\cos(\alpha-2\phi))\big]\end{aligned}$$

Multiplying this out and rearranging terms gives the final result

$$\frac{d\sigma_{\text{sc}}}{d\Omega} = k^4a^6\left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta - \frac{3}{4}\left(\frac{r}{1+r^2}\right)\sin^2\theta\cos(2\phi-\alpha)\right] \tag{4}$$

Alternatively, we could take the result of 10.1a

$$\frac{d\sigma}{d\Omega} = k^4a^6\left[\frac{5}{4} - |\hat{\epsilon}_0\cdot\hat{n}|^2 - \frac{1}{4}|\hat{n}\cdot\hat{n}_0\times\hat{\epsilon}_0|^2 - \hat{n}_0\cdot\hat{n}\right]$$

and substitute in the polarization

$$\hat{\epsilon}_0 = \frac{1}{\sqrt{1+r^2}}(\hat{\epsilon}_+ + re^{i\alpha}\hat{\epsilon}_-)$$

We take explicitly

$$\hat{n}_0 = \hat{z}, \qquad \hat{\epsilon}_\pm = \frac{1}{\sqrt{2}}(\hat{x} \pm i\hat{y}), \qquad \hat{n}_0\times\hat{\epsilon}_\pm = \mp i\hat{\epsilon}_\pm$$

as well as

$$\hat{n} = \sin\theta(\hat{x}\cos\phi + \hat{y}\sin\phi) + \cos\theta\hat{z}$$

so that

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= k^4a^6\left[\frac{5}{4} - \frac{\sin^2\theta}{2(1+r^2)}|e^{i\phi} + re^{i(\alpha-\phi)}|^2 - \frac{\sin^2\theta}{8(1+r^2)}|e^{i\phi} - re^{i(\alpha-\phi)}|^2 - \cos\theta\right] \\ &= k^4a^6\bigg[\frac{5}{4} - \cos\theta - \frac{1}{2}\sin^2\theta\left(1 + \frac{2r}{1+r^2}\cos(\alpha-2\phi)\right) \\ &\qquad - \frac{1}{8}\sin^2\theta\left(1 - \frac{2r}{1+r^2}\cos(\alpha-2\phi)\right)\bigg] \\ &= k^4a^6\left[\frac{5}{8}(2-\sin^2\theta) - \cos\theta - \frac{3}{4}\left(\frac{r}{1+r^2}\right)\sin^2\theta\cos(\alpha-2\phi)\right]\end{aligned}$$

This is identical to the $l = 1$ partial wave result (4).

10.3 A solid uniform sphere of radius R and conductivity σ acts as a scatterer of a plane-wave beam of unpolarized radiation of frequency ω, with $\omega R/c \ll 1$. The conductivity is large enough that the skin depth δ is small compared to R.

a) Justify and use a magnetostatic scalar potential to determine the magnetic field around the sphere, assuming the conductivity is infinite. (Remember that $\omega \neq 0$.)

We first note that for harmonic fields ($\omega \neq 0$) both the magnetic field and electric field must vanish inside a perfect conductor. Furthermore, there are no source currents outside the solid sphere. As a result of $\vec{J} = 0$, and since we are in the long wavelength limit $kR \ll 1$ (so we may work with a quasi-static magnetic field with $\vec{\nabla} \cdot \vec{B} \approx 0$), we may use a magnetostatic scalar potential $\vec{B} = -\vec{\nabla}\Phi_M$, at least in the vicinity (but always outside) of the sphere. Immediately outside the sphere, we may take a Legendre expansion

$$\begin{aligned}\Phi_M &= -B_0 z + \sum_l \frac{\alpha_l}{r^{l+1}} P_l(\cos\theta) \\ &= -B_0 r P_1(\cos\theta) + \sum_l \frac{\alpha_l}{r^{l+1}} P_l(\cos\theta)\end{aligned}$$

Note that we have taken the incident magnetic field to point along the z direction. (Since electromagnetic waves are transverse, this means the incident wave is actually traveling in the x-y plane.) We now use the fact that the perpendicular magnetic field must vanish at the surface $r = R$ of the conducting sphere. This gives

$$0 = B_r\big|_{r=R} = -\left.\frac{\partial \Phi_M}{\partial r}\right|_{r=R} = B_0 P_1(\cos\theta) + \sum_l \frac{(l+1)\alpha_l}{R^{l+2}} P_l(\cos\theta)$$

