Physics 506 **Winter 2008**

Homework Assignment #2 — Solutions

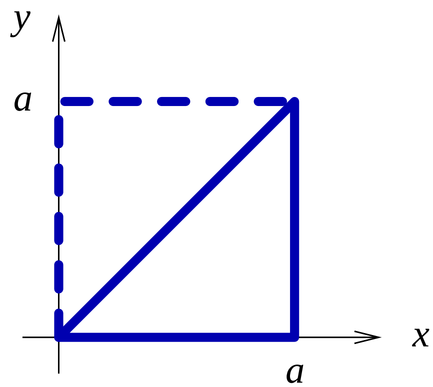
Textbook problems: Ch. 8: 8.5, 8.6, 8.7

8.5 A waveguide is constructed so that the cross section of the guide forms a right triangle with sides of length a, a, $\sqrt{2}a$, as shown. The medium inside has $\mu_r = \epsilon_r = 1$.

a) Assuming infinite conductivity for the walls, determine the possible modes of propagation and their cutoff frequencies.

In general, to solve a problem like this, we need to consider the Dirichlet or Neumann problem for a boundary without any 'standard' (ie rectangular or circular) symmetry. In particular, this means there is no natural coordinate system to use for the two-dimensional Helmholtz equation $[\nabla_t^2 + \gamma^2]\psi = 0$ that both allows for separation of variables *and* respects the symmetry of the boundary surface (which would allow a simple specification of the boundary data). A general problem of this form (with no simple boundary symmetry) is quite unpleasant to solve.

In this case, however, we can think of the triangle as 'half' of a square.



In particular, the key step to this problem is to note that the triangle may be obtained from the square by imposing reflection symmetry along the $x = y$ diagonal. This symmetry is a $\mathbb{Z}_2$ reflection on the coordinates of the form

$$\mathbb{Z}_2: \quad x \to y, \;\; y \to x$$

Eigenfunctions $\psi(x, y)$ can then be classified as either $\mathbb{Z}_2$-even or $\mathbb{Z}_2$-odd

$$\mathbb{Z}_2: \quad \psi(x, y) \to \pm\psi(y, x)$$

The odd functions vanish along the diagonal, so they automatically satisfy Dirichlet conditions $\psi(x = y) = 0$ on the diagonal. Similarly, the even functions have vanishing normal derivative on the diagonal and hence automatically satisfy Neumann conditions. We will use this fact to construct TM and TE modes for the triangle.

We begin with the TM modes. Using rectangular coordinates, it is natural to write solutions of the Helmholtz equation $[\partial_x^2 + \partial_y^2 + \gamma^2]\psi = 0$ as $\psi \sim e^{i(k_x x + k_y y)}$ where $k_x^2 + k_y^2 = \gamma^2$. This means we may expand the eigenfuctions in terms of sines and cosines. For TM modes satisfying the Dirichlet condition $\psi_S = 0$, we start with eigenfunctions on the square

$$\psi \sim \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}$$

which automatically satisfy the boundary conditions on the four walls of the square. This gives

$$\gamma_{mn} = \frac{\pi}{a}\sqrt{m^2+n^2}$$

so the cutoff frequencies are

$$\omega_{mn} = \frac{\pi}{\sqrt{\mu_0\epsilon_0}a}\sqrt{m^2+n^2} = \frac{\pi c}{a}\sqrt{m^2+n^2} \tag{1}$$

In order to satisfy the Dirichlet condition on the diagonal, we take the $\mathbb{Z}_2$-odd combination

$$\text{(TM)} \qquad \psi_{mn} = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a} - \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{a}$$

It is simple to verify that $\psi(x,0) = \psi(a,y) = \psi(x,x) = 0$, so that all boundary conditions on the triangle are indeed satisfied. The cutoff frequencies are given by (1). Note here that the $\mathbb{Z}_2$ projection removes the $m = n$ modes and also antisymmetrizes m with n. As a result, the integer labels m and n may be taken to satisfy the condition $m > n > 0$.

