Physics 506 **Winter 2008**

Homework Assignment #3 — Solutions

Textbook problems: Ch. 8: 8.18, 8.20
Ch. 9: 9.1, 9.3

8.18 *a*) From the use of Green's theorem in two dimensions show that the TM and TE modes in a waveguide defined by the boundary-value problems (8.34) and (8.36) are orthogonal in the sense that

$$\int_A E_{z\,\lambda} E_{z\,\mu} da = 0 \qquad \text{for } \lambda \neq \mu$$

for TM modes, and a corresponding relation for H_z for TE modes.

Orthogonality is a general property of the eigenfunctions of the wave equation. The general two-dimensional equation is given by

$$[\nabla_t^2 + \gamma_\lambda^2]\psi_\lambda = 0$$

where either

$$\psi_\lambda|_S = 0 \qquad \text{TM modes}$$

or

$$\left.\frac{\partial \psi_\lambda}{\partial n}\right|_S = 0 \qquad \text{TE modes}$$

To prove orthogonality, note that ψ_λ and ψ_μ satisfy the equations

$$[\nabla_t^2 + \gamma_\lambda^2]\psi_\lambda = 0, \qquad [\nabla_t^2 + \gamma_\mu^2]\psi_\mu = 0$$

Multiplying the first by ψ_μ and the second by ψ_λ and subtracting gives

$$(\gamma_\mu^2 - \gamma_\lambda^2)\psi_\mu\psi_\lambda = \psi_\mu \nabla_t^2 \psi_\lambda - \psi_\lambda \nabla_t^2 \psi_\mu$$

Integrating this over the cross-sectional area, and using Green's theorem yields

$$\begin{aligned}(\gamma_\mu^2 - \gamma_\lambda^2)\int_A \psi_\mu\psi_\lambda\, da &= \int_A [\psi_\mu \nabla_t^2 \psi_\lambda - \psi_\lambda \nabla_t^2 \psi_\mu]\, da \\ &= -\oint_C \left[\psi_\mu \frac{\partial \psi_\lambda}{\partial n} - \psi_\lambda \frac{\partial \psi_\mu}{\partial n}\right] dl\end{aligned}$$

where we have used an *inward pointing* normal direction. We now note that the right hand side vanishes for either TM or TE boundary conditions. Thus, provided $\gamma_\mu^2 \neq \gamma_\lambda^2$, we end up with

$$\int_A \psi_\mu \psi_\lambda\, da = 0 \qquad (\gamma_\mu^2 \neq \gamma_\lambda^2)$$

For non-degenerate eigenvalues, we conclude that

$$\int_A \psi_\mu \psi_\lambda \, da = 0 \qquad \text{for } \mu \neq \lambda$$

so long as ψ_μ and ψ_λ are both TM modes (or are both TE modes). Note that $\psi_\mu = E_{z,\mu}$ for TM modes, while $\psi_\mu = H_{z,\mu}$ for TE modes.

For degenerate eigenvalues, linearity of the wave equation guarantees that we may find an orthogonal basis using, e.g., a Gram-Schmidt orthogonalization process.

b) Prove that the relations (8.131)–(8.134) form a consistent set of normalization conditions for the fields, including the circumstances when λ is a TM mode and μ is a TE mode.

We start with relation (8.131), which states

$$\int_A \vec{E}_{t,\lambda} \cdot \vec{E}_{t,\mu} \, da = \delta_{\lambda,\mu}$$

where $\vec{E}_{t,\lambda}$ may be either a TM or a TE mode. To handle this expression, we note that the transverse fields for TM and TE modes are given by

$$\begin{array}{llll} \text{TM:} & \vec{E}_t = \dfrac{ik}{\gamma^2}\vec{\nabla}_t E_z, & \vec{H}_t = \dfrac{1}{Z}\hat{z}\times\vec{E}_t & Z = \dfrac{k}{\epsilon\omega} \\ \text{TE:} & \vec{E}_t = -\dfrac{i\mu\omega}{\gamma^2}\hat{z}\times\vec{\nabla}_t H_z, & \vec{H}_t = \dfrac{1}{Z}\hat{z}\times\vec{E}_t & Z = \dfrac{\mu\omega}{k} \end{array} \tag{1}$$

Hence for two TM modes, we end up with

$$\begin{aligned} \int_A \vec{E}_{t,\lambda} \cdot \vec{E}_{t,\mu} \, da &= -\frac{k^2}{\gamma_\mu^2 \gamma_\lambda^2} \int_A \vec{\nabla}_t E_{z,\lambda} \cdot \vec{\nabla}_t E_{z,\mu} \, da \\ &= -\frac{k^2}{\gamma_\mu^2 \gamma_\lambda^2} \left[-\oint_S E_{z,\lambda} \frac{\partial E_{z,\mu}}{\partial n} dl - \int_A E_{z,\lambda} \nabla_t^2 E_{z,\mu} \, da \right] \end{aligned}$$