Since the Legendre polynomials form an orthogonal set, this indicates that all α_l must vanish for $l \neq 1$, while

$$\alpha_1 = -\tfrac{1}{2} B_0 R^3$$

This gives

$$\Phi_M = -B_0\left(r + \frac{R^3}{2r^2}\right) P_1(\cos\theta) = -B_0 z\left(1 + \frac{R^3}{2r^3}\right)$$

The resulting magnetic field is

$$\vec{B} = -\vec{\nabla}\Phi_M = B_0\left[\hat{z} - \frac{R^3}{2}\frac{3\hat{r}(\hat{r}\cdot\hat{z}) - \hat{z}}{r^3}\right] \tag{5}$$

The second term is clearly that of a magnetic dipole of strength

$$\vec{m} = -\frac{2\pi R^3}{\mu_0}\vec{B}_0 \qquad (\vec{B}_0 = B_0\hat{z})$$

This agrees with the conducting sphere result of (10.13). When combined with the electric dipole term, this gives the long wavelength scattering cross section of (10.14).

b) Use the technique of Section 8.1 to determine the absorption cross section of the sphere. Show that it varies as $(\omega)^{1/2}$ provided σ is independent of frequency.

We start with the power loss calculation

$$P_{\text{loss}} = \frac{1}{2\sigma\delta}\int |\hat{n}\times\vec{H}|^2 da$$

where

$$\hat{n}\times\vec{H} = \frac{1}{\mu_0}\hat{r}\times\vec{B} = \frac{B_0}{\mu_0}\left[1+\frac{R^3}{2r^3}\right]_{r=R}\hat{r}\times\hat{z} = -\frac{3B_0}{2\mu_0}\sin\theta\hat{\phi}$$

We have used $\vec{B}$ given in (5), and evaluated the field at the surface of the conductor. Integrating this over the sphere gives

$$P_{\text{loss}} = \frac{1}{2\sigma\delta}\frac{9|B_0|^2}{4\mu_0^2}\int \sin^2\theta\, R^2 d\cos\theta\, d\phi = \frac{3\pi|B_0|^2R^2}{\sigma\delta\mu_0^2}$$

For normalization, note that the incident flux is

$$I_0 = \frac{1}{2Z_0}|\vec{E}_0|^2 = \frac{c^2}{2\sqrt{\mu_0/\epsilon_0}}|\vec{B}_0|^2 = \frac{Z_0}{2\mu_0^2}|B_0|^2$$

This gives an absorption cross section

$$\sigma_{\text{abs}} = \frac{P_{\text{loss}}}{I_0} = \frac{6\pi R^2}{\sigma\delta Z_0}$$

Using $\delta = \sqrt{2/\mu_0\sigma\omega}$ gives

$$\sigma_{\text{abs}} = 6\pi R^2\sqrt{\frac{\epsilon_0\omega}{2\sigma}}$$

which is clearly proportional to $(\omega)^{1/2}$ provide σ is independent of frequency.

10.8 Consider the scattering of a plane wave by a nonpermeable sphere of radius a and very good, but not perfect, conductivity following the spherical multipole field approach of Section 10.4. Assume that $ka \ll 1$ and that the skin depth $\delta < a$.

a) Show from the analysis of Section 8.1 that

$$\frac{Z_s}{Z_0} = \frac{k\delta}{2}(1-i)$$

The 'surface impedance' Z_s is defined as

$$\vec{E}_{\parallel} = Z_s\hat{n}\times\vec{H}$$

where both $\vec{E}$ and $\vec{H}$ are measured immediately outside the surface of the conductor. Comparing this with the analysis of Section 8.1

$$\vec{E}_{\parallel} = \sqrt{\frac{\mu_c \omega}{2\sigma}}(1-i)\hat{n} \times \vec{H}_{\parallel}$$

gives

$$Z_s = \sqrt{\frac{\mu_c \omega}{2\sigma}}(1-i) = \frac{\delta \mu_c \omega}{2}(1-i)$$

where $\delta = \sqrt{2/\mu_c \sigma \omega}$ is the skin depth. Using $\omega = ck$ gives

$$\frac{Z_s}{Z_0} = \frac{k\delta}{2}\frac{\mu_c}{\mu_0}(1-i)$$

which reduces to

$$\frac{Z_s}{Z_0} = \frac{k\delta}{2}(1-i) \tag{6}$$

for a nonpermeable conductor ($\mu_c = \mu_0$).

