Physics 506 **Winter 2008**

Homework Assignment #4 — Solutions

Textbook problems: Ch. 9: 9.6, 9.11, 9.16, 9.17

9.6 *a)* Starting from the general expression (9.2) for $\vec{A}$ and the corresponding expression for Φ, expand both $R = |\vec{x} - \vec{x}\,'|$ and $t' = t - R/c$ to first order in $|\vec{x}\,'|/r$ to obtain the electric dipole potentials for arbitrary time variation

$$\Phi(\vec{x},t) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r^2}\vec{n}\cdot\vec{p}_{\mathrm{ret}} + \frac{1}{cr}\vec{n}\cdot\frac{\partial\vec{p}_{\mathrm{ret}}}{\partial t}\right]$$
$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi r}\frac{\partial\vec{p}_{\mathrm{ret}}}{\partial t}$$

where $\vec{p}_{\mathrm{ret}} = \vec{p}(t' = t - r/c)$ is the dipole moment evaluated at the retarded time measured from the origin.

We start with the scalar potential, which is given by

$$\Phi(\vec{x}\,) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{x}\,', t - |\vec{x} - \vec{x}\,'|/c)}{|\vec{x} - \vec{x}\,'|}d^3x' \qquad (1)$$

We now use the expansion

$$|\vec{x} - \vec{x}\,'| \approx r - \hat{n}\cdot\vec{x}\,'$$

as well as

$$t' = t - \frac{|\vec{x} - \vec{x}\,'|}{c} \approx t - \frac{r}{c} + \frac{\hat{n}\cdot\vec{x}\,'}{c} = t_{\mathrm{ret}} + \frac{\hat{n}\cdot\vec{x}\,'}{c}$$

where $t_{\mathrm{ret}} = t - r/c$. Since ρ is a function of time t', we make the expansion

$$\begin{aligned}\rho(\vec{x}\,', t') &= \rho(\vec{x}\,', t_{\mathrm{ret}}) + \frac{\hat{n}\cdot\vec{x}\,'}{c}\frac{\partial\rho(\vec{x}\,', t_{\mathrm{ret}})}{\partial t} + \cdots\\ &= \rho_{\mathrm{ret}} + \frac{\hat{n}\cdot\vec{x}\,'}{c}\frac{\partial\rho_{\mathrm{ret}}}{\partial t} + \cdots\end{aligned}$$

As a result, the expansion of (1) becomes

$$\begin{aligned}\Phi(\vec{x}\,) &= \frac{1}{4\pi\epsilon_0 r}\int\left[\rho_{\mathrm{ret}} + \hat{n}\cdot\vec{x}\,'\left(\frac{1}{r}\rho_{\mathrm{ret}} + \frac{1}{c}\frac{\partial\rho_{\mathrm{ret}}}{\partial t}\right) + \cdots\right]d^3x'\\ &= \frac{1}{4\pi\epsilon_0 r}\left[Q + \hat{n}\cdot\left(\frac{1}{r}\vec{p}_{\mathrm{ret}} + \frac{1}{c}\frac{\partial\vec{p}_{\mathrm{ret}}}{\partial t}\right) + \cdots\right]\end{aligned}$$

where we have used the expressions for charge and electric dipole moment

$$Q = \int \rho_{\text{ret}}\, d^3x', \qquad \vec{p}_{\text{ret}} = \int \vec{x}\,'\rho_{\text{ret}}\, d^3x'$$

Note that, by charge conservation, Q is independent of time, so the subscript Q_{ret} is superfluous. Dropping the static Coulomb potential (which does not radiate) then gives

$$\Phi(\vec{x}\,) \approx \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r^2}\hat{n}\cdot\vec{p}_{\text{ret}} + \frac{1}{cr}\hat{n}\cdot\frac{\partial\vec{p}_{\text{ret}}}{\partial t}\right] \tag{2}$$

For the vector potential, the expansion is even simpler. We only need to keep the lowest order behavior

$$\vec{A}(\vec{x}\,) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{x}\,', t - |\vec{x}-\vec{x}\,'|/c)}{|\vec{x}-\vec{x}\,'|}d^3x' = \frac{\mu_0}{4\pi r}\int\left[\vec{J}_{\text{ret}} + \cdots\right]\, d^3x'$$

Using integration by parts, we note that

$$\int J_{\text{ret}\,i}\, d^3x' = \int\frac{\partial x_i'}{\partial x_j'}J_{\text{ret}\,j}\, d^3x = -\int x_i'(\vec{\nabla}\cdot\vec{J}_{\text{ret}})\, d^3x' = \int x_i'\frac{\partial\rho_{\text{ret}}}{\partial t}d^3x' = \frac{\partial p_{\text{ret}\,i}}{\partial t}$$

Hence

$$\vec{A}(\vec{x}\,') \approx \frac{\mu_0}{4\pi r}\frac{\partial\vec{p}_{\text{ret}}}{\partial t} \tag{3}$$

b) Calculate the dipole electric and magnetic fields directly from these potentials and show that

$$\begin{aligned}
\vec{B}(\vec{x},t) &= \frac{\mu_0}{4\pi}\left[-\frac{1}{cr^2}\vec{n}\times\frac{\partial\vec{p}_{\text{ret}}}{\partial t} - \frac{1}{c^2r}\vec{n}\times\frac{\partial^2\vec{p}_{\text{ret}}}{\partial t^2}\right]\\
\vec{E}(\vec{x},t) &= \frac{1}{4\pi\epsilon_0}\left\{\left(1+\frac{r}{c}\frac{\partial}{\partial t}\right)\left[\frac{3\vec{n}(\vec{n}\cdot\vec{p}_{\text{ret}})-\vec{p}_{\text{ret}}}{r^3}\right] + \frac{1}{c^2r}\vec{n}\times\left(\vec{n}\times\frac{\partial^2\vec{p}_{\text{ret}}}{\partial t^2}\right)\right\}
\end{aligned}$$