The surface term vanishes because of Dirichlet boundary conditions, while the area term may be simplified using $\nabla_t^2 E_{z,\mu} = -\gamma_\mu^2 E_{z,\mu}$. Hence we arrive at

$$\int_A \vec{E}_{t,\lambda} \cdot \vec{E}_{t,\mu} \, da = -\frac{k^2}{\gamma_\lambda^2} \int_A E_{z,\lambda} E_{z,\mu} \, da = 0 \quad \text{for } \lambda \neq \mu \tag{2}$$

When properly normalized for $\lambda = \mu$, this gives (8.131) for two TM modes. The case of two TE modes is similar. We have

$$\begin{aligned} \int_A \vec{E}_{t,\lambda} \cdot \vec{E}_{t,\mu} \, da &= -\frac{\mu^2\omega^2}{\gamma_\mu^2 \gamma_\lambda^2} \int_A (\hat{z}\times\vec{\nabla}_t H_{z,\lambda}) \cdot (\hat{z}\times\vec{\nabla}_t H_{z,\mu}) \, da \\ &= -\frac{\mu^2\omega^2}{\gamma_\mu^2 \gamma_\lambda^2} \int_A \left[\vec{\nabla}_t H_{z,\lambda} \cdot \vec{\nabla}_t H_{z,\mu} - (\hat{z}\cdot\vec{\nabla}_t H_{z,\lambda})(\hat{z}\cdot\vec{\nabla}_t H_{z,\mu}) \right] da \\ &= -\frac{\mu^2\omega^2}{\gamma_\mu^2 \gamma_\lambda^2} \int_A \vec{\nabla}_t H_{z,\lambda} \cdot \vec{\nabla}_t H_{z,\mu} \, da \end{aligned} \tag{3}$$

we we have noted that $\hat{z}\cdot\vec{\nabla}_t = 0$ identically (since the transverse gradient is orthogonal to $\hat{z}$). The proof of orthogonality of two TE modes then follows using the same integration method that was used above for the TM modes (but with E_z replaced by H_z, and with $\partial H_z/\partial n$ vanishing on the boundary). Finally, for one TE mode and one TM mode, we have

$$\begin{aligned}\int_A \vec{E}_{t,\lambda}\cdot\vec{E}_{t,\mu}\,da &= \frac{\mu\omega k}{\gamma_\mu^2\gamma_\lambda^2}\int_A \vec{\nabla}_t E_{z,\lambda}\cdot(\hat{z}\times\vec{\nabla}_t H_{z,\mu})\,da\\ &= -\frac{\mu\omega k}{\gamma_\mu^2\gamma_\lambda^2}\int_A [\vec{\nabla}_t E_{z,\lambda}\times\vec{\nabla}_t H_{z,\mu}]\cdot\hat{z}\,da\\ &= -\frac{\mu\omega k}{\gamma_\mu^2\gamma_\lambda^2}\int_A \vec{\nabla}_t\times(E_{z,\lambda}\vec{\nabla}_t H_{z,\mu})\cdot\hat{z}\,da\\ &= -\frac{\mu\omega k}{\gamma_\mu^2\gamma_\lambda^2}\oint_S E_{z,\lambda}\vec{\nabla}_t H_{z,\mu}\cdot d\vec{l} = 0\end{aligned}$$

This integral vanishes because $E_{z,\lambda}$ vanishes on the boundary. As a result, all TE modes are orthogonal to all TM modes. Proper normalization then results in (8.131).

We now turn to relation (8.132), which states

$$\int_A \vec{H}_{t,\lambda}\cdot\vec{H}_{t,\mu}\,da = \frac{1}{Z_\lambda^2}\delta_{\lambda,\mu}$$

The best way to prove this is to note from (1) that

$$\vec{H}_{t,\lambda} = \frac{1}{Z_\lambda}\hat{z}\times\vec{E}_{t,\lambda}$$

for either TM or TE modes, provided Z_λ is chosen accordingly. In this case

$$\begin{aligned}\int_A \vec{H}_{t,\lambda}\cdot\vec{H}_{t,\mu}\,da &= \frac{1}{Z_\mu Z_\lambda}\int_A (\hat{z}\times\vec{E}_{t,\lambda})(\hat{z}\times\vec{E}_{t,\mu})\,da\\ &= \frac{1}{Z_\mu Z_\lambda}\int_A \left[\vec{E}_{t,\lambda}\cdot\vec{E}_{t,\mu} - (\hat{z}\cdot\vec{E}_{t,\lambda})(\hat{z}\cdot\vec{E}_{t,\mu})\right]\,da\\ &= \frac{1}{Z_\mu Z_\lambda}\int_A \vec{E}_{t,\lambda}\cdot\vec{E}_{t,\mu}\,da = \frac{1}{Z_\mu Z_\lambda}\delta_{\lambda,\mu} = \frac{1}{Z_\lambda^2}\delta_{\lambda,\mu}\end{aligned}$$

Here we have made use of the fact that $\hat{z}\cdot\vec{E}_t$ vanishes because $\vec{E}_t$ is transverse to the $\hat{z}$ direction. The last line follows from applying (8.131), which we proved above.

