Physics 506 **Winter 2008**

Homework Assignment #5 — Solutions

Textbook problems: Ch. 9: 9.22, 9.23, 9.24
Ch. 10: 10.1

9.22 A spherical hole of radius a in a conducting medium can serve as an electromagnetic resonant cavity.

a) Assuming infinite conductivity, determine the transcendental equations for the characteristic frequencies ω_{lm} of the cavity for TE and TM modes.

Because of the spherical symmetry, it is natural to describe the modes of the spherical cavity in terms of a vector spherical wave expansion. These waves fall into either TE or TM modes, depending on whether $\vec{r}\cdot\vec{E}=0$ or $\vec{r}\cdot\vec{H}=0$, respectively. The TE (or magnetic multipole) modes are given by

$$\vec{H}=-\frac{i}{k}\vec{\nabla}\times[j_l(kr)\vec{X}_{lm}],\qquad \vec{E}=Z_0 j_l(kr)\vec{X}_{lm}\tag{1}$$

where we have chosen the spherical Bessel function $j_l(kr)$ since it is regular at $r=0$. FOr a perfect conductor, we impose the boundary conditions $H_\perp=0$ and $E_\parallel=0$ at $r=a$. More precisely, we demand

$$\hat{r}\cdot\vec{H}\Big|_{r=a}=0,\qquad \hat{r}\times\vec{E}\Big|_{r=a}=0$$

These are equivalent to the condition $j_l(ka)=0$, and leads to the quantization $k_{nlm}=x_{ln}/a$ where x_{ln} is the n-th zero of the spherical Bessel function j_l. The TE_{nlm} frequencies are thus

$$\omega_{nlm}=\frac{x_{ln}c}{a},\qquad j_l(x_{ln})=0,\qquad l\ge 1,\quad |m|\le l$$

Each frequency specified by l and n is $(2l+1)$-fold degenerate, with azimuthal quantum number labeled by m.

The TM (or electric multipole) modes are similar, although the boundary conditions are somewhat more involved. The modes themselves are given by

$$\vec{H}=j_l(kr)\vec{X}_{lm},\qquad \vec{E}=Z_0\frac{i}{k}\vec{\nabla}\times[j_l(kr)\vec{X}_{lm}]\tag{2}$$

This time, the $H_\perp=0$ boundary condition is automatic, while the $E_\parallel=0$ condition gives

$$\vec{r}\times(\vec{\nabla}\times[j_l(kr)\vec{X}_{lm}])\Big|_{r=a}=0$$

This vector quantity may be simplified using

$$\vec{r}\times(\vec{\nabla}\times\vec{V}) = \vec{\nabla}(\vec{r}\cdot\vec{V})-\vec{V}-(\vec{r}\cdot\vec{\nabla})\vec{V} = \vec{\nabla}(\vec{r}\cdot\vec{V})-\left(1+r\frac{\partial}{\partial r}\right)\vec{V} = \vec{\nabla}(\vec{r}\cdot\vec{V})-\frac{\partial}{\partial r}r\vec{V}$$

Using $\vec{V} = j_l(kr)\vec{X}_{lm}$ with $\vec{r}\cdot\vec{X}_{lm} = 0$ gives

$$\vec{r}\times(\nabla\times[j_l(kr)\vec{X}_{lm}]) = -\frac{\partial}{\partial r}(rj_l(kr))\vec{X}_{lm} \tag{3}$$

Hence the $E_\parallel = 0$ boundary condition leads to the TM_{nlm} frequencies

$$\omega_{nlm} = \frac{y_{ln}c}{a}, \qquad \frac{d}{dx}[xj_l(x)]\bigg|_{z=y_{ln}} = 0, \qquad l\geq 1, \quad |m|\leq l$$

The y_{ln} correspond to zeros of $[xj_l(x)]'$ or equivalently $j_l(x)+xj_l'(x)$.

b) Calculate numerical values for the wavelength λ_{lm} in units of the radius a for the four lowest modes for TE and TM waves.

The numerical values for the wavelengths are obtained from the zeros x_{ln} and y_{ln}. For TE_{nlm} modes, the first four zeros of $j_l(x)$ are

$$x_{11} = 4.4934, \qquad x_{21} = 5.7635, \qquad x_{31} = 6.9879, \qquad x_{12} = 7.7253$$

Since $k_{nlm} = x_{ln}/a$ and $\lambda_{nlm} = 2\pi/k_{nlm}$, we end up with $\lambda_{nlm}/a = 2\pi/x_{ln}$ or

$$\frac{\lambda_{11m}}{a} = 1.398, \qquad \frac{\lambda_{12m}}{a} = 1.090, \qquad \frac{\lambda_{13m}}{a} = 0.899, \qquad \frac{\lambda_{21m}}{a} = 0.813$$

All these modes are $(2l+1)$-fold degenerate.

