Physics 506 **Winter 2008**

Homework Assignment #1 — Solutions

Textbook problems: Ch. 8: 8.2, 8.4

8.2 A transmission line consisting of two concentric circular cylinders of metal with conductivity σ and skin depth δ, as shown, is filled with a uniform lossless dielectric (μ, ϵ). A TEM mode is propagated along this line. Section 8.1 applies.

a) Show that the time-averaged power flow along the line is

$$P = \sqrt{\frac{\mu}{\epsilon}}\pi a^2 |H_0|^2 \ln\left(\frac{b}{a}\right)$$

where H_0 is the peak value of the azimuthal magnetic field at the surface of the inner conductor.

A TEM mode is essentially a two-dimensional electrostatic problem. Thus we start by finding the electric field between the two cylinders. By elementary means, it should be clear that

$$\vec{E}_t = \frac{A}{\rho}\hat{\rho}$$

where A is a constant that will be determined shortly. Assuming wave propagation in the $+z$ direction, we use $\vec{B}_t = \sqrt{\mu\epsilon}\,\hat{z}\times\vec{E}_t$ to obtain the magnetic field

$$\vec{H}_t = \sqrt{\frac{\epsilon}{\mu}}\frac{A}{\rho}\hat{\phi}$$

This indicates that the magnitude of the magnetic field at the inner conductor is $H(a) = \sqrt{\epsilon/\mu}(A/a)$. Defining this as H_0 gives

$$\vec{E}_t = \sqrt{\frac{\mu}{\epsilon}}H_0\frac{a}{\rho}\hat{\rho}, \qquad \vec{H}_t = H_0\frac{a}{\rho}\hat{\phi} \tag{1}$$

The (harmonic) Poynting vector is then

$$\vec{S} = \tfrac{1}{2}\vec{E}\times\vec{H}^* = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}|H_0|^2\frac{a^2}{\rho^2}\hat{z}$$

so the power flow is

$$P = \int_A \hat{z}\cdot\vec{S}\,da = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}|H_0|^2\int_a^b \frac{a^2}{\rho^2}\,2\pi\rho\,d\rho = \pi\sqrt{\frac{\mu}{\epsilon}}|H_0|^2 a^2 \ln\left(\frac{b}{a}\right) \tag{2}$$

b) Show that the transmitted power is attenuated along the line as

$$P(z) = P_0 e^{-2\gamma z}$$

where

$$\gamma = \frac{1}{2\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\frac{\left(\frac{1}{a}+\frac{1}{b}\right)}{\ln\left(\frac{b}{a}\right)}$$

We compute the attenuation coefficient according to

$$\gamma = -\frac{1}{2P}\frac{dP}{dz} \tag{3}$$

The power P was calculated in part a. For the power loss per unit length of the waveguide, we use

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\oint_C |\hat{n}\times\vec{H}|^2\,dl = \frac{1}{2\sigma\delta}|H_0|^2\oint_C \frac{a^2}{\rho^2}\,dl$$

Note that there are two boundaries, one at $\rho = a$ (with circumference $2\pi a$) and the other at $\rho = b$ (with circumference $2\pi b$). This gives

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}|H_0|^2[2\pi a + (a/b)^2 2\pi b] = \frac{\pi}{\sigma\delta}|H_0|^2\frac{a}{b}(a+b) \tag{4}$$

Inserting this power loss expression and the power (2) into (3) yields

$$\gamma = \frac{1}{2\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\frac{a+b}{ab\ln(b/a)} = \frac{1}{2\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\frac{\left(\frac{1}{a}+\frac{1}{b}\right)}{\ln\left(\frac{b}{a}\right)}$$

c) The characteristic impedance Z_0 of the line is defined as the ratio of the voltage between the cylinders to the axial current flowing in one of them at any position z. Show that for this line

$$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\ln\left(\frac{b}{a}\right)$$

Since $Z_0 = |\Delta V|/I$, we need to compute the voltage difference between the cylinders as well as the current. For the voltage difference, we have

$$\Delta V = -\int_a^b \vec{E}\cdot d\vec{l} = -\sqrt{\frac{\mu}{\epsilon}}H_0\int_a^b \frac{a}{\rho}\,d\rho = -\sqrt{\frac{\mu}{\epsilon}}H_0 a\ln\left(\frac{b}{a}\right)$$

where we have used (1) for the electric field. In addition, the current is given by integrating the surface current density. For the inside conductor, we have

$$\vec{K} = \hat{n}\times\vec{H} = \hat{\rho}\times\left(H_0\frac{a}{\rho}\hat{\phi}\right)_{\rho=a} = H_0\hat{z}$$