The power flow relation (8.133)

$$\frac{1}{2}\int_A (\vec{E}_{t,\lambda}\times\vec{H}_{t,\mu})\cdot\hat{z}\,da = \frac{1}{2Z_\lambda}\delta_{\lambda,\mu}$$

follows similarly. Specifically, we have

$$
\begin{aligned}
\frac{1}{2}\int_A (\vec{E}_{t,\lambda}\times\vec{H}_{t,\mu})\cdot\hat{z}\,da &= \frac{1}{2Z_\mu}\int_A \hat{z}\cdot[\vec{E}_{t,\lambda}\times(\hat{z}\times\vec{E}_{t,\mu})]\,da \\
&= \frac{1}{2Z_\mu}\int_A \left[\vec{E}_{t,\lambda}\cdot\vec{E}_{t,\mu}-(\hat{z}\cdot\vec{E}_{t,\lambda})(\hat{z}\cdot\vec{E}_{t,\mu})\right]\,da \\
&= \frac{1}{2Z_\mu}\int_A \vec{E}_{t,\lambda}\cdot\vec{E}_{t,\mu}\,da = \frac{1}{2Z_\mu}\delta_{\lambda,\mu} = \frac{1}{2Z_\lambda}\delta_{\lambda,\mu}
\end{aligned}
$$

The relation (8.134) essentially normalizes the modes for the TM and TE case. Examination of (2) for TM modes and (3) for TE modes indicates that the proper normalization is

$$
\begin{aligned}
&\text{TM:} \qquad \int_A E_{z,\lambda}E_{z,\mu}\,da = -\frac{\gamma_\lambda^2}{k_\lambda^2}\delta_{\lambda,\mu} \\
&\text{TE:} \qquad \int_A E_{z,\lambda}E_{z,\mu}\,da = -\frac{\gamma_\lambda^2}{\mu^2\omega^2}\delta_{\lambda,\mu} = -\frac{\gamma_\lambda^2}{k_\lambda^2 Z_\lambda^2}
\end{aligned}
\tag{4}
$$

8.20 An infinitely long rectangular waveguide has a coaxial line terminating in the short side of the guide with the thin central conductor forming a semicircular loop of radius R whose center is a height h above the floor of the guide, as shown in the accompanying cross-sectional view. The half-loop is in the plane $z=0$ and its radius R is sufficiently small that the current can be taken as having a constant value I_0 everywhere on the loop.

a) Prove that to the extent that the current is constant around the half-loop, the TM modes are not excited. Give a physical explanation of this lack of excitation.

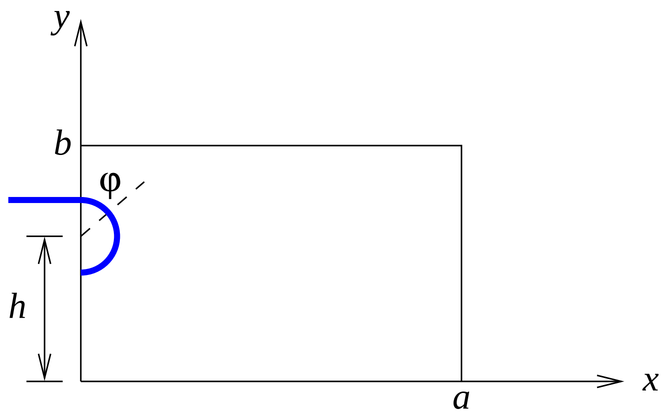
For a current density $\vec{J}$, we use

$$
A_{mn}^{(\pm)} = -\frac{Z_{mn}}{2}\int_V \vec{J}\cdot\vec{E}_{mn}^{(\mp)}d^3x \tag{5}
$$

to compute the mode expansion coefficients $A_{mn}^{(\pm)}$. Since the current density corresponds to that of a wire loop, it is actually easier to convert the volume integral into a line integral along the wire

$$
\int_V \vec{J}d^3x \qquad \Rightarrow \qquad \int I_0 d\vec{l}
$$

We may parametrize the position $\vec{l}(\varphi)$ along the wire according to the angular position φ along the loop