b) In the long-wavelength limit, show that for $l = 1$ the coefficients $\alpha_{\pm}(l)$ and $\beta_{\pm}(l)$ in (10.65) are

$$\alpha_{\pm}(l) \approx -\frac{2i}{3}(ka)^3 \left[\frac{\left(1 - \frac{\delta}{a}\right) - i\frac{\delta}{a}}{\left(1 + \frac{\delta}{2a}\right) + i\frac{\delta}{2a}} \right]$$

$$\beta_{\pm}(1) \approx \frac{4i}{3}(ka)^3$$

For a sphere of radius a and surface impedance Z_s, the coefficients $\alpha_{\pm}(l)$ are given by

$$\alpha_{\pm}(l) = -2\frac{j_l - i\frac{Z_s}{Z_0}\frac{1}{x}\frac{d}{dx}(xj_l)}{h_l^{(1)} - i\frac{Z_s}{Z_0}\frac{1}{x}\frac{d}{dx}(xh_l^{(1)})}$$

where $x = ka$. For $l = 1$, the explicit functions are

$$j_1 = \frac{\sin x}{x^2} - \frac{\cos x}{x}, \qquad h_1^{(1)} = -\frac{e^{ix}}{x}\left(1 + \frac{i}{x}\right)$$

In the long wavelength limit, $x \ll 1$, these may be expanded as

$$j_1 \approx \frac{x}{3}, \qquad \frac{1}{x}\frac{d}{dx}(xj_1) \approx \frac{2}{3}$$

$$h_1^{(1)} \approx -\frac{i}{x^2}, \qquad \frac{1}{x}\frac{d}{dx}(xh_1^{(1)}) \approx \frac{i}{x^3}$$

Hence

$$\alpha_\pm(1) \approx -2\frac{\frac{x}{3} - i\frac{2}{3}\frac{Z_s}{Z_0}}{-\frac{i}{x^2} + \frac{1}{x^3}\frac{Z_s}{Z_0}} = -\frac{2i}{3}x^3\frac{x - 2i\frac{Z_s}{Z_0}}{x + i\frac{Z_s}{Z_0}} \tag{7}$$

Noting from (6) that

$$\frac{Z_s}{Z_0} = x\frac{\delta}{2a}(1-i)$$

we see that

$$\alpha_\pm(1) \approx -\frac{2i}{3}x^3\frac{1 - i\frac{\delta}{a}(1-i)}{1 + i\frac{\delta}{2a}(1-i)} = -\frac{2i}{3}(ka)^3\frac{1 - \frac{\delta}{a} - i\frac{\delta}{a}}{1 + \frac{\delta}{2a} + i\frac{\delta}{2a}} \tag{8}$$

Incidentally, note that, had we used the expression

$$\alpha_\pm(l) + 1 = -\left[\frac{h_l^{(2)} - i\left(\frac{Z_s}{Z_0}\right)\frac{1}{x}\frac{d}{dx}(xh_l^{(2)})}{h_l^{(1)} - i\left(\frac{Z_s}{Z_0}\right)\frac{1}{x}\frac{d}{dx}(xh_l^{(1)})}\right]$$

we would have gotten the same result. However, in this case, it would have been necessary to expand all terms up to and including $\mathcal{O}(x^3)$.

For the $\beta_\pm(1)$ coefficients, we note that they may be obtained from (7) by the substitution $Z_s/Z_0 \to Z_0/Z_s$

$$\beta_\pm(1) \approx -\frac{2i}{3}x^3\frac{x - 2i\frac{Z_0}{Z_s}}{x + i\frac{Z_0}{Z_s}} = \frac{4i}{3}x^3\frac{1 + \frac{i}{2}x\frac{Z_s}{Z_0}}{1 - ix\frac{Z_s}{Z_0}}$$

In this case, the leading order term is especially simple

$$\beta_\pm(1) \approx \frac{4i}{3}(ka)^3 \tag{9}$$

c) Write out explicitly the differential scattering cross section, correct to *first* order in δ/a and lowest order in ka.