For the magnetic field, we use $\vec{B} = \vec{\nabla}\times\vec{A}$, where the vector potential is given by (3). It is important to note that the electric dipole $\vec{p}_{\text{ret}}$ in (3) is actually a function of retarded time

$$\vec{p}_{\text{ret}} = \vec{p}(t - r/c)$$

Application of the chain rule then gives

$$\frac{\partial\vec{p}_{\text{ret}}}{\partial r} = -\frac{1}{c}\frac{\partial\vec{p}_{\text{ret}}}{\partial t}$$

Since $\vec{\nabla}r = \hat{n}$, the magnetic field turns out to be

$$\vec{B} = \frac{\mu_0}{4\pi}\vec{\nabla}\times\left(\frac{1}{r}\frac{\partial\vec{p}_{\text{ret}}}{\partial t}\right) = \frac{\mu_0}{4\pi}\hat{n}\times\left(-\frac{1}{r^2}\frac{\partial\vec{p}_{\text{ret}}}{\partial t} - \frac{1}{cr}\frac{\partial^2\vec{p}_{\text{ret}}}{\partial t^2}\right) \tag{4}$$

The expression for the electric field is a bit more involved. Using (2) and (3), we obtain

$$
\begin{aligned}
\vec{E} &= -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \\
&= -\frac{1}{4\pi\epsilon_0}\vec{\nabla}\left(\frac{\vec{x}}{r^3}\cdot\vec{p}_{\text{ret}} + \frac{\vec{x}}{cr^2}\cdot\frac{\partial\vec{p}_{\text{ret}}}{\partial t}\right) - \frac{\mu_0}{4\pi r}\frac{\partial^2\vec{p}_{\text{ret}}}{\partial t^2} \\
&= -\frac{1}{4\pi\epsilon_0}\left[\frac{\vec{p}_{\text{ret}}}{r^3} - 3\frac{\vec{x}(\vec{x}\cdot\vec{p}_{\text{ret}})}{r^5} - \frac{\vec{x}}{cr^4}\left(\vec{x}\cdot\frac{\partial\vec{p}_{\text{ret}}}{\partial t}\right) + \frac{1}{cr^2}\frac{\partial\vec{p}_{\text{ret}}}{\partial t}\right. \\
&\qquad\left. - \frac{2\vec{x}}{cr^4}\left(\vec{x}\cdot\frac{\partial\vec{p}_{\text{ret}}}{\partial t}\right) - \frac{\vec{x}}{c^2r^3}\left(\vec{x}\cdot\frac{\partial^2\vec{p}_{\text{ret}}}{\partial t^2}\right)\right] - \frac{1}{4\pi\epsilon_0}\frac{1}{c^2r}\frac{\partial^2\vec{p}_{\text{ret}}}{\partial t^2} \qquad (5) \\
&= -\frac{1}{4\pi\epsilon_0}\left[\frac{\vec{p}_{\text{ret}} - 3\hat{n}(\hat{n}\cdot\vec{p}_{\text{ret}})}{r^3} + \frac{1}{cr^2}\frac{\partial}{\partial t}(\vec{p}_{\text{ret}} - 3\hat{n}(\hat{n}\cdot\vec{p}_{\text{ret}}))\right. \\
&\qquad\left. + \frac{1}{c^2r}\frac{\partial^2}{\partial t^2}(\vec{p}_{\text{ret}} - \hat{n}(\hat{n}\cdot\vec{p}_{\text{ret}}))\right] \\
&= \frac{1}{4\pi\epsilon_0}\left[\left(1 + \frac{r}{c}\frac{\partial}{\partial t}\right)\frac{3\hat{n}(\hat{n}\cdot\vec{p}_{\text{ret}}) - \vec{p}_{\text{ret}}}{r^3} + \frac{1}{c^2r}\hat{n}\times\left(\hat{n}\times\frac{\partial^2\vec{p}_{\text{ret}}}{\partial t^2}\right)\right]
\end{aligned}
$$

c) Show explicitly how you can go back and forth between these results and the harmonic fields of (9.18) by the substitutions $-i\omega \leftrightarrow \partial/\partial t$ and $\vec{p}e^{ikr-i\omega t} \leftrightarrow \vec{p}_{\text{ret}}(t')$.