In this case, we have

$$\vec{l}(\varphi) = \hat{x} R \sin\varphi + \hat{y}(h + R\cos\varphi)$$

so that

$$d\vec{l} = R(\hat{x}\cos\varphi - \hat{y}\sin\varphi)d\varphi$$

For a rectangular waveguide, the normalized TM_{mn} mode is given by

$$\begin{aligned}\vec{E}^{(\pm)}_{mn} =& \frac{2}{\sqrt{ab}}\left[\frac{\pi}{\gamma_{mn}}\left(\hat{x}\frac{m}{a}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) + \hat{y}\frac{n}{b}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\right)\right.\\ &\left.\mp \hat{z}\frac{i\gamma_{mn}}{k_{mn}}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\right]e^{\pm i k_{mn} z}\end{aligned}$$

Using (5), we then have

$$\begin{aligned}A^{(\pm)}_{mn} &= -\frac{Z_{mn}}{2}\int I_0 \vec{E}^{(\mp)}_{mn}\cdot d\vec{l}\\ &= -\frac{I_0 Z_{mn} R}{2}\int_0^\pi \vec{E}^{(\mp)}_{mn}\cdot(\hat{x}\cos\varphi - \hat{y}\sin\varphi)d\varphi\\ &= -\frac{I_0 Z_{mn}\pi R}{\gamma_{mn}\sqrt{ab}}\int_0^\pi\left[\frac{m}{a}\cos\varphi\cos\left(\frac{m\pi R\sin\varphi}{a}\right)\sin\left(\frac{n\pi(h+R\cos\varphi)}{b}\right)\right.\\ &\qquad\left. - \frac{n}{b}\sin\varphi\sin\left(\frac{m\pi R\sin\varphi}{a}\right)\cos\left(\frac{n\pi(h+R\cos\varphi)}{b}\right)\right]d\varphi\end{aligned} \tag{6}$$

Although this expression looks rather horrendous, the integrand is in fact a total derivative. To see this, we note that

$$\begin{aligned}\frac{d}{d\varphi}\left(\frac{m\pi R\sin\varphi}{a}\right) &= \frac{m\pi R}{a}\cos\varphi\\ \frac{d}{d\varphi}\left(\frac{n\pi(h+R\cos\varphi)}{b}\right) &= -\frac{n\pi R}{b}\sin\varphi\end{aligned}$$

so that

$$\begin{aligned}
A_{mn}^{(\pm)} &= -\frac{I_0 Z_{mn}}{\gamma_{mn}\sqrt{ab}} \int_0^\pi \frac{d}{d\varphi} \left[\sin\left(\frac{m\pi R \sin\varphi}{a} \right) \sin\left(\frac{n\pi(h + R\cos\varphi)}{b} \right) \right] d\varphi \\
&= -\frac{I_0 Z_{mn}}{\gamma_{mn}\sqrt{ab}} \left[\sin\left(\frac{m\pi R \sin\varphi}{a} \right) \sin\left(\frac{n\pi(h + R\cos\varphi)}{b} \right) \right]_0^\pi \\
&= 0
\end{aligned}$$

This demonstrates that the TM_{mn} modes are not excited by this semi-circular loop. In fact, this conspiracy between $\hat{x}$ and $\hat{y}$ components of the current and the electric field in (6) to form a total derivative suggests that there is a more general proof that TM modes are not excited by current loops contained in the x-y plane.

Recall that, for a TM mode, the transverse electric field is given by a transverse gradient

$$\vec{E}_{t,\lambda} = \frac{ik_\lambda}{\gamma_\lambda^2} \vec{\nabla}_t \psi_\lambda$$

so that

$$\vec{E}_\lambda^{(\pm)} = C_\lambda \left[\frac{ik_\lambda}{\gamma_\lambda^2} \vec{\nabla}_t \psi_\lambda \pm \hat{z}\psi_\lambda \right] e^{\pm ik_\lambda z}$$

where C_λ is an appropriate normalization constant. Using (5) for a wire source in the transverse plane gives

$$\begin{aligned}
A_\lambda^{(\pm)} &= -\frac{Z_\lambda}{2} \int_V \vec{J} \cdot \vec{E}_\lambda^{(\mp)} d^3x \\
&= -\frac{I_0 Z_\lambda}{2} \int_{\vec{x}_0}^{\vec{x}_1} \vec{E}_\lambda^{(\mp)} \cdot d\vec{l} \\
&= -\frac{ik_\lambda I_0 Z_\lambda C_\lambda}{2\gamma_\lambda^2} e^{\pm ik_\lambda z_0} \int_{\vec{x}_0}^{\vec{x}_1} \vec{\nabla}_t \psi_\lambda \cdot d\vec{l} \\
&= -\frac{ik_\lambda I_0 Z_\lambda C_\lambda}{2\gamma_\lambda^2} e^{\pm ik_\lambda z_0} [\psi_\lambda(\vec{x}_1) - \psi_\lambda(\vec{x}_0)]
\end{aligned}$$

Note that the wire lies in the $z = z_0$ plane, and that $\vec{x}_0$ and $\vec{x}_1$ are the initial and final endpoints along the wire. Since this is a TM mode, we recall that ψ satisfies Dirichlet boundary conditions, $\psi|_S = 0$. Therefore, as long as the wire source starts and ends at the waveguide walls, we immediately see that $A_\lambda^{(\pm)}$ vanishes, regardless of the shape of the wire or the waveguide. Note that this general proof depends on the wire being restricted to a transverse plane, and carrying a constant current throughout.