For TM_{nlm} modes, the first four zeros of $[xj_l(x)]'$ are

$$y_{11} = 2.7437, \qquad y_{21} = 3.8702, \qquad y_{31} = 4.9734, \qquad y_{41} = 6.0619$$

with corresponding wavelengths

$$\frac{\lambda_{11m}}{a} = 2.290, \qquad \frac{\lambda_{12m}}{a} = 1.623, \qquad \frac{\lambda_{13m}}{a} = 1.263, \qquad \frac{\lambda_{14m}}{a} = 1.036$$

Note that the next mode, given by $y_{12} = 6.1168$ is nearly degenerate with y_{41}.

c) Calculate explicitly the electric and magnetic fields inside the cavity for the lowest TE and lowest TM mode.

The lowest TE and TM modes both have $l = 1$. Thus we begin with an overview of $l = 1$ vector spherical harmonics

$$\vec{X}_{1m} = \frac{1}{\sqrt{2}}\vec{L}Y_{1m}$$

It is natural to write the angular momentum operator $\vec{L}$ in terms of raising and lowering components

$$L_+ = L_x + iL_y, \qquad L_- = L_x - iL_y, \qquad L_z$$

Using

$$\begin{aligned} L_+ Y_{lm} &= \sqrt{l(l+1) - m(m+1)} Y_{l,m+1} \\ L_- Y_{lm} &= \sqrt{l(l+1) - m(m-1)} Y_{l,m-1} \\ L_z Y_{lm} &= m Y_l m \end{aligned}$$

for $l = 1$ gives

$$\begin{aligned} &X_{11}^+ = 0, && X_{11}^z = \frac{1}{\sqrt{2}} Y_{11}, && X_{11}^- = Y_{10} \\ &X_{10}^+ = Y_{11}, && X_{10}^z = 0, && X_{10}^- = Y_{1,-1} \\ &X_{1,-1}^+ = Y_{10}, && X_{1,-1}^z = -\frac{1}{\sqrt{2}} Y_{1,-1}, && X_{1,-1}^- = 0 \end{aligned} \tag{4}$$

A vector with components (V_+, V_-, V_z) can be converted to spherical coordinates (V_r, V_θ, V_ϕ) according to

$$\begin{aligned} V_r &= \frac{1}{2}(V_+ e^{-i\phi} + V_- e^{i\phi}) \sin\theta + V_z \cos\theta \\ V_\theta &= \frac{1}{2}(V_+ e^{-i\phi} + V_- e^{i\phi}) \cos\theta - V_z \sin\theta \\ V_\phi &= -\frac{i}{2}(V_+ e^{-i\phi} - V_- e^{i\phi}) \end{aligned}$$

Using the explicit form of the spherical harmonics then gives

$$\begin{aligned} &X_{11}^r = 0, && X_{11}^\theta = \sqrt{\frac{3}{16\pi}} e^{i\phi}, && X_{11}^\phi = i\sqrt{\frac{3}{16\pi}} \cos\theta e^{i\phi} \\ &X_{10}^r = 0, && X_{10}^\theta = 0, && X_{10}^\phi = i\sqrt{\frac{3}{8\pi}} \sin\theta \\ &X_{1,-1}^r = 0, && X_{1,-1}^\theta = \sqrt{\frac{3}{16\pi}} e^{-i\phi}, && X_{1,-1}^\phi = -i\sqrt{\frac{3}{16\pi}} \cos\theta e^{-i\phi} \end{aligned}$$

We are now ready to examine the explicit electric and magnetic fields. From the expression (1) for TE_{nlm} modes, we have

$$\vec{E}_{11m} = Z_0 j_1(kr) \vec{X}_{1m}, \qquad \vec{H}_{11m} = -\frac{i}{Z_0 k} \vec{\nabla} \times \vec{E}_{11m}$$

The $m = 0$ mode is the most straightforward to write down

$$\begin{aligned} \vec{E}_{110} &= iZ_0 \sqrt{\frac{3}{8\pi}} j_1(kr) \sin\theta \hat{\phi} \\ \vec{H}_{110} &= \frac{1}{kr} \sqrt{\frac{3}{8\pi}} \Big(2j_1(kr) \cos\theta \hat{r} - [krj_0(kr) - j_1(kr)] \sin\theta \hat{\theta} \Big) \end{aligned} \tag{5}$$

Note that we have used the spherical Bessel function identity

$$j_l'(\zeta) = j_{l-1}(\zeta) - \frac{l+1}{\zeta} j_l(\zeta)$$

Even more explicitly, we have

$$\begin{aligned} j_1(\zeta) &= \frac{\sin\zeta}{\zeta^2} - \frac{\cos\zeta}{\zeta} \\ [\zeta j_1(\zeta)]' = \zeta j_0(\zeta) - j_1(\zeta) &= -\left(\frac{1}{\zeta^2} - 1\right)\sin\zeta + \frac{\cos\zeta}{\zeta} \end{aligned}$$