The analysis for TE modes is similar. However, for Neumann conditions, we take cosine combinations as well as a $\mathbb{Z}_2$-even eigenfunction. This gives

$$\text{(TE)} \qquad \psi_{mn} = \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{a} + \cos\frac{n\pi x}{a}\cos\frac{m\pi y}{a}$$

with identical cutoff frequencies as in (1). This time, however, the labels m and n may be taken to satisfy $m \geq n \geq 0$ (except $m = n = 0$ is not allowed).

b) For the lowest modes of each type calculate the attenuation constant, assuming that the walls have large, but finite, conductivity. Compare the result with that for a square guide of side a made from the same material.

The attenuation coefficients are determined by power and power loss. We begin with TM modes. For the power, we need to compute

$$\int_A |\psi|^2\, da = \int_A \left[\sin k_m x \sin k_n y - \sin k_n x \sin k_m y\right]^2 da \tag{2}$$

It is perhaps easiest to compute this by integrating over the square and then dividing by two for the triangle. This is because the integration separates into x and y integrals, and we may use orthogonality

$$\int_0^a \sin k_i x \sin k_j x \, dx = \frac{a}{2}\delta_{i,j} \qquad \left(\text{where } k_j = \frac{j\pi}{a}\right)$$

This gives

$$\int_A |\psi|^2 \, da = \frac{1}{2} \times 2 \left(\frac{a}{2}\right)^2 = \frac{a^2}{4}$$

The factor of $1/2$ is for the triangle, while the factor of 2 is because two non-vanishing terms arise when squaring the integrand in (2). (Recall that $m \neq n$ for TM modes.) This gives an expression for the power

$$\begin{aligned} P &= \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}} \left(\frac{\omega}{\omega_{mn}}\right)^2 \left(1 - \frac{\omega_{mn}^2}{\omega^2}\right)^{1/2} \int_A |\psi|^2 \, da \\ &= \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}} \left(\frac{\omega}{\omega_{mn}}\right)^2 \left(1 - \frac{\omega_{mn}^2}{\omega^2}\right)^{1/2} \frac{A}{2} \end{aligned}$$

where $A = a^2/2$ is the area of the triangle. Calculating the power loss involves integrating a normal derivative

$$\oint_C \left|\frac{\partial \psi}{\partial n}\right|^2 dl$$

We break this into three parts: along $y = 0$, along $x = a$ and along the diagonal $x = y$. Along the $y = 0$ wall, we have $\hat{n} = \hat{y}$ and

$$\left.\frac{\partial \psi}{\partial y}\right|_{y=0} = \frac{\pi}{a}\left[n \sin k_m x - m \sin k_n x\right]$$

As a result

$$\int_0^a \left|\frac{\partial \psi}{\partial y}\right|^2 dx = \left(\frac{\pi}{a}\right)^2 \frac{a}{2}(m^2 + n^2) = \frac{\pi^2}{2a}(m^2 + n^2) \tag{3}$$

A similar calculation, or use of symmetry, will result in an identical expression for the integral along the $x = a$ wall. For the diagonal, we use $\hat{n} = \frac{1}{\sqrt{2}}(\hat{x} - \hat{y})$ to compute

$$\begin{aligned} \frac{\partial \psi}{\partial n} = \frac{1}{\sqrt{2}}\left[\frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial y}\right]_{y=x} &= \sqrt{2}\frac{\pi}{a}\left[m \cos k_m x \sin k_n x - n \cos k_n x \sin k_m x\right] \\ &= \frac{\sqrt{2}}{2}\frac{\pi}{a}\left[(m - n)\sin k_{m+n} x - (m + n)\sin k_{m-n} x\right] \end{aligned}$$