A simple physical explanation for why the TM modes are not excited is that a current loop in the x-y plane will generate a magnetic field in the $\hat{z}$ direction. However, TM modes by their very nature do not have any magnetic fields in the $\hat{z}$ direction. Therefore, the fields that are excited by the current in the wire are orthogonal to the fields of the TM modes. The TE modes, however, will be excited, as they are the ones with magnetic fields in the $\hat{z}$ direction.

b) Determine the amplitude for the lowest TE mode in the guide and show that its value is independent of the height h.

The normalized TE$_{mn}$ mode is given by

$$\vec{E}_{mn}^{(\pm)} = \frac{2\pi}{\gamma_{mn}\sqrt{ab}}\left[-\hat{x}\frac{n}{b}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)+\hat{y}\frac{m}{a}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\right] \times e^{\pm ik_{mn}z}$$

Note that this must be multiplied by $1/\sqrt{2}$ if either $m=0$ or $n=0$. In this case, we find

$$\begin{aligned}
A_{mn}^{(\pm)} &= -\frac{I_0 Z_{mn} R}{2}\int_0^\pi \vec{E}_{mn}^{(\mp)}\cdot(\hat{x}\cos\varphi - \hat{y}\sin\varphi)d\varphi \\
&= \frac{I_0 Z_{mn}\pi R}{\gamma_{mn}\sqrt{ab}}\int_0^\pi\left[\frac{n}{b}\cos\varphi\cos\left(\frac{m\pi R\sin\varphi}{a}\right)\sin\left(\frac{n\pi(h+R\cos\varphi)}{b}\right)\right. \\
&\qquad\left. + \frac{m}{a}\sin\varphi\sin\left(\frac{m\pi R\sin\varphi}{a}\right)\cos\left(\frac{n\pi(h+R\cos\varphi)}{b}\right)\right]d\varphi
\end{aligned}$$

Unlike in the TM case, here there is nothing particularly nice about this integral. We focus on the lowest TE mode, namely TE$_{10}$. Using $\gamma_{10}=\pi/a$, and including the additional normalization factor of $1/\sqrt{2}$, we have

$$A_{10}^{(\pm)} = \frac{I_0 Z_{10} R}{\sqrt{2ab}}\int_0^\pi \sin\varphi\sin\left(\frac{\pi R\sin\varphi}{a}\right)d\varphi$$

Noting the integral representation for the Bessel function

$$J_{2n+1}(x) = \frac{1}{\pi}\int_0^\pi \sin(x\sin\theta)\sin[(2n+1)\theta]d\theta$$

we see that

$$A_{10}^{(\pm)} = \frac{\pi I_0 Z_{10} R}{\sqrt{2ab}}J_1\left(\frac{\pi R}{a}\right)$$

This is explicitly independent of the height h. The reason for this is that the fields (both electric and magnetic) of the TE$_{10}$ mode are independent of the height. Expansion of the Bessel function for $R \ll a$ gives

$$A_{10}^{(\pm)} \approx \frac{\pi^2 I_0 Z_{10} R^2}{2\sqrt{2a^3 b}} \tag{7}$$

c) Show that the power radiated in either direction in the lowest TE mode is

$$P = \frac{I_0^2}{16} Z \frac{a}{b} \left(\frac{\pi R}{a}\right)^4$$

where Z is the wave impedance of the TE_{10} mode. Here assume $R \ll a, b$.

The average radiated power is given by

$$P_\lambda^{(\pm)} = \frac{1}{2}\int (\vec{E}^{(\pm)} \times \vec{H}^{(\pm)*}) \cdot \hat{z}\, da = \frac{1}{2}|A_\lambda^{(\pm)}|^2 \int (\vec{E}_\lambda^{(\pm)} \times \vec{H}_\lambda^{(\pm)*}) \cdot \hat{z}\, da$$

Since the power is computed along the $\hat{z}$ direction, we only need the transverse components of the fields

$$P_\lambda^{(\pm)} = \pm\frac{1}{2}|A_\lambda^{(\pm)}|^2 \int (\vec{E}_{t,\lambda} \times \vec{H}_{t,\lambda}) \cdot \hat{z}\, da = \pm\frac{1}{2Z_\lambda}|A_\lambda^{(\pm)}|^2$$

where we used orthonormality [Jackson equation (8.133)] in the last step. For the TE_{10} mode, we substitute in (7) to find

$$P_{10}^{(\pm)} = \pm\frac{\pi^4 I_0^2 Z_{10} R^4}{16a^3 b} = \pm\frac{I_0^2}{16} Z_{10} \frac{a}{b}\left(\frac{\pi R}{a}\right)^4$$

The power flows to the right on the right of the source, and to the left on the left of the source.