Making the substitution

$$
\vec{p}_{\text{ret}} \to \vec{p}e^{ikr} \qquad \text{and} \qquad \frac{\partial}{\partial t} \to -i\omega
$$

the magnetic field (4) becomes

$$
\begin{aligned}
\vec{H} &= \frac{1}{4\pi}\hat{n}\times\left(-\frac{1}{r^2}(-i\omega)\vec{p} - \frac{1}{cr}(-\omega^2)\vec{p}\right)e^{ikr} \\
&= \frac{\omega^2}{4\pi cr}(\hat{n}\times\vec{p})\left(1 - \frac{c}{i\omega r}\right)e^{ikr} = \frac{ck^2}{4\pi}(\hat{n}\times\vec{p})\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)
\end{aligned}
$$

while the electric field (5) becomes

$$
\begin{aligned}
\vec{E} &= \frac{1}{4\pi\epsilon_0}\left[\left(1 + \frac{r}{c}(-i\omega)\right)\frac{3\hat{n}(\hat{n}\cdot\vec{p}) - \vec{p}}{r^3} + \frac{1}{c^2r}(-\omega^2)\hat{n}\times(\hat{n}\times\vec{p})\right]e^{ikr} \\
&= \frac{1}{4\pi\epsilon_0}\left[\frac{e^{ikr}}{r^3}(1 - ikr)(3\hat{n}(\hat{n}\cdot\vec{p}) - \vec{p}) - k^2\frac{e^{ikr}}{r}\hat{n}\times(\hat{n}\times\vec{p})\right]
\end{aligned}
$$

To go in the other direction, we simply read these equations backwards.

9.11 Three charges are located along the z axis, a charge $+2q$ at the origin, and charges $-q$ at $z = \pm a\cos\omega t$. Determine the lowest nonvanishing multipole moments, the angular distribution of radiation, and the total power radiated. Assume that $ka \ll 1$.

We start by specifying the charge and current densities

$$\begin{aligned}\rho &= q[2\delta(z) - \delta(z - a\cos\omega t) - \delta(z + a\cos\omega t)]\delta(x)\delta(y)\\ \vec{J} &= \hat{z}qa\omega\sin\omega t[\delta(z - a\cos\omega t) - \delta(z + a\cos\omega t)]\delta(x)\delta(y)\end{aligned} \tag{6}$$

It should be clear that these moving charges do not directly correspond to time harmonic sources of the form

$$\rho e^{-i\omega t}, \qquad \vec{J}e^{-i\omega t}$$

Thus we must use some of the techniques discussed in problem 9.1 in Fourier decomposing the source charge and current distributions. Essentially we find it easiest to take the approach of 9.1a, which is to compute the time-dependent multipole moments first before Fourier decomposing in frequency.

Assuming that $ka \ll 1$, we may directly compute the first few multipole moments. Working with Cartesian tensors, we have

$$\vec{p}(t) = \int \vec{x}\,\rho\,d^3x = -q(a\cos\omega t - a\cos\omega t) = 0$$

and

$$\vec{m}(t) = \frac{1}{2}\int \vec{x}\times\vec{J}\,d^3x = 0$$

In fact, all magnetic multipole moments vanish since the charges are undergoing linear motion. The electric quadrupole moment is non-vanishing, however

$$Q_{ij}(t) = \int (3x_ix_j - r^2\delta_{ij})\rho(t)\,d^3x = -qa^2\cos^2\omega t(3\delta_{i3}\delta_{j3} - \delta_{ij})$$

The non-vanishing moments are then

$$Q_{33}(t) = -2Q_{11}(t) = -2Q_{22}(t) = -4qa^2\cos^2\omega t$$

Note that this may be written as

$$Q_{33}(t) = -2qa^2[1 + \cos(2\omega t)] = \Re[-2qa^2(1 + e^{-2i\omega t})]$$

Since the zero frequency term does not radiate, this indicates that we may assume a *harmonic* quadrupole moment

$$Q_{33} = -2Q_{11} = -2Q_{22} = -2qa^2 \tag{7}$$

which oscillates with angular frequency 2ω. The angular distribution of radiation is then given by

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^6}{512\pi^2}|Q_{33}|^2 \sin^2\theta\cos^2\theta = \frac{Z_0 q^2}{128\pi^2}(ck)^2(ka)^4\sin^2\theta\cos^2\theta$$

Using $ck = 2\omega$ (since the harmonic frequency is 2ω), we find

$$\frac{dP}{d\Omega} = \frac{Z_0 q^2\omega^2}{32\pi^2}(ka)^4\sin^2\theta\cos^2\theta \tag{8}$$

Integrating this over the solid angle gives a total power

$$P = \frac{Z_0 q^2\omega^2}{60\pi}(ka)^4 \tag{9}$$

Alternatively, we may apply the multipole expansion formalism to write down all multipole coefficients. Using the $ka \ll 1$ approximation, these expansion coefficients are given by

$$\begin{aligned} a_E(l,m) &\approx \frac{ck^{l+2}}{i(2l+1)!!}\sqrt{\frac{l+1}{l}}Q_{lm} \\ a_M(l,m) &\approx \frac{ik^{l+2}}{(2l+1)!!}\sqrt{\frac{l+1}{l}}M_{lm} \end{aligned} \tag{10}$$

where

$$Q_{lm} = \int r^l Y^*_{lm}\rho\, d^3x, \qquad M_{lm} = -\frac{1}{l+1}\int r^l Y^*_{lm}\vec{\nabla}\cdot(\vec{r}\times\vec{J})\, d^3x$$

To proceed, we convert the charge and current densities (6) to spherical coordinates

$$\begin{aligned} \rho &= \frac{q}{2\pi r^2}\big(\delta(r) - \delta(r - a\cos\omega t)[\delta(\cos\theta - 1) + \delta(\cos\theta + 1)]\big) \\ \vec{J} &= \hat{r}\frac{qa\omega}{2\pi r^2}\sin\omega t\,\delta(r - a\cos\omega t)[\delta(\cos\theta - 1) + \delta(\cos\theta + 1)] \end{aligned}$$