This gives

$$\int_0^{\sqrt{a}} \left|\frac{\partial\psi}{\partial n}\right|^2 dl = \sqrt{2}\int_0^a \left|\frac{\partial\psi}{\partial n}\right|^2 dx = \sqrt{2}\frac{1}{2}\left(\frac{\pi}{a}\right)^2 \frac{a}{2}\left[(m-n)^2+(m+n)^2\right]$$
$$= \sqrt{2}\frac{\pi^2}{2a}(m^2+n^2)$$

Combining this diagonal with (3) for the sides, we obtain

$$\oint_C \left|\frac{\partial\psi}{\partial n}\right|^2 dl = C\frac{\pi^2}{2a^2}(m^2+n^2) = \frac{C}{2}\gamma_{mn}^2$$

where $C = a + a + \sqrt{2}a$ is the circumference of the triangle. This gives a TM mode power loss of

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{mn}}\right)^2 \frac{1}{\mu^2\omega_{mn}^2}\oint_C \left|\frac{\partial\psi}{\partial n}\right|^2 dl$$
$$= \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{mn}}\right)^2 \frac{1}{\mu^2\omega_{mn}^2}\frac{C}{2}\gamma_{mn}^2 = \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{mn}}\right)^2 \frac{\epsilon}{\mu}\frac{C}{2}$$

The attenuation coefficient is thus

$$\beta_{mn} = -\frac{1}{2P}\frac{dP}{dz} = \frac{1}{\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{-1/2}\frac{C}{2A}$$

so that the geometrical factor $\xi_{mn} = 1$ is trivial. Note that the energy loss calculation along the diagonal of the triangle gives the same result as along the square edges. As a result, the geometrical factor $\xi_{mn} = 1$ is independent of whether the waveguide is square or right triangular. This is why the triangular TM result is identical to the square TM result, at least up to the ratios $C/A = 2(2+\sqrt{2})/a \approx 6.83/a$ for the triangle and $C/A = 4/a$ for the square.

The power loss for the TE modes is somewhat harder to deal with because of the possibility of special cases. Consider

$$\psi = \cos k_m x \cos k_n y + \cos k_n x \cos k_m y \tag{4}$$

where $m \geq n \geq 0$. If $n = 0$, we end up with

$$\psi = \cos k_m x + \cos k_m y \qquad (m > 0)$$

In this case

$$\int_A |\psi|^2 da = \frac{1}{2}\int_0^a dx \int_0^a dy \left[\cos k_m x + \cos k_m y\right]^2 = \frac{1}{2}\times 2(\tfrac{1}{2}a^2) = \frac{a^2}{2} = A$$

while the perimeter integrals are

$$\int_0^a dx\big|\psi(y=0)\big|^2 = \int_0^a dx\big[1+\cos k_m x\big]^2 = a(1+\tfrac{1}{2}) = \frac{3a}{2}$$
$$\sqrt{2}\int_0^a dx\big|\psi(y=x)\big|^2 = \sqrt{2}\int_0^a dx\big[2\cos k_m x\big]^2 = 4\sqrt{2}(\tfrac{1}{2}a) = 2\sqrt{2}a$$

which gives

$$\oint_C \big|\psi\big|^2 dl = (3+2\sqrt{2})a$$

and

$$\begin{aligned}
\int_0^a dx\big|\hat{n}\times\vec{\nabla}_t\psi\big|^2 &= \int_0^a dx\big|\hat{y}\times\vec{\nabla}_t\psi\big|^2 = \int_0^a dx\big|-\hat{z}\partial_x\psi\big|^2_{y=0} \\
&= \int_0^a dx\frac{\pi^2}{a^2}m^2\big|\sin k_m x\big|^2 = \frac{\pi^2}{2a}m^2 \\
\sqrt{2}\int_0^a dx\big|\hat{n}\times\vec{\nabla}_t\psi\big|^2_{y=x} &= \sqrt{2}\int_0^a dx\big|\tfrac{1}{\sqrt{2}}\hat{z}(\partial_y+\partial_x)\psi\big|^2_{y=x} \\
&= \frac{\sqrt{2}}{2}\int_0^a dx\frac{\pi^2}{a^2}m^2\big|2\sin k_m x\big|^2 = \sqrt{2}\frac{\pi^2}{a}m^2
\end{aligned}$$

which gives

$$\oint_C \big|\hat{n}\times\vec{\nabla}_t\psi\big|^2 dl = (1+\sqrt{2})\frac{\pi^2}{a}m^2 = (1+\sqrt{2})a\gamma_{m0}^2$$