In terms of the $\alpha_\pm(l)$ and $\beta_\pm(l)$ coefficients, the differential scattering cross section is

$$\frac{d\sigma_{\rm sc}}{d\Omega} = \frac{\pi}{2k^2}\left|\sum_l \sqrt{2l+1}(\alpha_\pm(l)\vec{X}_{l,\pm1} \pm i\beta_\pm(l)\hat{n}\times\vec{X}_{l,\pm1})\right|^2$$

In the long wavelength limit, we restrict to $l=1$ to obtain

$$\frac{d\sigma_{\rm sc}}{d\Omega} = \frac{3\pi}{2k^2}\left|\alpha_\pm(1)\vec{X}_{1,\pm1} \pm i\beta_\pm(1)\hat{n}\times\vec{X}_{1,\pm1}\right|^2$$

Using the explicit form of the vector spherical harmonics given in (2) and (3), we see that

$$\begin{aligned}\frac{d\sigma_{\rm sc}}{d\Omega} &= \frac{9}{32k^2}\left|(\alpha_\pm(1)+\beta_\pm(1)\cos\theta)\hat\theta \pm i(\beta_\pm(1)+\alpha_\pm(1)\cos\theta)\hat\phi\right|^2\\ &= \frac{9}{32k^2}\left[|\alpha_\pm(1)+\beta_\pm(1)\cos\theta|^2+|\beta_\pm(1)+\alpha_\pm(1)\cos\theta|^2\right]\\ &= \frac{9}{32k^2}\left[(|\alpha_\pm(1)|^2+|\beta_\pm(1)|^2)(1+\cos^2\theta)+4\Re(\alpha_\pm(1)\beta_\pm(1)^*)\cos\theta\right]\end{aligned}$$

To first order in δ/a, the expressions (8) and (9) give

$$\alpha_\pm(1)\approx -\frac{2i}{3}(ka)^3(1-\frac{3\delta}{2a}(1+i)),\qquad \beta_\pm(1)\approx\frac{4i}{3}(ka)^3$$

Hence

$$\begin{aligned}\frac{d\sigma_{\rm sc}}{d\Omega} &\approx \frac{1}{8k^2}(ka)^6\left[(|1-\tfrac{3\delta}{2a}(1+i)|^2+4)(1+\cos^2\theta)-8\Re(1-\tfrac{3\delta}{2a}(1+i))\cos\theta\right]\\ &= \frac{k^4a^6}{8}\left[(5-\tfrac{3\delta}{a})(1+\cos^2\theta)-8(1-\tfrac{3\delta}{2a})\cos\theta\right]\\ &= k^4a^6\left[\tfrac{5}{8}(1+\cos^2\theta)-\cos\theta-\tfrac{3\delta}{8a}(1+\cos^2\theta-4\cos\theta)\right]\end{aligned}$$

Note that this reduces to the perfectly conducting sphere result in the limit $\delta=0$.

d) Using (10.61), evaluate the absorption cross section. Show that to first order in δ it is $\sigma_{\rm abs}\approx 3\pi(k\delta)a^2$. How different is the value if $\delta=a$?

The absorption cross section is given by

$$\sigma_{\rm abs}=\frac{\pi}{2k^2}\sum_l(2l+1)[2-|\alpha(l)+1|^2-|\beta(l)+1|^2]$$

In the long wavelength limit we restrict to $l=1$ and write

$$\sigma_{\rm abs}=-\frac{3\pi}{2k^2}[2\Re\alpha(1)+2\Re\beta(1)+|\alpha(1)|^2+\beta(1)|^2]$$

For small $\alpha\sim(ka)^3\ll 1$ and $\beta\sim(ka)^3\ll 1$, we only need to retain the linear terms

$$\sigma_{\rm abs}\approx-\frac{3\pi}{k^2}\Re[\alpha(1)+\beta(1)]$$

From (9) we see that $\beta(1)$ is pure imaginary, and does not contribute to the absorption. Using (8) for $\alpha(1)$ gives

$$\sigma_{\rm abs}\approx -2\pi ka^3\Im\frac{1-\frac{\delta}{a}-i\frac{\delta}{a}}{1+\frac{\delta}{2a}+i\frac{\delta}{2a}}=3\pi k\delta a^2\left[1+\frac{\delta}{a}+\frac{\delta^2}{2a^2}\right]^{-1}$$

To lowest order in δ, this is simply

$$\sigma_{\text{abs}} \approx 3\pi(k\delta)a^2$$

On the other hand, for $\delta = a$, we find

$$\sigma_{\text{abs}} \approx 3\pi(k\delta)a^2 \times \left(\frac{2}{5}\right)$$

Hence the true value of the absorption cross section for $\delta = a$ is 2/5 as large as the simple first order approximation. (This is all done in the long wavelength approximation, of course. Note furthermore that when $\delta = a$, the skin depth is comparable to the size of the sphere. In this case, we can hardly expect to trust the analysis of Section 8.1.)