The $m = 1$ mode is given by

$$\begin{aligned} \vec{E}_{111} &= Z_0\sqrt{\frac{3}{16\pi}} j_1(kr) e^{i\phi}(\hat{\theta} + i\cos\theta\hat{\phi}) \\ \vec{H}_{111} &= \frac{1}{kr}\sqrt{\frac{3}{16\pi}} e^{i\phi}\left(-2j_1(kr)\sin\theta\hat{r} - [krj_0(kr) - j_1(kr)](\cos\theta\hat{\theta} + i\hat{\phi})\right) \end{aligned} \tag{6}$$

while the $m = -1$ mode is given by

$$\begin{aligned} \vec{E}_{11,-1} &= Z_0\sqrt{\frac{3}{16\pi}} j_1(kr) e^{-i\phi}(\hat{\theta} - i\cos\theta\hat{\phi}) \\ \vec{H}_{11,-1} &= \frac{1}{kr}\sqrt{\frac{3}{16\pi}} e^{-i\phi}\left(2j_1(kr)\sin\theta\hat{r} + [krj_0(kr) - j_1(kr)](\cos\theta\hat{\theta} - i\hat{\phi})\right) \end{aligned} \tag{7}$$

We now turn to the lowest TM mode, which is the TM_{11m} mode with fields given by (2)

$$\vec{H}_{11m} = j_1(kr)\vec{X}_{1m}, \qquad \vec{E}_{11m} = \frac{iZ_0}{k}\vec{\nabla}\times\vec{H}_{11m}$$

It ought to be clear the the roles of $\vec{E}$ and $\vec{H}$ are interchanged between the TE and TM modes. In particular, the TM_{11m} fields may be obtained from the TE_{11m} fields of (5), (6) and (7) through the substitution

$$\vec{E} \to Z_0\vec{H}, \qquad Z_0\vec{H} \to -\vec{E}$$

(This is essentially the action of electric-magnetic duality.) Explicitly, the TM_{11m}

modes correspond to

$$\vec{H}_{110} = i\sqrt{\frac{3}{8\pi}} j_1(kr)\sin\theta\hat{\phi}$$

$$\vec{E}_{110} = \frac{Z_0}{kr}\sqrt{\frac{3}{8\pi}}\left(2j_1(kr)\cos\theta\hat{r} - [krj_0(kr) - j_1(kr)]\sin\theta\hat{\theta}\right)$$

$$\vec{H}_{111} = \sqrt{\frac{3}{16\pi}} j_1(kr)e^{i\phi}(\hat{\theta} + i\cos\theta\hat{\phi})$$

$$\vec{E}_{111} = \frac{Z_0}{kr}\sqrt{\frac{3}{16\pi}} e^{i\phi}\left(-2j_1(kr)\sin\theta\hat{r} - [krj_0(kr) - j_1(kr)](\cos\theta\hat{\theta} + i\hat{\phi})\right)$$

$$\vec{H}_{11,-1} = \sqrt{\frac{3}{16\pi}} j_1(kr)e^{-i\phi}(\hat{\theta} - i\cos\theta\hat{\phi})$$

$$\vec{E}_{11,-1} = \frac{Z_0}{kr}\sqrt{\frac{3}{16\pi}} e^{-i\phi}\left(2j_1(kr)\sin\theta\hat{r} + [krj_0(kr) - j_1(kr)](\cos\theta\hat{\theta} - i\hat{\phi})\right)$$

Note, however, that the wavenumbers k_{nlm} are quantized differently for the TE versus the TM modes.

9.23 The spherical resonant cavity of Problem 9.22 has nonpermeable walls of large, but finite, conductivity. In the approximation that the skin depth δ is small compared to the cavity radius a, show that the Q of the cavity, defined by equation (8.86), is given by

$$Q = \frac{a}{\delta} \qquad \text{for all TE modes}$$

$$Q = \frac{a}{\delta}\left(1 - \frac{l(l+1)}{x_{lm}^2}\right) \qquad \text{for TM modes}$$

where $x_{lm} = (a/c)\omega_{lm}$ for TM modes.

In order to calculate the Q factor, we need to obtain both the stored energy and the power loss at the walls. We start with the simpler case of TE modes, given by (1). The energy density for harmonic fields is

$$u = \frac{\epsilon_0}{4}|\vec{E}|^2 + \frac{\mu_0}{4}|\vec{H}|^2$$

However, the energy is equally distributed between $\vec{E}$ and $\vec{H}$. Thus for TE modes we may immediately write down

$$u = \frac{\epsilon_0}{2}|\vec{E}|^2 = \frac{\mu_0}{2} j_l(kr)^2 |\vec{X}_{lm}|^2$$

The stored energy is given by integrating this over the volume of the sphere

$$U = \frac{\mu_0}{2}\int j_l(kr)^2|\vec{X}_{lm}|^2\, r^2 dr d\Omega = \frac{\mu_0}{2}\int_0^a j_l(kr)^2 r^2 dr$$