For the magnetic multipoles, we see that since $\vec{J} \sim \hat{r}$ the cross product vanishes, $\vec{r}\times\vec{J} = 0$. Thus all magnetic multipoles vanish

$$a_M(l,m) = 0$$

We are thus left with the electric multipoles

$$\begin{aligned} Q_{lm}(t) &= \frac{q}{2\pi}\int r^l Y^*_{lm}\big(\delta(r) - \delta(r - a\cos\omega t) \\ &\qquad\qquad \times [\delta(\cos\theta - 1) + \delta(\cos\theta + 1)]\big)dr\,d\cos\theta\, d\phi \\ &= q\delta_{m,0}\left(2\delta_{l,0}Y^*_{00} - [Y^*_{l0}(0,0) + Y^*_{l0}(\pi,0)](a\cos\omega t)^l\right) \end{aligned}$$

Using

$$Y_{00} = \sqrt{\frac{1}{4\pi}}, \qquad [Y_{l0}(0,0) + Y_{l0}(\pi,0)] = \sqrt{\frac{2l+1}{4\pi}}[P_l(1) + P_l(-1)]$$

then gives

$$Q_{l0}(t) = -2q\sqrt{\frac{2l+1}{4\pi}}(a\cos\omega t)^l \qquad l = 2,4,6,\ldots$$

A Fourier decomposition gives both positive and negative frequency modes

$$\begin{aligned} Q_{l0}(t) &= -2q\sqrt{\frac{2l+1}{4\pi}}\left(\frac{a}{2}\right)^l \left(e^{i\omega t} + e^{-i\omega t}\right)^l \\ &= -2q\sqrt{\frac{2l+1}{4\pi}}\left(\frac{a}{2}\right)^l \sum_{k=0}^{l}\binom{l}{k} e^{i(l-2k)\omega t} \qquad l = 2,4,6,\ldots \end{aligned}$$

However, since the real part of $e^{\pm in\omega t}$ does not care about the sign of $in\omega t$ we may group such terms together to eliminate negative frequencies

$$Q_{l0}(t) = -4q\sqrt{\frac{2l+1}{4\pi}}\left(\frac{a}{2}\right)^l \Re\left[\frac{1}{2}\binom{l}{l/2} + \sum_{n=2,4,\ldots,l}\binom{l}{(l-n)/2}e^{-in\omega t}\right]$$

where $l = 2,4,6,\ldots$. Note that the zero frequency mode does not radiate, and hence may be ignored. In general, the l-th mode radiates at frequencies $l\omega, (l-2)\omega, (l-4)\omega, \ldots$.

The lowest non-vanishing moment is the electric quadrupole moment

$$Q_{20} = -qa^2\sqrt{\frac{5}{4\pi}}$$

and its harmonic frequency is 2ω. Note, in particular, that this agrees with (7) when converted to a Cartesian tensor. Using (10), this yields a multipole coefficient

$$a_E(2,0) = iqk(ck)(ka)^2\sqrt{\frac{1}{120\pi}} \tag{11}$$

We now turn to the angular distribution of the radiation. For a pure multipole of order (l,m), the angular distribution of radiated power is

$$\begin{aligned} \frac{dP(l,m)}{d\Omega} = \frac{Z_0}{2k^2 l(l+1)}|a(l,m)|^2\Big[&\tfrac{1}{2}(l-m)(l+m+1)|Y_{l,m+1}|^2 \\ &+ \tfrac{1}{2}(l+m)(l-m+1)|Y_{l,m-1}|^2 + m^2|Y_{lm}|^2\Big] \end{aligned}$$

This simplifies considerably for $m = 0$

$$\frac{dP(l,0)}{d\Omega} = \frac{Z_0}{2k^2}|a(l,0)|^2|Y_{l,1}|^2$$

Since the lowest multipole is the electric quadrupole, we substitute in $l = 2$ and $m = 0$ to obtain

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}|a_E(2,0)|^2\frac{15}{8\pi}\sin^2\theta\cos^2\theta \tag{12}$$

Using (11) then gives

$$\frac{dP}{d\Omega} = \frac{Z_0 q^2}{128\pi^2}(ck)^2(ka)^4\sin^2\theta\cos^2\theta = \frac{Z_0 q^2\omega^2}{32\pi^2}(ka)^4\sin^2\theta\cos^2\theta$$

Note that we have used $ck = 2\omega$, since the harmonic frequency is 2ω. The total radiated power is given by

$$P = \frac{Z_0}{2k^2}\sum_{l,m}[|a_E(l,m)|^2 + |a_M(l,m)|^2]$$

For the electric quadrupole, this gives

$$P = \frac{Z_0}{2k^2}|a_E(2,0)|^2 = \frac{Z_0 q^2\omega^2}{60\pi}(ka)^4 \tag{13}$$

The next non-vanishing multipole would be $l = 4$, which radiates at frequencies 2ω and 4ω. However, this will be subdominant, so long as $ka \ll 1$. Note that the angular distribution (12) and the total radiated power (13) agree with those found earlier, namely (8) and (9).

9.16 A thin linear antenna of length d is excited in such a way that the sinusoidal current makes a full wavelength of oscillation as shown in the figure.

a) Calculate exactly the power radiated per unit solid angle and plot the angular distribution of radiation.