9.1 A common textbook example of a radiating system (see Problem 9.2) is a configuration of charges fixed relative to each other but in rotation. The charge density is obviously a function of time, but it is not in the form of (9.1).

a) Show that for rotating charges one alternative is to calculate *real* time-dependent multipole moments using $\rho(\vec{x}, t)$ directly and then compute the multipole moments for a given harmonic frequency with the convention of (9.1) by inspection or Fourier decomposition of the time-dependent moments. Note that care must be taken when calculating $q_{lm}(t)$ to form linear combinations that are real before making the connection.

For a rotating set of charges, where the rotation is along the z axis, the charge density may be written as

$$\rho = \rho(r, \theta, \phi - \omega_0 t)$$

where ω_0 is the angular frequency of rotation. Using this, we first examine the time-dependent multipole moments

$$\begin{aligned} q_{lm}(t) &= \int r^l Y_{lm}^*(\theta, \phi)\rho(r, \theta, \phi - \omega_0 t) d^3x \\ &= \int r^l Y_{lm}^*(\theta, \phi' + \omega_0 t)\rho(r, \theta, \phi') d^3x \end{aligned}$$

where in the second line we have made the substitution $\phi = \phi' + \omega_0 t$. We now note that the azimuthal behavior of the spherical harmonics goes as

$$Y_{lm}(\theta, \phi) \sim e^{im\phi}$$

Hence

$$Y_{lm}(\theta, \phi' + \omega_0 t) = Y_{lm}(\theta, \phi')e^{im\omega_0 t}$$

This allows us to isolate the time dependence of $q_{lm}(t)$ as

$$q_{lm}(t) = \overline{q}_{lm}e^{-im\omega_0 t} \tag{8}$$

where $\overline{q}_{lm}$ is the static multipole moment calculated in the body frame [ie with $\rho(r, \theta, \phi)$]. This expression is almost of the analogous form as (9.1), in the sense that the harmonic time dependence is given by a complex exponential. One interesting difference, however, is that (9.1) involves a pure frequency ω of the form

$$\rho(\vec{x}, t) = \rho(\vec{x})e^{-i\omega t}$$

while (8) involves a different frequency

$$\omega_m = m\omega_0$$

for each different value of m. This demonstrates that a rotating set of charges generally radiates at the fundamental frequency ω_0 as well as all higher harmonics.

Another important difference, however, is that $m < 0$ components of (8) appear to have a negative frequency. This is somewhat artificial, since the harmonic prescription we are using is to take the real part of the complex time-dependent quantities. In particular

$$\Re(e^{-im\omega t}) = \Re(e^{+im\omega t}) = \cos(m\omega t)$$

This indicates that modes $q_{lm}(t)$ and $q_{l,-m}(t)$ radiate at the same frequency $m\omega_0$. To avoid negative frequencies, we may use the identity

$$Y_{l,-m}(\theta, \phi) = (-1)^m Y^*_{lm}(\theta, \phi)$$

to show that $\overline{q}_{l,-m} = (-1)^m \overline{q}^*_{l,m}$. This allows us to rewrite (8) as

$$q_{lm}(t) = \begin{cases} \overline{q}_{lm}e^{-im\omega_0 t} & m > 0 \\ \overline{q}_{l0} & m = 0 \\ (-1)^m[\overline{q}_{l|m|}e^{-i|m|\omega_0 t}]^* & m < 0 \end{cases}$$

Note that the $m = 0$ term has zero frequency, and hence does not radiate. At this stage, we still have not specified what to do with the $m < 0$ multipoles. We note, however, that since ultimately we only care about the real parts of these complex expressions, it does not matter much whether we take a complex conjugate or not. Hence we can drop the complex conjugate in the $m < 0$ expression above. In this case, both $q_{lm}(t)$ and $q_{l,-m}(t)$ can be expressed using $\overline{q}_{l|m|}$, at least up to a possible minus sign. To see how to deal with this sign, we note that $q_{lm}(t)$ is essentially the coefficient of $Y_{lm}(\theta,\phi)$ in the spherical harmonic expansion. The product $q_{lm}(t)Y_{lm}(\theta,\phi)$ then has a simple $m \to -m$ behavior

$$q_{l,-m}(t)Y_{l,-m}(\theta,\phi) = [q_{lm}(t)Y_{lm}(\theta,\phi)]^*$$

Linearly superposing the $+m$ and $-m$ moments then gives

$$q_{lm}(t)Y_{lm}(\theta,\phi) + q_{l,-m}(t)Y_{l,-m}(\theta,\phi) = \Re[2\overline{q}_{lm}Y_{lm}(\theta,\phi)e^{-im\omega_0 t}]$$