Note that we have used orthonormality of the vector spherical harmonics to simplify the integral. We now use the normalization integral for spherical Bessel functions

$$\int_0^a j_l(x_{lm}\rho/a) j_l(x_{ln}\rho/a)\rho^2 d\rho = \tfrac{1}{2}a^3[j_l'(x_{ln})]^2\delta_{mn}$$

to obtain

$$U_{lmn} = \frac{\mu_0 a^3}{4} j_l'(x_{ln})^2 \tag{8}$$

The power loss is given in terms of the tangential magnetic field at the conducting surface

$$P = \frac{1}{2\sigma\delta}\int |\hat{r}\times\vec{H}|^2 da$$

Using $\vec{H} = -(i/k)\vec{\nabla}\times j_l(kr)\vec{X}_{lm}$ from (1) as well as the vector identity (3) gives

$$\begin{aligned} P_{lmn} &= \frac{1}{2\sigma\delta}\int_{r=a}\left(\frac{1}{kr}\frac{d}{dr}r j_l(kr)\right)^2 |\vec{X}_{lm}|^2 r^2 d\Omega \\ &= \frac{1}{2\sigma\delta k^2}([r j_l(kr)]')^2\bigg|_{r=a} \\ &= \frac{1}{2\sigma\delta k^2}(j_l(ka) + ka j_l'(ka))^2 = \frac{a^2}{2\sigma\delta} j_l'(x_{ln})^2 \end{aligned} \tag{9}$$

where in the last line we made use of the fact that $ka = x_{ln}$ and that $j_l(x_{ln}) = 0$. Combining (8) and (9) then gives the Q factor for TE modes

$$Q_{lmn} = \omega\frac{U_{lmn}}{P_{lmn}} = \frac{\mu_0\sigma\omega\delta a}{2} = \frac{a}{\delta}$$

where we made use of the definition of the skin depth $\delta = \sqrt{2/\mu_0\sigma\omega}$.

The calculation for TM modes is similar. However, the appropriate spherical Bessel function normalization integral needs to be modified for integrating to zeros of $[x j_l(x)]'$. Here we simply state that the appropriate normalization integral may be written as

$$\int_0^a j_l(\alpha_m\rho/a) j_l(\alpha_n\rho/a)\rho^2 d\rho = \tfrac{1}{2}a^3\left(1 + \frac{p(p-1) - l(l+1)}{\alpha_n^2}\right)[j_l(\alpha_n)]^2\delta_{mn}$$

where α_n is the n-th positive zero of

$$[x^p j_l(x)]' = 0$$

The fields for the TM modes are given in (2), while the characteristic frequencies are given in terms of zeros of $[x j_l(x)]'$. We thus set $p = 1$ in the above normalization integral and use the notation y_{ln} to denote the n-th zero of $[x j_l(x)]' = 0$. The expression for the TM stored energy then becomes

$$U_{lmn} = \frac{\mu_0}{2}\int_0^a j_l(kr)^2 r^2 dr = \frac{\mu_0 a^3}{4}\left(1 - \frac{l(l+1)}{y_{ln}^2}\right) j_l(y_{ln})^2$$

The power loss is

$$P_{lmn} = \frac{1}{2\sigma\delta}\int |\hat{r}\times\vec{H}|^2 da = \frac{1}{2\sigma\delta}\int_{r=a} j_l(kr)^2|\hat{r}\times\vec{X}_{lm}^2|r^2 d\Omega = \frac{a^2}{2\sigma\delta}j_l(y_{mn})^2$$

As a result, the Q factor for a TM_{lmn} mode is

$$Q_{lmn} = \omega\frac{U_{lmn}}{P_{lmn}} = \frac{\mu_0\sigma\omega\delta a}{2}\left(1-\frac{l(l+1)}{y_{ln}^2}\right) = \frac{a}{\delta}\left(1-\frac{l(l+1)}{y_{ln}^2}\right)$$

9.24 Discuss the normal modes of oscillation of a perfectly conducting solid sphere of radius a in free space.

a) Determine the characteristic equations for the eigenfrequencies for TE and TM modes of oscilation. Show that the roots for ω always have a negative imaginary part, assuming a time dependence of $e^{-i\omega t}$.

Setting up this perfectly conducting sphere problem is similar to what we did for the spherical hole problem. However, an important feature of the sphere in free space is that the volume of the 'resonant cavity' is unbounded (ie it is all of space outside of the radius a). An important physical consequence of this is that oscillating electromagnetic fields will radiate out to infinity. Since power is 'lost' to infinity, these so-called normal modes are actually unstable in the sense that they decay away after a while. Such modes are generally denoted 'quasi-normal modes', and are described by a complex frequency ω. For

$$\omega = \omega_0 - \frac{i}{2}\gamma \tag{10}$$

the electric and magnetic fields behave as

$$\vec{E} \sim e^{-i\omega t} = e^{-\gamma t/2}e^{-i\omega_0 t}$$

Hence the imaginary part of the quasi-normal mode frequency governs the decay of the fields. Since energy is proportional to the square of the fields, the energy decays as $e^{-\gamma t}$. Note that $\gamma \geq 0$ is essential for this to make sense. If γ were negative, then the mode would grow exponentially with time. Clearly this would violate energy considerations. In fact, so long as radiation is emitted and reaches infinity, the mode must necessarily decay. In this case, we may argue that γ is strictly positive. In terms of the frequency ω in (10), energy conservation then demands that ω always has a negative imaginary part.