Note that the current flows in opposite directions in the top and bottom half of this antenna. As a result, we may write the source current density as

$$\vec{J}(z) = \hat{z}I\sin(kz)\delta(x)\delta(y)\Theta(d/2 - |z|) \tag{14}$$

where

$$k = \frac{2\pi}{d}$$

In the radiation zone, the vector potential is given by

$$\begin{aligned}\vec{A}(\vec{x}\,) &= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{J}(\vec{x}\,')e^{-ik\hat{n}\cdot\vec{x}\,'}d^3x' \\ &= \hat{z}\frac{\mu_0 I}{4\pi}\frac{e^{ikr}}{r}\int_{-d/2}^{d/2}\sin(kz)e^{-ikz\cos\theta}dz\end{aligned}$$

Since the source current is odd under $z \to -z$, this integral may be written as

$$\begin{aligned}
\vec{A} &= -\hat{z}\frac{i\mu_0 I}{4\pi}\frac{e^{ikr}}{r}\int_0^{d/2} 2\sin(kz)\sin(kz\cos\theta)\,dz \\
&= -\hat{z}\frac{i\mu_0 I}{4\pi}\frac{e^{ikr}}{r}\int_0^{d/2} [\cos((1-\cos\theta)kz) - \cos((1+\cos\theta)kz)]\,dz \\
&= -\hat{z}\frac{i\mu_0 I}{4\pi}\frac{e^{ikr}}{kr}\left[\frac{1}{1-\cos\theta}\sin((1-\cos\theta)kz) - \frac{1}{1+\cos\theta}\sin((1+\cos\theta)kz)\right]_0^{d/2} \\
&= -\hat{z}\frac{i\mu_0 I}{2\pi}\frac{e^{ikr}}{kr}\frac{\sin(\pi\cos\theta)}{\sin^2\theta}
\end{aligned}$$

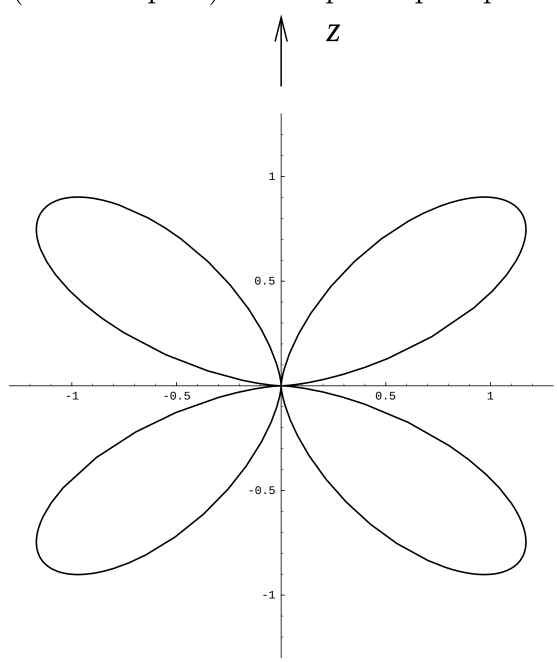
In the radiation zone, the magnetic field is

$$\vec{H} = \frac{ik}{\mu_0}\hat{n}\times\vec{A} = -\hat{\phi}\frac{I}{2\pi}\frac{e^{ikr}}{r}\frac{\sin(\pi\cos\theta)}{\sin\theta}$$

where we have used $\hat{n}\times\hat{z} \equiv \hat{r}\times\hat{z} = -\hat{\phi}\sin\theta$. This gives rise to a radiated power

$$\frac{dP}{d\Omega} = \frac{Z_0 r^2}{2}|\vec{H}|^2 = \frac{Z_0|I|^2}{8\pi^2}\frac{\sin^2(\pi\cos\theta)}{\sin^2\theta} \tag{15}$$

This looks almost (but not quite) like a quadrupole pattern.



b) Determine the total power radiated and find a numerical value for the radiation resistance.

The total radiated power is given by integrating the angular distribution over the solid angle

$$P = \frac{Z_0|I|^2}{8\pi^2}2\pi\int_{-1}^{1}\frac{\sin^2(\pi\cos\theta)}{1-\cos^2\theta}d\cos\theta = \frac{Z_0|I|^2}{4\pi}\int_{-1}^{1}\frac{\sin^2(\pi x)}{1-x^2}dx \approx \frac{Z_0|I|^2}{4\pi}\times 1.557$$

Comparing this with $P = \frac{1}{2} R_{\rm rad}|I|^2$ gives a radiation resistance

$$R_{\rm rad} = \frac{Z_0}{2\pi} \times 1.557 = 93.4\,\Omega \tag{16}$$

9.17 Treat the linear antenna of Problem 9.16 by the multipole expansion method.

a) Calculate the multipole moments (electric dipole, magnetic dipole, and electric quadrupole) exactly and in the long-wavelength approximation.