10.9 In the scattering of light by a gas very near the critical point the scattered light is observed to be "whiter" (i.e., its spectrum is less predominantly peaked toward the blue) than far from the critical point. Show that this can be understood by the fact that the volumes of the density fluctuations become large enough that Rayleigh's law fails to hold. In particular, consider the lowest order approximation to the scattering by a uniform dielectric sphere of radius a whose dielectric constant ϵ_r differs only slightly from unity.

a) Show that for $ka \gg 1$, the differential cross section is sharply peaked in the forward direction and the total scattering cross section is approximately

$$\sigma \approx \frac{\pi}{2}(ka)^2|\epsilon_r - 1|^2 a^2$$

with a k^2, rather than k^4, dependence on frequency.

Since ϵ_r differs only slightly from unity, we may use the first Born approximation. The scattering amplitude then has the form

$$\frac{\vec{\epsilon}^* \cdot \vec{A}_{\text{sc}}^{(1)}}{D_0} = \frac{k^2}{4\pi}\int e^{i\vec{q}\cdot\vec{x}}\left[\vec{\epsilon}^* \cdot \vec{\epsilon}_0 \frac{\delta\epsilon}{\epsilon_0} + (\hat{n}\times\vec{\epsilon}^*)\cdot(\hat{n}_0\times\vec{\epsilon}_0)\frac{\delta\mu}{\mu_0}\right] d^3x$$

where $\vec{q} = k(\hat{n}_0 - \hat{n})$, so that

$$q^2 = k^2(2 - 2\cos\theta) = (2k)^2 \sin^2\frac{\theta}{2} \tag{10}$$

Here θ is the angle between $\hat{n}$ and $\hat{n}_0$ (ie the incident and scattered waves). For the dielectric sphere, we set $\delta\mu = 0$. Noting that

$$\frac{\delta\epsilon}{\epsilon_0} = \begin{cases} \epsilon_r - 1 & r < a \\ 0 & r > a \end{cases}$$

we end up with

$$\frac{\vec{\epsilon}^* \cdot \vec{A}_{\rm sc}^{(1)}}{D_0} = \frac{k^2}{4\pi}(\epsilon_r - 1)(\vec{\epsilon}^* \cdot \vec{\epsilon}_0) \int_{r<a} e^{i\vec{q}\cdot\vec{x}} d^3x$$

The integral can be performed in spherical coordinates

$$\begin{aligned}\int_{r<a} e^{i\vec{q}\cdot\vec{x}} d^3x &= \int_{r<a} e^{iqr\cos\gamma} r^2 dr d\cos\gamma d\phi \\ &= 2\pi \int_0^a dr \int_{-1}^1 d\cos\gamma\, r^2 e^{iqr\cos\gamma} \\ &= \frac{4\pi}{q}\int_0^a r\sin(qr)dr = \frac{4\pi}{q^3}[\sin(qa) - qa\cos(qa)]\end{aligned}$$

As a result

$$\begin{aligned}\frac{\vec{\epsilon}^* \cdot \vec{A}_{\rm sc}^{(1)}}{D_0} &= \frac{(ka)^2}{q}(\epsilon_r - 1)(\vec{\epsilon}^* \cdot \vec{\epsilon}_0)\frac{\sin(qa) - qa\cos(qa)}{(qa)^2} \\ &= \frac{(ka)^2}{q}(\epsilon_r - 1)(\vec{\epsilon}^* \cdot \vec{\epsilon}_0) j_1(qa)\end{aligned}$$

where j_1 is the $l = 1$ spherical Bessel function

$$j_1(\zeta) = \frac{\sin\zeta}{\zeta^2} - \frac{\cos\zeta}{\zeta}$$

The differential cross section is then

$$\frac{d\sigma}{d\Omega} = \left|\frac{\vec{\epsilon}^* \cdot \vec{A}_{\rm sc}^{(1)}}{D_0}\right|^2 = k^4 a^6 |\epsilon_r - 1|^2 \left(\frac{j_1(qa)}{qa}\right)^2 |\vec{\epsilon}^* \cdot \vec{\epsilon}_0|^2$$

where q is given by (10). The unpolarized cross section is

$$\frac{d\sigma}{d\Omega} = k^4 a^6 |\epsilon_r - 1|^2 \left(\frac{j_1(qa)}{qa}\right)^2 \frac{1 + \cos^2\theta}{2} \tag{11}$$