In order to actually work out the quasi-normal mode frequencies, we note that TE modes are given by the analog of (1) for the exterior problem with outgoing radiation

$$\vec{H} = -\frac{i}{k}\vec{\nabla}\times h_l^{(1)}(kr)\vec{X}_{lm}, \qquad \vec{E} = Z_0 h_l^{(1)}(kr)\vec{X}_{lm}$$

Here we have used physical outgoing radiation boundary conditions to select the first spherical Hankel function $h_l^{(1)}$. The TE boundary conditions are identical to what we found above, namely

$$\hat{r} \cdot \vec{H}\Big|_{r=a} = 0, \qquad \hat{r} \times \vec{E}\Big|_{r=a} = 0$$

This corresponds to the equation

$$h_l^{(1)}(ka) = 0 \qquad \text{(TE modes)}$$

Unlike in the case of the spherical Bessel functions $j_l(\zeta)$ and $n_l(\zeta)$, the spherical Hankel functions do not admit any real zeros. One way to see this is to note that $h_l^{(1)}(\zeta)$ is defined as the complex combination

$$h_l^{(1)}(\zeta) = j_l(\zeta) + i n_l(\zeta)$$

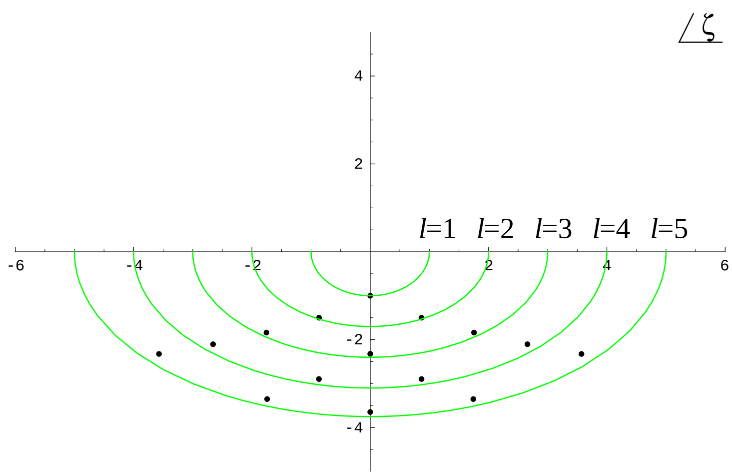
If ζ were real, then the only way for $h_l^{(1)}(\zeta)$ to vanish is if both real and imaginary parts [ie $j_l(\zeta)$ and $n_l(\zeta)$] were to simultaneously vanish for the same ζ. However, it is easy to see that the zeros of j_l and n_l never coincide. Therefore, the zeros of $h_l^{(1)}$ are always complex. In fact, $h_l^{(1)}$ has precisely l zeros in the complex plane. To show this, we note that $h_l^{(1)}(\zeta)$ may be written as a complex polynomial in $1/\zeta$ times the outgoing spherical wave factor $e^{i\zeta}/\zeta$. In particular

$$h_l^{(1)}(\zeta) = (-i)^{l+1} \frac{e^{i\zeta}}{\zeta} \sum_{s=0}^{l} \frac{(l+s)!}{s!(l-s)!} \left(\frac{i}{2\zeta}\right)^s$$

Ignoring the irregular point at infinity, the zeros of $h_l^{(1)}$ then correspond to the zeros of the polynomial

$$P_l(\zeta) = \sum_{s=0}^{l} \frac{(2l-s)!}{(l-s)!s!} (-2i\zeta)^s$$

Since this is a polynomial of degree l, it admits precisely l complex zeros. In fact, it can be shown that all these zeros have negative imaginary part, and approximately lie along an arc in the lower half complex ζ plane. The zeros of $h_l^{(1)}(\zeta)$ are plotted for small values of l as



The TE$_{nlm}$ frequencies are thus

$$\omega_{nlm} = \frac{x_{ln} c}{a}, \qquad h_l^{(1)}(x_{ln}) = 0, \qquad l \geq 1, \quad |m| \leq l, \quad n = 1, 2, \ldots, l$$

where x_{ln} denotes the n-th zero of the spherical Hankel function $h_l^{(1)}$.