Using

$$P = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}\left(\frac{\omega}{\omega_{mn}}\right)^2\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{1/2}\int_A |\psi|^2 da$$

and

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{mn}}\right)^2\oint_C\left[\frac{1}{\gamma_{mn}^2}\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)|\hat{n}\times\vec{\nabla}_t\psi|^2 + \frac{\omega_{mn}^2}{\omega^2}|\psi|^2\right]dl$$

with the above integrals gives an attenuation coefficient

$$\begin{aligned}
\beta_{m0} &= -\frac{1}{2P}\frac{dP}{dz} \\
&= \frac{1}{2\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\left(1-\frac{\omega_{m0}^2}{\omega^2}\right)^{-1/2}\left[(1+\sqrt{2})\left(1-\frac{\omega_{m0}^2}{\omega^2}\right)+\frac{\omega_{m0}^2}{\omega^2}(3+2\sqrt{2})\right]\frac{a}{A} \\
&= \frac{1}{\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\left(1-\frac{\omega_{m0}^2}{\omega^2}\right)^{-1/2}\left[\frac{1+\sqrt{2}}{2+\sqrt{2}}+\frac{\omega_{m0}^2}{\omega^2}\right]\frac{C}{2A}
\end{aligned}$$

where $C = (2+\sqrt{2})a$ and $A = a^2/2$. Here the geometrical factors are

$$\xi_{m0} = \frac{1+\sqrt{2}}{2+\sqrt{2}}, \qquad \eta_{m0} = 1 \qquad (m > n = 0)$$

For the rectangular waveguide, one has instead

$$\xi_{m0} = \frac{a}{a+b} \to \frac{1}{2}, \qquad \eta_{m0} = \frac{2b}{a+b} \to 1 \qquad \text{when } b \to a$$

This is different because the power loss calculation is no longer universal, giving different coefficients along the diagonal as along the square edges. The remaining TE cases to consider are modes (4) where $m = n > 0$ and $m > n > 0$. Here we simply state the results. For $m = n > 0$ we have

$$\psi = \cos k_m x \cos k_m y$$

(we have removed an unimportant factor of two) so that

$$\begin{aligned}
\int_A |\psi|^2 da &= \frac{a^2}{8} = \frac{A}{4} \\
\oint_C |\psi|^2 dl &= \left(1 + \frac{3\sqrt{2}}{8}\right) a \\
\oint_C |\hat{n} \times \vec{\nabla}_t \psi|^2 dl &= \left(1 + \frac{\sqrt{2}}{4}\right) \frac{\pi^2}{a} m^2 = \left(\frac{1}{2} + \frac{\sqrt{2}}{8}\right) a\gamma_{mm}^2
\end{aligned}$$

This gives

$$\xi_{mm} = \frac{4+\sqrt{2}}{4+2\sqrt{2}}, \qquad \eta_{mm} = 1 \qquad (m = n > 0)$$

On the other hand, for the general case $m > n > 0$ we find

$$\begin{aligned}
\int_A |\psi|^2 da &= \frac{a^2}{4} = \frac{A}{2} \\
\oint_C |\psi|^2 dl &= (2+\sqrt{2})a = C \\
\oint_C |\hat{n} \times \vec{\nabla}_t \psi|^2 dl &= (2+\sqrt{2})\frac{\pi^2}{2a}(m^2+n^2) = \frac{C}{2}\gamma_{mn}^2
\end{aligned}$$

which yields

$$\xi_{mn} = 1, \qquad \eta_{mn} = 1 \qquad (m > n > 0)$$

In all cases, $\eta_{mn} = 1$, which is the same for the triangle or the square waveguide. For ξ_{mn}, the factor is essentially a geometric combination of contributions along the perimeter of either 1 or 1/2 depending on the particular mode and its degeneracies.