Hence

$$I = \oint_C |K|\, dl = 2\pi a H_0$$

Taking the ratio $Z_0 = |\Delta V|/I$ results in

$$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}} \ln\left(\frac{b}{a}\right)$$

d) Show that the series resistance and inductance per unit length of the line are

$$R = \frac{1}{2\pi\sigma\delta}\left(\frac{1}{a}+\frac{1}{b}\right)$$
$$L = \left\{\frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right) + \frac{\mu_c\delta}{4\pi}\left(\frac{1}{a}+\frac{1}{b}\right)\right\}$$

where μ_c is the permeability of the conductor. The correction to the inductance comes from the penetration of the flux into the conductors by a distance of order δ.

We may obtain the series resistance from the power loss

$$\tfrac{1}{2}|I|^2 R = -\frac{dP}{dz}$$

where R denotes the resistance per unit length. Using $-dP/dz$ from (4) as well as the current computed above, we find

$$R = \frac{2}{|I|^2}\left(-\frac{dP}{dz}\right) = \frac{1}{2\pi\sigma\delta}\frac{a+b}{ab}$$

For the inductance per unit length, we compute the energy per unit length stored in the magnetic field. Inside the volume of the waveguide, we have

$$U_{\text{vol}} = \int_A \frac{\mu}{4}|\vec{H}|^2 da = \frac{\mu}{4}|H_0|^2 \int_a^b \frac{a^2}{\rho^2}\, 2\pi\rho\, d\rho = \frac{\mu}{2}|H_0|^2 \pi a^2 \ln\left(\frac{b}{a}\right)$$

In addition, since some of the magnetic field penetrates the conducting walls, we use the approximation

$$H(\zeta) = H_\parallel e^{-\zeta/\delta} e^{i\zeta/\delta}$$

where ζ is the distance into the conductor. Assuming the skin depth is much less than the thickness of the conductor as well as the radius of curvature, we approximate

$$U_{\text{wall}} = C\int_0^\infty \frac{\mu_c}{4}|H(\xi)|^2\, d\xi = \frac{\mu_c}{4} C|H_\parallel|^2 \int_0^\infty e^{-2\xi/\delta} d\xi = \frac{\mu_c}{8} C\delta |H_\parallel|^2$$

where C is the circumference of the wall. On the inside wall, we have $C = 2\pi a$ and $H_\| = H_0$, while on the outside wall, we have $C = 2\pi b$ and $H_\| = H_0(a/b)$. Hence

$$U_{\text{walls}} = \frac{\mu_c}{8}\delta|H_0|^2[2\pi a + 2\pi b(a/b)^2] = \frac{\mu_c}{4}\pi\delta|H_0|^2\frac{a}{b}(a+b)$$

Using

$$\tfrac{1}{4}L|I|^2 = U_{\text{vol}} + U_{\text{walls}}$$

we end up with

$$L = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right) + \frac{\mu_c\delta}{4\pi}\frac{a+b}{ab}$$

8.4 Transverse electric and magnetic waves are propagated along a hollow, right circular cylinder with inner radius R and conductivity σ.

a) Find the cutoff frequencies of the various TE and TM modes. Determine numerically the lowest cutoff frequency (the dominant mode) in terms of the tube radius and the ratio of cutoff frequencies of the next four higher modes to that of the dominant mode. For this part assume that the conductivity of the cylinder is infinite.

The eigenvalue equation for either TE or TM modes is

$$[\nabla_t^2 + \gamma^2]\psi(\rho, \phi) = 0$$

where $\psi(R, \phi) = 0$ for TM modes or $d\psi(\rho, \phi)/d\rho|_{\rho=R} = 0$ for TE modes. Writing $\psi(\rho, \phi) = \psi(\rho)e^{\pm im\phi}$, the radial equation (in cylindrical coordinates) becomes

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \gamma^2 - \frac{m^2}{\rho^2}\right)\psi(\rho) = 0$$

which is solved by Bessel functions. Avoiding the Neumann function which blows up at $\rho = 0$, we have

$$\psi(\rho, \phi) \sim J_m(\gamma\rho)e^{\pm im\phi}$$

The boundary conditions then place conditions on γ. For TM modes (Dirichlet conditions), we demand $J_m(\gamma R) = 0$. Hence

$$\text{(TM)} \qquad \gamma_{mn} = \frac{x_{mn}}{R} \qquad \text{or} \qquad \omega_{mn} = \frac{x_{mn}}{\sqrt{\mu\epsilon}R}$$

where x_{mn} is the n-th zero of J_m. For TE modes (Neumann conditions), on the other hand, we demand $J'_m(\gamma R) = 0$. Hence