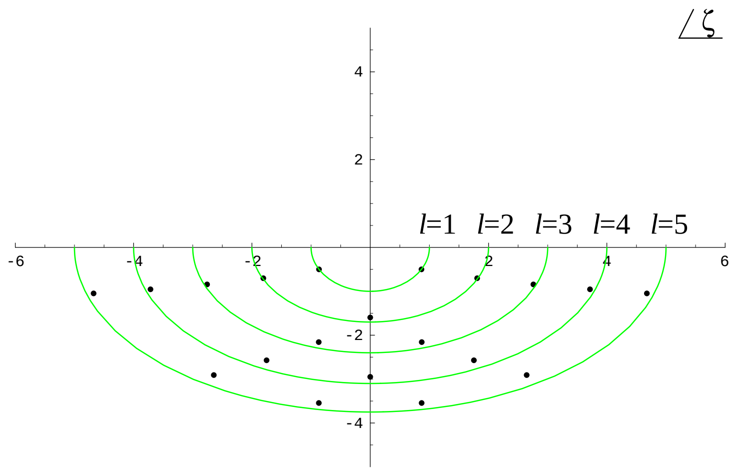
The TM modes may also be worked out in a similar fashion. In particular, the analog of (2) for the exterior problem is

$$\vec{H} = h_l^{(1)}(kr)\vec{X}_{lm}, \qquad \vec{E} = Z_0 \frac{i}{k} \vec{\nabla} \times [h_l^{(1)}(kr)\vec{X}_{lm}]$$

This time, the conducting sphere boundary conditions lead to

$$\frac{d}{dx}[x h_l^{(1)}(x)]\Big|_{x=ka} = 0 \qquad \text{(TM modes)}$$

This time, there are $l+1$ zeros, which also approximately lie along arcs in the lower half complex ζ plane



Hence the TM$_{nlm}$ frequencies are

$$\omega_{nlm} = \frac{y_{ln} c}{a}, \qquad \frac{d}{dx}[x h_l^{(1)}(x)]\Big|_{x=y_{ln}} = 0, \qquad l \geq 1, \quad |m| \leq l, \quad n = 1, 2, \ldots, l+1$$

b) Calculate the eigenfrequencies for the $l = 1$ and $l = 2$ TE and TM modes. Tabulate the wavelength (defined in terms of the real part of the frequency) in units of the radius a and the decay time (defined as the time taken for the *energy* to fall to e^{-1} of its initial value) in units of the transit time (a/c) for each of the modes.

For $l = 1$ and $l = 2$, the spherical Hankel functions are explicitly

$$h_1^{(1)}(\zeta) = -\frac{e^{i\zeta}}{\zeta}\left(1 + \frac{i}{\zeta}\right), \qquad h_2^{(1)}(\zeta) = \frac{ie^{i\zeta}}{\zeta}\left(1 + \frac{3i}{\zeta} - \frac{3}{\zeta^2}\right)$$

The zeros of $h_l^{(1)}$ $(l = 1, 2)$ are then

$$x_{11} = -i$$
$$x_{21} = \frac{\sqrt{3}}{2} - \frac{3i}{2}, \qquad x_{22} = -\frac{\sqrt{3}}{2} - \frac{3i}{2}$$

while the zeros of $[\zeta h_l^{(1)}(\zeta)]'$ $(l = 1, 2)$ are

$$y_{11} = \frac{\sqrt{3}}{2} - \frac{i}{2}, \qquad y_{12} = -\frac{\sqrt{3}}{2} - \frac{i}{2}$$
$$y_{21} \approx -1.596i, \qquad y_{22} \approx 1.807 - 0.702i, \qquad y_{23} \approx -1.807 - 0.702i$$

Since the complex frequencies are given by these zeros multiplied by c/a, we end up with

Mode$_{nlm}$	λ/a	$\tau/(a/c)$
TE$_{11m}$	∞	1/2
TE$_{12m}$	$4\pi/\sqrt{3}$	1/3
TM$_{11m}$	$4\pi/\sqrt{3}$	1
TM$_{12m}$	∞	0.313
TM$_{22m}$	3.476	0.712

where the wavelength λ and the energy decay time τ is given by

$$\omega = \frac{2\pi c}{\lambda} - \frac{i}{2\tau}$$

10.1 *a*) Show that for arbitrary initial polarization, the scattering cross section of a perfectly conducting sphere of radius a, summed over outgoing polarizations, is given in the long-wavelength limit by

$$\frac{d\sigma}{d\Omega}(\vec{\epsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{4} - |\vec{\epsilon}_0 \cdot \hat{n}|^2 - \frac{1}{4}|\hat{n} \cdot (\hat{n}_0 \times \vec{\epsilon}_0)|^2 - \hat{n}_0 \cdot \hat{n}\right]$$

where $\hat{n}_0$ and $\hat{n}$ are the directions of the incident and scattered radiations, respectively, while $\vec{\epsilon}_0$ is the (perhaps complex) unit polarization vector of the incident radiation ($\vec{\epsilon}_0{}^* \cdot \vec{\epsilon}_0 = 1$; $\hat{n}_0 \cdot \vec{\epsilon}_0 = 0$).