8.6 A resonant cavity of copper consists of a hollow, right circular cylinder of inner radius R and length L, with flat end faces.

a) Determine the resonant frequencies of the cavity for all types of waves. With $(1/\sqrt{\mu\epsilon}R)$ as a unit of frequency, plot the lowest four resonant frequencies of each type as a function of R/L for $0 < R/L < 2$. Does the same mode have the lowest frequency for all R/L?

This cavity is essentially covered in Section 8.7 of the textbook. It is also similar to the waveguide problem 8.4, but with endcaps to form a resonant cavity. The normal modes are either TM or TE modes. The TM modes are given by

$$\psi(\rho,\phi) = E_0 J_m(\gamma_{mn}\rho)e^{\pm im\phi}, \qquad \gamma_{mn} = \frac{x_{mn}}{R} \tag{5}$$

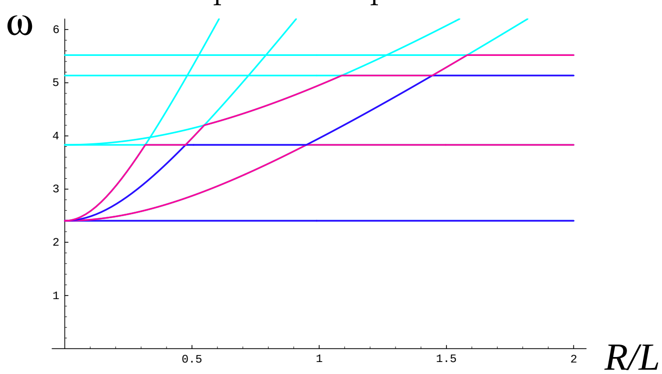
where x_{mn} are the zeros of the Bessel functions J_m. The resonant frequencies are thus

$$\text{(TM)} \qquad \omega_{mnp} = \frac{1}{\sqrt{\mu\epsilon}R}\sqrt{x_{mn}^2 + \left(\frac{p\pi R}{L}\right)^2} \qquad (p \geq 0)$$

with

$$x_{01} = 2.405, \qquad x_{11} = 3.832, \qquad x_{21} = 5.136, \qquad x_{02} = 5.520$$

The lowest four resonant frequncies are plotted as follows



Note that the $p = 0$ modes are independent of R/L. Clearly the same mode does not always have the lowest frequency. The cross-over points are accidental degeneracies, and are not related to any particular symmetry of the cylinder.

The TE modes are given by

$$\psi(\rho,\phi) = H_0 J_m(\gamma_{mn}\rho)e^{\pm im\phi}, \qquad \gamma_{mn} = \frac{x'_{mn}}{R}$$

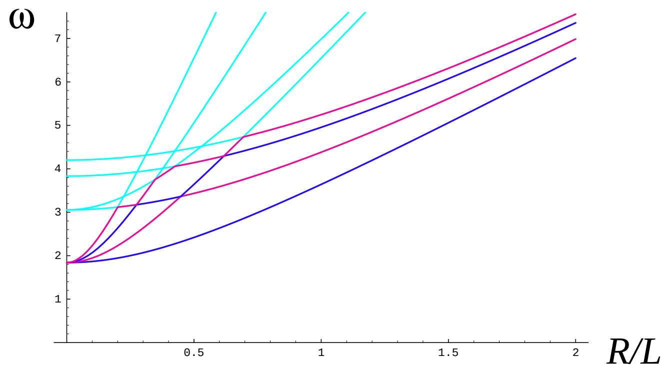
where x'_{mn} are the zeros of J'_m. The TE resonant frequencies are

$$\text{(TE)} \qquad \omega_{mnp} = \frac{1}{\sqrt{\mu\epsilon}R}\sqrt{x'^2_{mn} + \left(\frac{p\pi R}{L}\right)^2} \qquad (p > 0)$$

with

$$x'_{11} = 1.841, \qquad x'_{21} = 3.054, \qquad x'_{01} = 3.832, \qquad x'_{31} = 4.201$$