$$\text{(TE)} \qquad \gamma_{mn} = \frac{x'_{mn}}{R} \qquad \text{or} \qquad \omega_{mn} = \frac{x'_{mn}}{\sqrt{\mu\epsilon}R}$$

where x'_{mn} is the n-th zero of J'_m. Sorting through the zeros of J_m and J'_m, the lowest five modes are given by

mode	$\sqrt{\mu\epsilon}R\omega_{mn}$	$\omega_{mn}/\omega_{\text{dominant}}$
TE_{11}	1.841	1
TM_{01}	2.405	1.306
TE_{21}	3.054	1.659
TE_{02} and TM_{11}	3.832	2.081

Note that the TE_{02} and TM_{11} modes are degenerate. This is a special case where the Bessel identity $J'_0(\zeta) = -J_1(\zeta)$ demonstrates that $x'_{0,n+1} = x_{1n}$.

b) Calculate the attenuation constants of the waveguide as a function of frequency for the lowest two distinct modes and plot them as a function of frequency.

The computation of the attenuation coefficients involves computing both power P and power loss $-dP/dz$. We first consider TM modes. The power is given by

$$P = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\left(\frac{\omega}{\omega_{mn}}\right)^2\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{1/2}\int_A |\psi|^2\,da \tag{5}$$

Using $\psi = J_m(\gamma\rho)e^{\pm im\phi}$ gives

$$\int_A |\psi|^2\,da = 2\pi\int_0^R J_m(x_{mn}\rho/R)^2\rho\,d\rho = 2\pi[\tfrac{1}{2}R^2J_{m+1}(x_{mn})^2] = \pi R^2 J_{m+1}(x_{mn})^2$$

where the expression in the square brackets comes from the Bessel function orthogonality relation

$$\int_0^a J_\nu(x_{\nu m}\rho/a)J_\nu(x_{\nu n}\rho/a)\rho\,d\rho = \tfrac{1}{2}a^2J_{\nu+1}(x_{\nu m})^2\delta_{mn}$$

Hence

$$P = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\left(\frac{\omega}{\omega_{mn}}\right)^2\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{1/2}\pi R^2 J_{m+1}(x_{mn})^2 \tag{6}$$

For a TM mode, the power loss is given by

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{mn}}\right)^2\oint_C \frac{1}{\mu^2\omega_{mn}^2}\left|\frac{\partial\psi}{\partial n}\right|^2 dl$$

In this case

$$\frac{\partial\psi}{\partial n} = -\left.\frac{\partial\psi}{\partial\rho}\right|_{\rho=R} = -\gamma_{mn}J'_m(x_{mn})e^{\pm im\phi}$$

Using $\gamma_{mn}^2 = \mu\epsilon\omega_{mn}^2$, we obtain

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\frac{\epsilon}{\mu}(2\pi R)J'_m(x_{mn})^2$$

We may now have some fun with Bessel functions. Using the recursion relation

$$J_{m+1}(\zeta) = \frac{m}{\zeta} J_m(\zeta) - J'_m(\zeta)$$

as setting $\zeta = x_{mn}$ to be a zero of J_m, we obtain

$$J_{m+1}(x_{mn}) = -J'_m(x_{mn})$$

This allows us to rewrite the power loss as

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta} \frac{\epsilon}{\mu} (2\pi R) J_{m+1}(x_{mn})^2 \tag{7}$$

Given (6) and (7), the TM_{mn} attenuation coefficient is obtained by setting

$$\beta_{mn} = -\frac{1}{2P}\frac{dP}{dz} = \frac{1}{2\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\left(1 - \frac{\omega_{mn}^2}{\omega^2}\right)^{-1/2} \frac{2\pi R}{\pi R^2} = \frac{1}{\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\left(1 - \frac{\omega_{mn}^2}{\omega^2}\right)^{-1/2} \frac{1}{R}$$

Note that $1/R = C/(2A)$ were $C = 2\pi R$ and $A = \pi R^2$ are the circumference and area of the cylindrical waveguide. Since $\delta = \delta_{mn}\sqrt{\omega_{mn}/\omega}$ (where δ_{mn} is the skin depth at the cutoff frequency ω_{mn}), we get the standard TM expression with the geometric factor $\xi_{mn} = 1$.