If all polarizations are specified, the conducting sphere scattering cross section is given by

$$\frac{d\sigma}{d\Omega}(\hat{n}, \vec{\epsilon}; \hat{n}_0, \vec{\epsilon}_0) = k^4 a^6 |\vec{\epsilon}^* \cdot \vec{\epsilon}_0 - \tfrac{1}{2}(\hat{n} \times \vec{\epsilon}^*) \cdot (\hat{n}_0 \times \vec{\epsilon}_0)|^2 \qquad (11)$$

What we would like to do is to sum this over both orthogonal outgoing polarizations. One way to do this is to introduce a linear polarization basis transverse to the outgoing direction $\hat{n}$. To do so, we first assume the scattering is not in the forward direction. Then the incoming direction $\hat{n}_0$ may be used to define orthogonal polarizations

$$\vec{\epsilon}^{\,1} = \frac{\hat{n}\times\hat{n}_0}{\sin\theta}, \qquad \vec{\epsilon}^{\,2} = \hat{n}\times\vec{\epsilon}^{\,1} = \frac{\hat{n}(\hat{n}\cdot\hat{n}_0)-\hat{n}_0}{\sin\theta}$$

where θ is the angle between $\hat{n}$ and $\hat{n}_0$. In particular, we may write $\sin^2\theta = 1-(\hat{n}\cdot\hat{n}_0)^2$. In this case, the cross section summed over outgoing polarizations becomes

$$\begin{aligned}
\frac{d\sigma}{d\Omega}(\hat{n};\hat{n}_0,\vec{\epsilon}_0) &= \frac{k^4a^6}{1-(\hat{n}\cdot\hat{n}_0)^2}\big[|(\hat{n}\times\hat{n}_0)\cdot\vec{\epsilon}_0 - \tfrac{1}{2}(\hat{n}\times(\hat{n}\times\hat{n}_0))\cdot(\hat{n}_0\times\vec{\epsilon}_0)|^2 \\
&\qquad + |(\hat{n}(\hat{n}\cdot\hat{n}_0)-\hat{n}_0)\cdot\vec{\epsilon}_0 - \tfrac{1}{2}(\hat{n}\times(\hat{n}(\hat{n}\cdot\hat{n}_0)-\hat{n}_0))\cdot(\hat{n}_0\times\vec{\epsilon}_0)|^2\big] \\
&= \frac{k^4a^6}{1-(\hat{n}\cdot\hat{n}_0)^2}\big[|(\hat{n}\times\hat{n}_0)\cdot\vec{\epsilon}_0 - \tfrac{1}{2}(\hat{n}(\hat{n}\cdot\hat{n}_0)-\hat{n}_0)\cdot(\hat{n}_0\times\vec{\epsilon}_0)|^2 \\
&\qquad + |(\hat{n}\cdot\hat{n}_0)(\hat{n}\cdot\vec{\epsilon}_0) - \tfrac{1}{2}(\hat{n}_0\times\hat{n})\cdot(\hat{n}_0\times\vec{\epsilon}_0)|^2\big] \\
&= \frac{k^4a^6}{1-(\hat{n}\cdot\hat{n}_0)^2}\big[|\hat{n}\cdot(\hat{n}_0\times\vec{\epsilon}_0) - \tfrac{1}{2}(\hat{n}\cdot\hat{n}_0)\hat{n}\cdot(\hat{n}_0\times\vec{\epsilon}_0)|^2 \\
&\qquad + |(\hat{n}\cdot\hat{n}_0)(\hat{n}\cdot\vec{\epsilon}_0) - \tfrac{1}{2}(\hat{n}\cdot\vec{\epsilon}_0)|^2\big] \\
&= \frac{k^4a^6}{1-(\hat{n}\cdot\hat{n}_0)^2}\big[|\hat{n}\cdot(\hat{n}_0\times\vec{\epsilon}_0)|^2(1-\tfrac{1}{2}(\hat{n}\cdot\hat{n}_0))^2 \\
&\qquad + |\hat{n}\cdot\vec{\epsilon}_0|^2(\tfrac{1}{2}-(\hat{n}\cdot\hat{n}_0))^2\big]
\end{aligned}$$

Note that we have used transversality of the initial polarization, $\hat{n}_0\cdot\vec{\epsilon}_0 = 0$. To proceed, we expand the squares and rewrite the above as

$$\begin{aligned}
\frac{d\sigma}{d\Omega}(\hat{n};\hat{n}_0,\vec{\epsilon}_0) = \frac{k^4a^6}{1-(\hat{n}\cdot\hat{n}_0)^2}\big[&(\tfrac{5}{4}-(\hat{n}\cdot\hat{n}_0))(|\hat{n}\cdot(\hat{n}_0\times\vec{\epsilon}_0)|^2+|\hat{n}\cdot\vec{\epsilon}_0|^2) \\
&-(1-(\hat{n}\cdot\hat{n}_0)^2)(\tfrac{1}{4}|\hat{n}\cdot(\hat{n}_0\times\vec{\epsilon}_0)|^2+|\hat{n}\cdot\vec{\epsilon}_0|^2\big]
\end{aligned} \tag{12}$$