In this case, the lowest four resonant frequencies are



b) If $R = 2$ cm, $L = 3$ cm, and the cavity is made of pure copper, what is the numerical value of Q for the lowest resonant mode?

For this geometry, it turns out the lowest mode is the TM_{010} mode. We thus calculate the Q factor for TM_{mn0} modes. We start with the stored energy

$$U = \frac{L\epsilon}{2}\int_A |\psi|^2 da \tag{6}$$

where ψ is given by (5). Note that this is double the $p \neq 0$ result. The power loss expression for TM_{mn0} modes is

$$P_{\text{loss}} = \frac{\epsilon}{\sigma\delta\mu}\left(1 + \xi_{mn}\frac{CL}{2A}\right)\int_A |\psi|^2 da \tag{7}$$

where we have taken $p = 0$ into account. Here $C = 2\pi R$ and $A = \pi R^2$ are the circumference and cross-sectional area of the cylinder. The geometrical factor ξ_{mn} is the same as the waveguide result, which was obtained in Problem 8.4 as $\xi_{mn} = 1$. More directly, we may start with the definition

$$\oint_C \left|\frac{d\psi}{dn}\right|^2 dl = \xi_{mn}\gamma^2_{mn}\frac{C}{A}\int_A |\psi|^2 da$$

Using (5), this statement is equivalent to

$$\begin{aligned} C\left(\gamma^2_{mn}J'_m(x_{mn})^2\right) &= \xi_{mn}\gamma^2_{mn}\frac{C}{A}\left(2\pi\int_0^R J_m(x_{mn}\rho/R)^2\rho\, d\rho\right) \\ &= \xi_{mn}\gamma^2_{mn}\frac{C}{A}\left(\pi R^2 J_{m+1}(x_{mn})^2\right) \end{aligned}$$

where the second line is obtained by the Bessel function normalization condition. This results in

$$\xi_{mn} = \left(\frac{J'_m(x_{mn})}{J_{m+1}(x_{mn})} \right)^2$$

However, using the Bessel recursion relation $J_{m+1}(\xi) = (m/\xi)J_m(\xi) - J'_m(\xi)$ and letting $\xi = x_{mn}$ be a zero of J_m, we obtain simply $J_{m+1}(x_{mn}) = -J'_m(x_{mn})$. This proves that the geometrical factor is simply $\xi_{mn} = 1$. Finally, using (6) and (7) with $\xi_{mn} = 1$ gives

$$Q_{mn0} = \omega_{mn} \frac{U}{P_{\text{loss}}} = \frac{\mu}{\mu_c} \frac{L}{\delta} \left(1 + \frac{CL}{2A} \right)^{-1} = \frac{\mu}{\mu_c} \frac{L}{\delta} \left(1 + \frac{L}{R} \right)^{-1}$$

Since copper is non-ferromagnetic, we may take $\mu_c = \mu_0$. Furthermore, we assume the interior of the cavity has $\mu = \mu_0$ and $\epsilon = \epsilon_0$. Substituting in $R = 2\,\text{cm}$, $L = 3\,\text{cm}$ then yields

$$Q_{mn0} = \frac{1.2 \times 10^{-2}\,\text{m}}{\delta}$$

We calculate the lowest resonant frequency to be

$$\omega_{010} = \frac{x_{01}c}{R} = \frac{2.405c}{R} = 3.61 \times 10^{10}\,\text{s}^{-1}$$

or $\nu_{010} = 5.94\,\text{GHz}$, where we have used $R = 3\,\text{cm}$. At this frequency, the skin depth for copper is

$$\delta = \frac{6.52 \times 10^{-2}\,\text{m}}{\sqrt{\nu_{\text{mnp}}(\text{Hz})}} = 8.6 \times 10^{-7}\,\text{m}$$