In order to calculate the exact multipole moments, we must use the full multipole expansion method. For a current source without intrinsic magnetization, this involves computing the electric and magnetic multipole coefficients

$$\begin{aligned} a_E(l,m) &= \frac{k^2}{i\sqrt{l(l+1)}} \int Y_{lm}^* \left[c\rho \frac{\partial}{\partial r}[r j_l(kr)] + ik(\vec{r}\cdot\vec{J}) j_l(kr) \right] d^3x \\ a_M(l,m) &= \frac{k^2}{i\sqrt{l(l+1)}} \int Y_{lm}^* \vec{\nabla}\cdot(\vec{r}\times\vec{J}) j_l(kr)\, d^3x \end{aligned} \tag{17}$$

However, we may begin with the long-wavelength approximation. In this case, the first few moments are given by the standard expressions

$$\vec{p} = \int \vec{x}\,\rho\, d^3x, \qquad \vec{m} = \frac{1}{2}\int \vec{x}\times\vec{J}\, d^3x$$

and

$$Q_{ij} = \int (3x_i x_j - r^2\delta_{ij})\rho\, d^3x$$

Although the length of the antenna is equal to the wavelength (so that the long-wavelength expansion is not particularly valid), we may still see what we get. Using the current density (14), we obtain a charge density

$$\rho = \frac{1}{i\omega}\vec{\nabla}\cdot\vec{J} = \frac{1}{i\omega}\frac{d\vec{J}}{dz} = -\frac{iI}{c}\cos(kz)\delta(x)\delta(y)\Theta(d/2-|z|)$$

where we used $\omega = ck$. The electric dipole moment is then

$$\vec{p} = \int \vec{x}\,\rho\, d^3x = -\hat{z}\frac{iI}{c}\int_{-d/2}^{d/2} z\cos(kz)\, dz = 0$$

Of course, a simple symmetry argument under $z \to -z$ demonstrates that this electric dipole must vanish. The magnetic dipole moment also vanishes since

$$\vec{m} = \frac{1}{2}\int \vec{x}\times\vec{J}\, d^3x = \frac{I}{2}\int_{-d/2}^{d/2} \vec{z}\times[\hat{z}\sin(kz)]\, dz = 0$$

We are left with an electric quadrupole moment

$$Q_{ij} = \int (3x_i x_j - r^2\delta_{ij})\rho\, d^3x = -\frac{iI}{c}\int_{-d/2}^{d/2} [3(z\delta_{i3})(z\delta_{j3}) - z^2\delta_{ij}]\cos(kz)\, dz$$

The only non-vanishing moments are

$$Q_{33} = -2Q_{11} = -2Q_{22} = -\frac{2iI}{c}\int_{-d/2}^{d/2} z^2\cos(kz)\, dz$$

The integral is straightforward, and the result is

$$Q_{33} = -2Q_{11} = -2Q_{22} = \frac{8\pi iI}{ck^3} \tag{18}$$

This is of course for the long-wavelength approximation.

In order to compute the exact multipole moments, we must go back to the expressions for $a_E(l,m)$ and $a_M(l,m)$ given in (17). Since these multipole coefficients are given as spherical quantities, it is convenient to convert the rectangular coordinate expressions for $\vec{J}$ and ρ into spherical coordinates

$$\begin{aligned} \rho &= -\frac{iI}{2\pi r^2 c}\cos(kr)[\delta(\cos\theta - 1) + \delta(\cos\theta + 1)] \\ \vec{J} &= \hat{r}\frac{I}{2\pi r^2}\sin(kr)[\delta(\cos\theta - 1) + \delta(\cos\theta + 1)] \end{aligned} \tag{19}$$

where we have dropped the step function $\Theta(d/2 - r)$ for simplicity of notation. (We will of course limit the r integration to the range $0 < r < d/2$.) Since in (17) we need expressions for $\vec{r}\cdot\vec{J}$ and $\vec{r}\times\vec{J}$, we compute

$$\begin{aligned} \vec{r}\cdot\vec{J} &= \frac{I}{2\pi r}\sin(kr)[\delta(\cos\theta - 1) + \delta(\cos\theta + 1)] \\ \vec{r}\times\vec{J} &= 0 \end{aligned}$$

The vanishing of the cross product immediately indicates that all magnetic multipole coefficients vanish

$$a_M(l,m) = 0$$

This is consistent with the vanishing of the magnetic dipole moment found above in the long-wavelength approximation. For the electric multipoles, we have

$$\begin{aligned} a_E(l,m) &= -\frac{k^2 I}{\sqrt{l(l+1)}}\int Y_{lm}^* \frac{1}{2\pi r^2}\left(\cos(kr)\frac{\partial}{\partial r}[rj_l(kr)] - kr\sin(kr)j_l(kr)\right) \\ &\qquad\qquad \times [\delta(\cos\theta - 1) + \delta(\cos\theta + 1)]r^2 dr d\cos\theta d\phi \\ &= -\frac{k^2 I}{\sqrt{l(l+1)}}\int Y_{lm}^* \frac{1}{2\pi}\frac{\partial}{\partial r}\left(r\cos(kr)j_l(kr)\right) \\ &\qquad\qquad \times [\delta(\cos\theta - 1) + \delta(\cos\theta + 1)]dr d\cos\theta d\phi \\ &= -\frac{k^2 I}{\sqrt{l(l+1)}}\delta_{m,0}[Y_{l0}^*(0,0) + Y_{l0}^*(\pi,0)]\Big[r\cos(kr)j_l(kr)\Big]_0^{d/2} \\ &= \frac{2\pi^2 I}{d\sqrt{l(l+1)}}\delta_{m,0}[Y_{l0}^*(0,0) + Y_{l0}^*(\pi,0)]j_l(\pi) \end{aligned}$$

where in the last line we have used $k = 2\pi/d$. Noting that

$$Y_{l0}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)$$

and that $P_l(1) = 1$ and $P_l(-1) = (-1)^l$, we see that

$$Y^*_{l0}(0,0) + Y^*_{l0}(\pi,0) = \sqrt{\frac{2l+1}{4\pi}} \times \begin{cases} 2 & l \text{ even} \\ 0 & l \text{ odd} \end{cases}$$