The second line cancels the denominator. However the first line needs a bit of work. We now use the fact that ϵ_0 is a unit polarization vector orthogonal to $\hat{n}_0$. As a result, the three vectors

$$\hat{n}_0, \qquad \vec{\epsilon}_0, \qquad \hat{n}_0\times\vec{\epsilon}_0 \tag{13}$$

form a normalized right-handed coordinate basis spanning the three-dimensional space. (There is a slight subtlety if $\vec{\epsilon}_0$ is complex, although the end result is okay, provided we are careful with magnitude squares.) The components of $\hat{n}$ expanded in this basis are

$$\hat{n}\cdot\hat{n}_0, \qquad \hat{n}\cdot\vec{\epsilon}_0, \qquad \hat{n}\cdot(\hat{n}_0\times\vec{\epsilon}_0)$$

and since $\hat{n}$ is a unit vector, the sum of the squares of these components must be one. In other words

$$(\hat{n}\cdot\hat{n}_0)^2 + |\hat{n}\cdot\vec{\epsilon}_0|^2 + |\hat{n}\cdot(\hat{n}_0\times\vec{\epsilon}_0)|^2 = 1$$

where we have been careful about complex quantities. Using this result, we see that the denominator in (12) can be completely eliminated, resulting in

$$\frac{d\sigma}{d\Omega}(\hat{n};\hat{n}_0,\vec{\epsilon}_0) = k^4a^6[\tfrac{5}{4} - (\hat{n}\cdot\hat{n}_0) - \tfrac{1}{4}|\hat{n}\cdot(\hat{n}_0\times\vec{\epsilon}_0)|^2 - |\hat{n}\cdot\vec{\epsilon}_0|^2] \qquad (14)$$

b) If the incident radiation is linearly polarized, show that the cross section is

$$\frac{d\sigma}{d\Omega}(\vec{\epsilon}_0,\hat{n}_0,\hat{n}) = k^4a^6\left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta - \frac{3}{8}\sin^2\theta\cos 2\phi\right]$$

where $\hat{n}\cdot\hat{n}_0 = \cos\theta$ and the azimuthal angle ϕ is measured from the direction of the linear polarization.

As stated, the scattering angle θ is given by $\hat{n}\cdot\hat{n}_0 = \cos\theta$. The azimuthal angle ϕ is the one between $\hat{n}$ and $\vec{\epsilon}_0$, measured in the plan perpendicular to $\hat{n}_0$. What this means is that, using the basis vectors (13) with $\vec{\epsilon}_0$ real, the components of $\hat{n}$ can be written as

$$\hat{n} = \hat{n}_0\cos\theta + \vec{\epsilon}_0\sin\theta\cos\phi + (\hat{n}_0\times\vec{\epsilon}_0)\sin\theta\sin\phi$$

or alternatively

$$\hat{n}\cdot\hat{n}_0 = \cos\theta, \qquad \hat{n}\cdot\vec{\epsilon}_0 = \sin\theta\cos\phi, \qquad \hat{n}\cdot(\hat{n}_0\times\vec{\epsilon}_0) = \sin\theta\sin\phi$$

Substituting this into (14) gives

$$\begin{aligned}\frac{d\sigma}{d\Omega}(\theta,\phi) &= k^4a^6[\tfrac{5}{4} - \cos\theta - \tfrac{1}{4}\sin^2\theta\sin^2\phi - \sin^2\theta\cos^2\phi]\\ &= k^4a^6[\tfrac{5}{4} - \cos\theta - \tfrac{1}{8}\sin^2\theta(1-\cos 2\phi) - \tfrac{1}{2}\sin^2\theta(1+\cos 2\phi)]\\ &= k^4a^6[\tfrac{5}{8}(1+\cos^2\theta) - \cos\theta - \tfrac{3}{8}\sin^2\theta\cos 2\phi]\end{aligned}$$

c) What is the ratio of scattered intensities at $\theta = \pi/2$, $\phi = 0$ and $\theta = \pi/2$, $\phi = \pi/2$? Explain physically in terms of the induced multipoles and their radiation patterns.

At $\theta = \pi/2$, we have

$$\frac{d\sigma}{d\Omega}(\pi/2,\phi) = k^4a^6[\tfrac{5}{8} - \tfrac{3}{8}\cos 2\phi]$$

Hence

$$\frac{d\sigma}{d\Omega}(\pi/2,0) = \tfrac{1}{4}k^4a^6, \qquad \frac{d\sigma}{d\Omega}(\pi/2,\pi/2) = k^4a^6$$

Scattering at 90° is fairly easy to understand physically. For $\phi = 0$, the scattered wave is lined up with the incident polarization ϵ_0. Since the polarization is given by the electric field vector, this indicates that the induced electric dipole of the sphere is lined up with the direction of the scattered wave. Since the radiation must be transverse, no dipole radiation can be emitted on axis, and in this case the scattering must be purely magnetic dipole in nature. On the other hand, for $\phi = \pi/2$, the scattered wave is lined up with the incident magnetic field, and hence the scattering must be purely electric dipole in nature. This demonstrates that the maximum strength of magnetic dipole scattering is a quarter that of electric dipole scattering. This is in fact evident by the factor of 1/2 in the magnetic dipole term in the cross section expression (11).