This demonstrates that, when summing over all multipoles for radiation, it is sufficient to sum over the positive frequency modes only while including an extra factor of two. In particular, we can take

$$q_{lm}^{\text{eff}} = \begin{cases} 2\overline{q}_{lm} & m > 0 \\ \overline{q}_{l0} & m = 0 \end{cases} \qquad \text{with frequencies } m\omega_0 \tag{9}$$

b) Consider a charge density $\rho(\vec{x},t)$ that is periodic in time with period $T = 2\pi/\omega_0$. By making a Fourier *series* expansion, show that it can be written as

$$\rho(\vec{x},t) = \rho_0(\vec{x}\,) + \sum_{n=1}^{\infty} \Re[2\rho_n(\vec{x}\,)e^{-in\omega_0 t}]$$

where

$$\rho_n(\vec{x}\,) = \frac{1}{T}\int_0^T \rho(\vec{x},t)e^{in\omega_0 t}dt$$

This shows explicitly how to establish connection with (9.1).

Recall that the complex Fourier series in the time variable t may be written as

$$\rho(\vec{x},t) = \sum_{n=-\infty}^{\infty} \rho_n(\vec{x}\,)e^{-in\omega_0 t}$$
$$\rho_n(\vec{x}\,) = \frac{1}{T}\int_0^T \rho(\vec{x},t)e^{in\omega_0 t}dt$$

Assuming that $\rho(\vec{x},t)$ is real (as it ought to be) we note that

$$\rho_{-n}(\vec{x}\,) = \rho_n(\vec{x}\,)^*$$

Hence

$$\begin{aligned}
\rho(\vec{x},t) &= \rho_0(\vec{x}\,) + \sum_{n=1}^{\infty}[\rho_n(\vec{x}\,)e^{-in\omega_0 t} + \rho_{-n}(\vec{x}\,)e^{in\omega_0 t}] \\
&= \rho_0(\vec{x}\,) + \sum_{n=1}^{\infty}[\rho_n(\vec{x}\,)e^{-in\omega_0 t} + (\rho_n(\vec{x}\,)e^{-in\omega_0 t})^*] \\
&= \rho_0(\vec{x}\,) + \sum_{n=1}^{\infty}\Re[2\rho_n(\vec{x}\,)e^{-in\omega_0 t}]
\end{aligned}$$

Taking the real part of a complex time harmonic quantity is of course what we want to make connection to (9.1). In particular, we show that the periodic in time charge distribution $\rho(\vec{x},t)$ may be treated as a collection of harmonic charge densities

$$\rho_n^{\text{eff}}(\vec{x}\,) = \begin{cases} 2\rho_n(\vec{x}\,) & n>0 \\ \rho_0(\vec{x}\,) & n=0 \end{cases} \qquad \text{with frequencies } n\omega_0 \tag{10}$$

Of course, $\rho_0(\vec{x}\,)$ is static and does not radiate. Note the similarity in form between this and (9).

c) For a single charge q rotating about the origin in the x-y plane in a circle of radius R at constant angular speed ω_0, calculate the $l=0$ and $l=1$ multipole moments by the methods of parts a and b and compare. In method b express the charge density $\rho_n(\vec{x}\,)$ in cylindrical coordinates. Are there higher multipoles, for example, quadrupole? At what frequencies?

For a single rotating charge q, the time dependent charge density may be written in spherical coordinates as

$$\rho(\vec{x},t) = \frac{q}{R^2}\delta(r-R)\delta(\cos\theta)\delta(\phi-\omega_0 t)$$

We start with the method of part a. Here we calculate the body-centric multipole moments

$$\begin{aligned}
\overline{q}_{lm} &= \int r^l Y_{lm}^*(\theta,\phi)\overline{\rho}(r,\theta,\phi)r^2 dr\,d\cos\theta\,d\phi \\
&= qR^l Y_{lm}^*(\pi/2,0) \\
&= qR^l\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(0)
\end{aligned} \tag{11}$$

The $l=0$ and $l=1$ moments are

$$\overline{q}_{00} = \sqrt{\frac{1}{4\pi}}q, \qquad \overline{q}_{11} = -\sqrt{\frac{3}{8\pi}}qR$$

so that, according to (9), we have

$$q_{00}^{\text{eff}} = \sqrt{\frac{1}{4\pi}}q, \qquad q_{11}^{\text{eff}} = -\sqrt{\frac{3}{2\pi}}qR$$

For the method of part b, we start by calculating the n-th Fourier mode

$$\begin{aligned}\rho_n(\vec{x}\,) &= \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} \rho(\vec{x},t)e^{in\omega_0 t}dt \\ &= \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} \frac{q}{R^2}\delta(r-R)\delta(\cos\theta)\delta(\phi-\omega_0 t)e^{in\omega_0 t}dt \\ &= \frac{q}{2\pi R^2}\delta(r-R)\delta(\cos\theta)e^{in\phi}\end{aligned}$$