Hence the non-vanishing electric multipoles are

$$a_E(l,m) = \frac{2\pi I}{d}\sqrt{\frac{(2l+1)\pi}{l(l+1)}} j_l(\pi)\delta_{m,0} \qquad (l \text{ even})$$

Taking into account the vanishing of all $a_M(l,m)$, we see that of the electric dipole, magnetic dipole, and electric quadrupole moments, only the latter is non-vanishing. Using $j_2(\pi) = 3/\pi^2$, we find explicitly

$$\text{electric quadrupole:} \qquad a_E(2,0) = \frac{I}{d}\sqrt{\frac{30}{\pi}} \approx 3.090\frac{I}{d} \qquad (\text{exact}) \tag{20}$$

We may also consider the long-wavelength approximation in the formalism of $a_E(l,m)$ and $a_M(l,m)$. In this case, we write the approximate versions of (17) as

$$\begin{aligned} a_E(l,m) &\approx \frac{ck^{l+2}}{i(2l+1)!!}\sqrt{\frac{l+1}{l}} Q_{lm} \\ a_M(l,m) &\approx \frac{ik^{l+2}}{(2l+1)!!}\sqrt{\frac{l+1}{l}} M_{lm} \end{aligned}$$

where

$$Q_{lm} = \int r^l Y^*_{lm} \rho \, d^3x, \qquad M_{lm} = -\frac{1}{l+1}\int r^l Y^*_{lm} \vec{\nabla}\cdot(\vec{r}\times\vec{J})\, d^3x$$

Since $\vec{r}\times\vec{J} = 0$, we see that the magnetic moments vanish regardless of whether the long-wavelength approximation is taken or not. For the electric moments, we use (19) to obtain

$$\begin{aligned} Q_{lm} &= -\frac{iI}{c}\int \frac{1}{2\pi r^2}\cos(kr) r^l Y^*_{lm}[\delta(\cos\theta - 1) + \delta(\cos\theta+1)] r^2 dr d\cos\theta d\phi \\ &= -\frac{iI}{c}\delta_{m,0}[Y^*_{l0}(0,0) + Y^*_{l0}(\pi,0)]\int_0^{d/2} r^l \cos(kr) dr \\ &= -\frac{2iI}{c}\sqrt{\frac{2l+1}{4\pi}}\delta_{m,0}\int_0^{d/2} r^l\cos(kr)dr \qquad (l \text{ even}) \\ &= -\frac{2iI}{ck^{l+1}}\sqrt{\frac{2l+1}{4\pi}}\delta_{m,0}\int_0^{\pi} x^l \cos x\, dx \qquad (l \text{ even}) \end{aligned}$$

so that

$$a_E(l,m) \approx -\frac{I}{d(2l+1)!!}\sqrt{\frac{4\pi(l+1)(2l+1)}{l}}\,\delta_{m,0}\int_0^\pi x^l \cos x\,dx \qquad (l \text{ even})$$

For $l = 2$, we use $\int_0^\pi x^2 \cos x\,dx = -2\pi$ to obtain

$$\text{electric quadrupole:} \qquad a_E(2,0) = \frac{I}{d}\sqrt{\frac{8\pi^3}{15}} \approx 4.067\frac{I}{d} \qquad (\text{long } \lambda \text{ approx}) \quad (21)$$

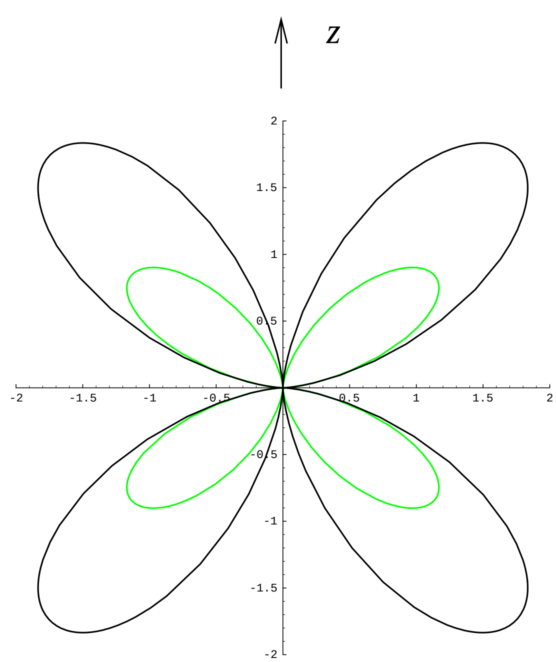
Note that this amplitude is larger than the exact expression (20). Of course, this approximation is not to be trusted, since the size of the antenna is equal to one whole wavelength.

b) Compare the shape of the angular distribution of radiated power for the lowest nonvanishing multipole with the exact distribution of Problem 9.16.