Note that in the long wavelength limit ($ka \ll 1$) we also have $qa \ll 1$. In this case, we use the small argument expansion of the spherical Bessel function

$$j_1(\zeta) \approx \frac{\zeta}{3} - \cdots \qquad (\zeta \to 0)$$

to obtain

$$\frac{d\sigma}{d\Omega} \approx \frac{a^2}{9}(ka)^4 |\epsilon_r - 1|^2 \frac{1 + \cos^2\theta}{2}$$

which agrees with the long wavelength dipole approximation when ϵ_r is close to unity.

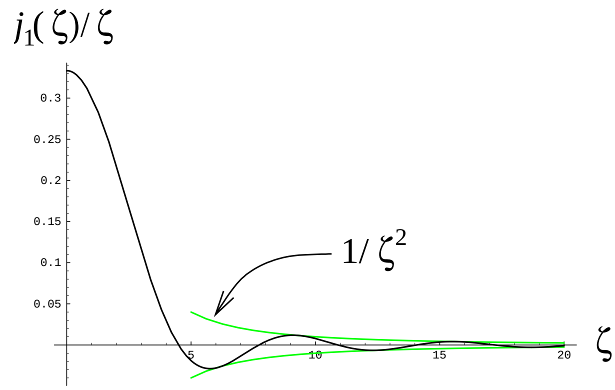
We are, of course, more interested in the short wavelength limit $ka \gg 1$. In this case, we note that the argument of the spherical Bessel function is

$$qa = 2ka \sin\frac{\theta}{2} \tag{12}$$

This quantity vanishes in the forward direction ($\theta = 0$), but otherwise is very large when $ka \gg 1$. In fact, the behavior of $j_1(\zeta)/\zeta$ is as follows

$$\frac{j_1(\zeta)}{\zeta} \sim \begin{cases} 1/3 & \zeta \ll 1 \\ -\cos\zeta/\zeta^2 & \zeta \gg 1 \end{cases}$$

This is peaked when $\zeta \approx 0$



As a result, the cross section (11) falls off as

$$\frac{d\sigma}{d\Omega} \sim \frac{1}{(qa)^4} = \frac{1}{[2ka\sin(\theta/2)]^4} \qquad (ka \gg 1)$$

away from the forward direction. Looking at the figure, we see that the cross section is large for $qa \lesssim 2$ but rapidly falls off for $qa \gtrsim 2$. From (12), we see that this forward peak corresponds to a cone with

$$\theta \lesssim \frac{2}{ka} \ll 1$$

With this in mind, we may make a rough estimate of the total cross section

$$\begin{aligned} \sigma &= k^4 a^6 |\epsilon_r - 1|^2 \int \left(\frac{j_1(qa)}{qa}\right)^2 \frac{1+\cos^2\theta}{2} d\Omega \\ &\approx k^4 a^6 |\epsilon_r - 1|^2 \left(\frac{1}{3}\right)^2 \times (\pi\theta^2)\Big|_{\theta=2/ka} \\ &= \frac{4\pi}{9} k^2 a^4 |\epsilon_r - 1|^2 \end{aligned}$$

We can make a better estimate by approximating the integral more carefully. Since the integrand is highly peaked at $\theta \approx 0$, we take

$$
\begin{aligned}
\int \left(\frac{j_1(qa)}{qa}\right)^2 \frac{1+\cos^2\theta}{2} d\Omega &\approx 2\pi \int_0^\pi \left(\frac{j_1(qa)}{qa}\right)^2 \sin\theta \, d\theta \\
&\approx 2\pi \int_0^\pi \left(\frac{j_1(ka\theta)}{ka\theta}\right)^2 \theta \, d\theta \\
&\approx \frac{2\pi}{(ka)^2} \int_0^\infty j_1(\zeta)^2 \frac{d\zeta}{\zeta} = \frac{\pi}{2(ka)^2}
\end{aligned}
$$

This gives an approximate value of the total cross section

$$
\sigma \approx \frac{\pi a^2}{2}(ka)^2 |\epsilon_r - 1|^2
$$