This gives a cavity Q of

$$Q_{010} = 1.4 \times 10^4$$

8.7 A resonant cavity consists of the empty space between two perfectly conducting, concentric spherical shells, the smaller having an outer radius a and the larger an inner radius b. As shown in Section 8.9, the azimuthal magnetic field has a radial dependence given by spherical Bessel functions, $j_l(kr)$ and $n_l(kr)$, where $k = \omega/c$.

a) Write down the transcendental equation for the characteristic frequencies of the cavity for arbitrary l.

According to the analysis of Section 8,9, the azimuthal magnetic field is given by

$$B_\phi(r, \theta) = \frac{u_l(r)}{r} P_l^1(\cos\theta)$$

where $u_l(r)$ satisfies

$$\left[\frac{d^2}{dr^2} + \frac{\omega^2}{c^2} - \frac{l(l+1)}{r^2} \right] u_l(r) = 0$$

The solution for $u_l(r)$ is of the form

$$u_l(r) = r[A_l j_l(kr) + B_l n_l(kr)]$$

where j_l and n_l are spherical Bessel functions and A_l and B_l are constants to be determined from the boundary conditions

$$\left.\frac{du_l(r)}{dr}\right|_{r=a} = \left.\frac{du_l(r)}{dr}\right|_{r=b} = 0$$

Since

$$\frac{du_l(r)}{dr} = A_l[j_l(kr) + kr\, j_l'(kr)] + B_l[n_l(kr) + kr\, n_l'(kr)]$$

the homogeneous boundary conditions may be given in terms of a matrix equation

$$\begin{pmatrix} j_l(ka) + ka\, j_l'(ka) & n_l(ka) + ka\, n_l'(ka) \\ j_l(kb) + kb\, j_l'(kb) & n_l(kb) + kb\, n_l'(kb) \end{pmatrix} \begin{pmatrix} A_l \\ B_l \end{pmatrix} = 0$$

This only admits non-trivial solutions when the determinant vanishes. This corresponds to the transcendental equation

$$\frac{j_l(ka) + ka j_l'(ka)}{n_l(ka) + ka\, n_l'(ka)} = \frac{j_l(kb) + kb\, j_l'(kb)}{n_l(kb) + kb\, n_l'(kb)} \tag{8}$$

where $k = \omega/c$.

b) For $l = 1$ use the explicit forms of the spherical Bessel functions to show that the characteristic frequencies are given by

$$\frac{\tan kh}{kh} = \frac{\left(k^2 + \frac{1}{ab}\right)}{k^2 + ab\left(k^2 - \frac{1}{a^2}\right)\left(k^2 - \frac{1}{b^2}\right)}$$

where $h = b - a$.

For $l = 1$, we have

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}, \qquad n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

So that

$$\frac{j_1(x) + xj_1'(x)}{n_1(x) + xn_1'(x)} = \frac{x\cos x - (1 - x^2)\sin x}{x\sin x + (1 - x^2)\cos x}$$

As a result, for $l = 1$, (8) may be rewritten as

$$\frac{ka\, \cos ka - (1 - (ka)^2)\sin ka}{ka\, \sin ka + (1 - (ka)^2)\cos ka} = \frac{kb\, \cos kb - (1 - (kb)^2)\sin kb}{kb\, \sin kb + (1 - (kb)^2)\cos kb}$$

which can be rearranged to give

$$\begin{aligned}&\big[(ka)(kb) + (1-(ka)^2)(1-(kb)^2)\big]\sin(kb-ka)\\&\qquad\qquad + \big[ka(1-(kb)^2) - kb(1-(ka)^2)\big]\cos(kb-ka) = 0\end{aligned}$$