The lowest multipole is the electric quadrupole. Using the simple multipole expansion in the long-wavelength approximation, the angular distribution is

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^6}{512\pi^2}|Q_0|^2 \sin^2\theta\cos^2\theta = \frac{Z_0|I|^2}{8\pi^2}\pi^2 \sin^2\theta\cos^2\theta \qquad (22)$$

where $Q_0 = Q_{33} = -2Q_{11} = -2Q_{22}$ is given in (18). This is plotted in black, and may be compared with the exact distribution (15), plotted in green in using the same scale



This shows that the angular distribution is similar to but not exactly the same as the exact one. Perhaps more importantly, however, the long-wavelength approximation seriously overestimates the radiated power.

We now turn to the exact quadrupole result of (20). For a pure multipole of order (l, m), the angular distribution of radiated power is

$$\frac{dP(l,m)}{d\Omega} = \frac{Z_0}{2k^2 l(l+1)} |a(l,m)|^2 \Big[\tfrac{1}{2}(l-m)(l+m+1)|Y_{l,m+1}|^2 + \tfrac{1}{2}(l+m)(l-m+1)|Y_{l,m-1}|^2 + m^2 |Y_{lm}|^2 \Big]$$

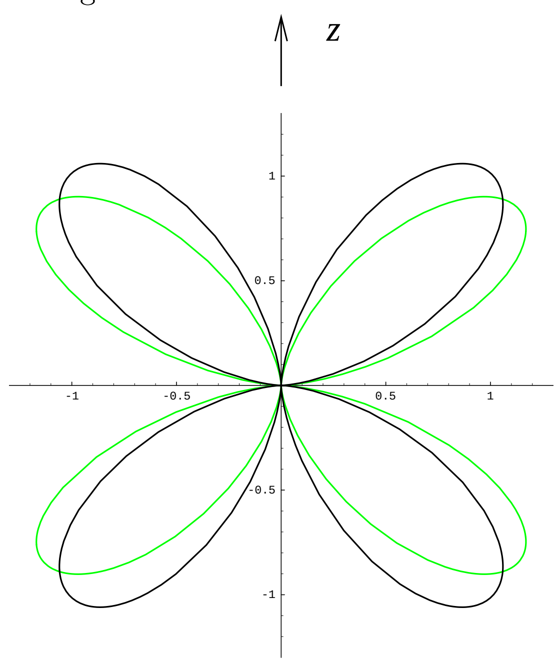
For the electric quadrupole ($l = 2, m = 0$), this simplifies to the expression given in (12)

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2} |a_E(2,0)|^2 \frac{15}{8\pi} \sin^2\theta \cos^2\theta$$

Using the exact calculation of $a_E(2,0)$ from (20) gives

$$\frac{dP}{d\Omega} = \frac{Z_0 |I|^2}{8\pi^2} \left(\frac{15}{2\pi}\right)^2 \sin^2\theta \cos^2\theta$$

This is plotted in black, and may be compared with the exact distribution (15), plotted in green in using the same scale



Note that, from (21), the approximate value of $a_E(2,0)$ in the long-wavelength approximation is given by

$$a_E^{\text{approx}}(2,0) = \frac{2\pi^2}{15} a_E^{\text{exact}}(2,0) \approx 1.316\, a_E^{\text{exact}}(2,0)$$

Hence the long-wavelength approximation would give an overestimate of the quadrupole radiated power

$$\frac{dP^{\text{approx}}}{d\Omega} = \frac{Z_0 |I|^2}{8\pi^2} \pi^2 \sin^2\theta \cos^2\theta$$

Furthermore, this expression is identical to (22), as it ought to be (since the long-wavelength expansion is physically the same, regardless of whether we use Cartesian or spherical multipole tensors).

c) Determine the total power radiated for the lowest multipole and the corresponding radiation resistance using both multipole moments from part a. Compare with Problem 9.16b. Is there a paradox here?

In terms of the multipole expansion coefficients, the total radiated power is given by

$$P = \frac{Z_0}{2k^2}\sum_{l,m}[|a_E(l,m)|^2 + |a_M(l,m)|^2] \tag{23}$$

which is a sum of squares without interference. (Of course, interference still shows up in the angular distribution.) Using the exact quadrupole factor (20), the total power radiated for the lowest multipole is

$$P = \frac{Z_0}{2k^2}|a_E(2,0)|^2 = Z_0|I|^2\frac{15}{4\pi^3} \qquad \Rightarrow \qquad R_{\text{rad}} = 91.2\,\Omega$$

which is around 2% less than the exact result for the total power radiated in all modes, (16). This shows that here the quadrupole really is the dominant mode.

Note that, if we had used the long-wavelength approximation (22), we would have found instead

$$P = \frac{Z_0|I|^2}{8}2\pi\int_{-1}^{1}\sin^2\theta\cos^2\theta\,d\cos\theta = Z_0|I|^2\frac{\pi}{15} \qquad \Rightarrow \qquad R_{\text{rad}} = 158\,\Omega$$

This radiated power (and radiation resistance) is *larger* than the exact expression of (16). Since the multipoles add incoherently in (23), the power cannot be reduced to the exact value of (16). However, this is not a paradox, since the long-wavelength approximation is invalid for the present case. In particular, this approximation involves the quasi-static source multipole moments

$$Q_{lm} = \int r^l Y_{lm}^* \rho\, d^3x \tag{24}$$

However, for an extended source, different charges (and currents) in different regions of the source may add destructively and reduce the true strength of the multipole coefficient compared to the approximate value obtained from (24).