The multipole moments calculated from $\rho_n(\vec{x})$ are

$$\begin{aligned}q_{lm}[\rho_n] &= \int r^l Y^*_{lm}(\theta,\phi)\rho_n(r,\theta,\phi)r^2 dr d\cos\theta d\phi \\ &= \frac{q}{2\pi R^2}\int r^l Y^*_{lm}(\theta,\phi)\delta(r-R)\delta(\cos\theta)e^{in\phi}r^2 dr d\cos\theta d\phi \\ &= qR^l\frac{1}{2\pi}\int_0^{2\pi} Y^*_{lm}(\pi/2,\phi)e^{in\phi}d\phi \\ &= qR^l\delta_{mn}Y^*_{lm}(\pi/2,0) \\ &= qR^l\delta_{mn}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(0)\end{aligned}$$

Note that the moments calculated from $\rho_n(\vec{x}\,)$ have $m=n$, but otherwise agree with (11). Since the effective charge density $\rho_n(\vec{x}\,)$ is doubled for $n>0$ according to (10), the effective moments $q_{lm}[\rho_n]$ are doubled as well. This effective doubling is consistent across parts a and b.

Finally, we note from (11) that all higher multipoles are present, so long as $P_l^m(0)$ is non-vanishing. By parity, this happens whenever $l+m$ is even. Thus the l-th multipole will radiate at frequences $l\omega_0, (l-2)\omega_0, (l-4)\omega_0, \ldots$.

9.3 Two halves of a spherical metallic shell of radius R and infinite conductivity are separated by a very small insulating gap. An alternating potential is applied between the two halves of the sphere so that the potentials are $\pm V\cos\omega t$. In the long-wavelength limit, find the radiation fields, the angular distribution of radiated power, and the total radiated power from the sphere.

In the long wavelength limit, the electric dipole approximation ought to be reasonable. In this case, we may first work out the multipole expansion of the source, and then extract the dipole term. For this problem, the source is essentially a harmonically ($e^{-i\omega t}$) varying version of the electrostatic problem with hemispheres at opposite potential. The long wavelength limit is also equivalent to the low frequency limit. Thus it is valid to think of the source as a quasi-static object. Using azimuthal symmetry, the potential then admits an expansion in Legendre polynomials

$$\Phi(r,\theta) = \sum_l \alpha_l\left(\frac{R}{r}\right)^{l+1} P_l(\cos\theta)$$

where

$$\alpha_l = \frac{2l+1}{2} \int_{-1}^{1} \Phi(R, \cos\theta) P_l(\cos\theta) d\cos\theta$$

For hemispheres at opposite potential $\pm V$ (times $e^{-i\omega t}$, which is to be understood), the expansion coefficients are

$$\alpha_l = (2l+1)V \int_0^1 P_l(x)\, dx \qquad \text{odd } l \text{ only}$$

The dipole term is dominant, and it is easy to see that $\alpha_1 = \frac{3}{2}V$. This gives rise to a dipole potential of the form

$$\Phi_{l=1} = \frac{3}{2}V \left(\frac{R}{r}\right)^2 P_1(\cos\theta) = \frac{3}{2}VR^2 \frac{z}{r^3}$$

Comparing this with the dipole expression

$$\Phi = \frac{1}{4\pi\epsilon_0} \frac{\vec{p}\cdot\vec{r}}{r^3}$$

allows us to read off an electric dipole moment

$$\vec{p} = 4\pi\epsilon_0(\tfrac{3}{2}VR^2\hat{z}) = 6\pi\epsilon_0 VR^2\hat{z}$$

Working in the radiation zone, this electric dipole gives

$$\vec{H} = \frac{ck^2}{4\pi}(\hat{r}\times\vec{p})\frac{e^{ikr}}{r} = -\frac{ck^2}{4\pi}6\pi\epsilon_0 VR^2\frac{e^{ikr}}{r}\sin\theta\hat{\phi} = -\frac{3}{2}Z_0^{-1}V(kR)^2\frac{e^{ikr}}{r}\sin\theta\hat{\phi}$$

and

$$\vec{E} = -Z_0\hat{r}\times\vec{H} = -\frac{3}{2}V(kR)^2\frac{e^{ikr}}{r}\sin\theta\hat{\theta}$$

The angular distribution of dipole radiation gives

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2}k^4|\vec{p}|^2\sin^2\theta = \frac{c^2 Z_0}{32\pi^2}k^4 36\pi^2\epsilon_0^2 V^2 R^4\sin^2\theta = \frac{9}{8}Z_0^{-1}V^2(kR)^4\sin^2\theta$$

and the total radiated power is

$$P = 3\pi Z_0^{-1}V^2(kR)^4$$