Defining $h = b - a$, we end up with

$$\begin{aligned}\tan kh &= \frac{kb(1-(ka)^2) - ka(1-(kb)^2)}{(ka)(kb) + (1-(ka)^2)(1-(kb)^2)}\\&= \frac{kh(1+(ka)(kb))}{(ka)(kb) + (1-(ka)^2)(1-(kb)^2)}\\&= kh\frac{k^2 + \dfrac{1}{ab}}{k^2 + ab\left(k^2 - \dfrac{1}{a^2}\right)\left(k^2 - \dfrac{1}{b^2}\right)}\end{aligned} \tag{9}$$

c) For $h/a \ll 1$, verify that the result of part b yields the frequency found in Section 8.9, and find the first order correction in h/a.

For $h/a \ll 1$, we write

$$b = a + h = a(1 + h/a)$$

and expand for small h/a. In this case, (9) becomes

$$\begin{aligned}\frac{\tan kh}{kh} &= \frac{k^2 + \dfrac{1}{a^2} - \dfrac{1}{a^2}\dfrac{h}{a} + \mathcal{O}((h/a)^2)}{k^4a^2 - k^2 + \dfrac{1}{a^2} + \left(k^4a^2 - \dfrac{1}{a^2}\right)\dfrac{h}{a} + \mathcal{O}((h/a)^2)}\\&= \frac{(ka)^2+1}{(ka)^4 - (ka)^2 + 1} - \frac{(ka)^2((ka)^4 + 2(ka)^2 - 2)}{((ka)^4 - (ka)^2 + 1)^2}\frac{h}{a} + \mathcal{O}((h/a)^2)\end{aligned}$$

Expanding the left-hand-side for $h/a \ll 1$ gives

$$\frac{\tan kh}{kh} = 1 + \mathcal{O}((h/a)^2)$$

As a result, the characteristic frequency equation becomes

$$0 = -\frac{(ka)^2((ka)^2-2)}{(ka)^4 - (ka)^2 + 1} - \frac{(ka)^2((ka)^4 + 2(ka)^2 - 2)}{((ka)^4 - (ka)^2 + 1)^2}\frac{h}{a} + \mathcal{O}((h/a)^2) \tag{10}$$

This demonstrates that the zeroth order solution is given by

$$ka = \sqrt{2} \qquad \Rightarrow \qquad \omega_1 = \sqrt{2}\frac{c}{a}$$

which agrees with the approximate result found in Section 8.9.

To get the first order correction, we write

$$ka = \sqrt{2} + \delta(ka)$$

where $\delta(ka)$ is the desired correction term. Substituting this into (10) and expanding to first order in $\delta(ka)$ gives

$$0 = -2\sqrt{2}\delta(ka) \left. \frac{(ka)^2}{(ka)^4 - (ka)^2 + 1} \right|_{ka=\sqrt{2}} - \left. \frac{(ka)^2((ka)^4 + 2(ka)^2 - 2)}{((ka)^4 - (ka)^2 + 1)^2} \right|_{ka=\sqrt{2}} \frac{h}{a}$$

or

$$0 = -2\sqrt{2}\delta(ka)\frac{2}{3} - \frac{4}{3}\frac{h}{a}$$

which gives

$$\delta(ka) = -\frac{1}{\sqrt{2}}\frac{h}{a}$$

As a result, we have

$$ka = \sqrt{2} - \frac{1}{\sqrt{2}}\frac{h}{a} + \cdots = \sqrt{2}(1 - h/2a + \cdots)$$

or

$$\omega_1 = \sqrt{2}\frac{c}{a}(1 - h/2a + \cdots) = \sqrt{2}\frac{c}{a(1 + h/2a + \cdots)} = \sqrt{2}\frac{c}{a + h/2 + \cdots}$$

Note that

$$a + h/2 = \frac{a+b}{2}$$

is the average of the radii of the inside and outside spheres.