For the TE mode, the power loss calculation is somewhat lengthier, as it involves both H_z and $\vec{H}_t$. We begin with the power, which is given by a similar expression as (5), however with a factor of $\sqrt{\mu/\epsilon}$ instead. The Bessel normalization integral is now

$$\int_0^R J_m(x'_{mn}\rho/R)^2 \rho\, d\rho = \tfrac{1}{2} R^2 (1 - m^2/x'^2_{mn}) J_m(x'_{mn})^2$$

which gives

$$P = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}\left(\frac{\omega}{\omega_{mn}}\right)^2 \left(1 - \frac{\omega_{mn}^2}{\omega^2}\right)^{1/2} \pi R^2 \left(1 - \frac{m^2}{x'^2_{mn}}\right) J_m(x'_{mn})^2 \tag{8}$$

This time, the power loss expression is

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{mn}}\right)^2 \oint_C \left[\frac{1}{\gamma_{mn}^2}\left(1 - \frac{\omega_{mn}^2}{\omega^2}\right)|\hat{n} \times \vec{\nabla}_t \psi|^2 + \frac{\omega_{mn}^2}{\omega^2}|\psi|^2\right] dl$$

There are two terms to evaluate. The simple one is

$$\oint_C |\psi|^2 dl = (2\pi R) J_m(x'_{mn})^2$$

For the gradient term, we note that $\hat{n} = -\hat{\rho}$ on the inside of the cylinder. And $\vec{\nabla}_t = \hat{\rho}\partial_\rho + (1/\rho)\hat{\phi}\partial_\phi$. Hence

$$\oint_C |\hat{n} \times \vec{\nabla}_t \psi|^2 dl = (2\pi R) \left| \frac{1}{\rho} \frac{\partial \psi}{\partial \phi} \right|^2 = (2\pi R) \frac{m^2}{R^2} J_m(x'_{mn})^2$$

Combining these two terms yields

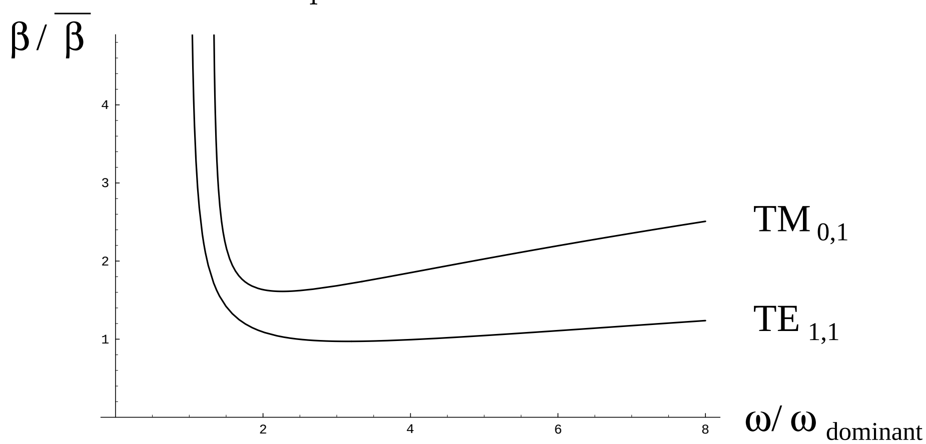
$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta} \left(\frac{\omega}{\omega_{mn}} \right)^2 (2\pi R) \left[\frac{m^2}{x'^2_{mn}} \left(1 - \frac{\omega_{mn}^2}{\omega^2} \right) + \frac{\omega_{mn}^2}{\omega^2} \right] J_m(x'_{mn})^2$$

Using this for the power loss and (8) for the power itself gives an attenuation coefficient

$$\begin{aligned} \beta_{mn} &= -\frac{1}{2P} \frac{dP}{dz} \\ &= \frac{1}{2\sigma\delta} \sqrt{\frac{\epsilon}{\mu}} \left(1 - \frac{\omega_{mn}^2}{\omega^2} \right)^{-1/2} \frac{2\pi R}{\pi R^2} \left[\frac{m^2}{x'^2_{mn}} \left(1 - \frac{\omega_{mn}^2}{\omega^2} \right) + \frac{\omega_{mn}^2}{\omega^2} \right] \left[1 - \frac{m^2}{x'^2_{mn}} \right]^{-1} \\ &= \frac{1}{\sigma\delta} \sqrt{\frac{\epsilon}{\mu}} \left(1 - \frac{\omega_{mn}^2}{\omega^2} \right)^{-1/2} \frac{1}{R} \left[\frac{m^2}{x'^2_{mn} - m^2} + \frac{\omega_{mn}^2}{\omega^2} \right] \end{aligned}$$

This demonstrates that the TE geometric factors are $\xi_{mn} = m^2/(x'^2_{mn} - m^2)$ and $\eta_{mn} = 1$.

The attenuation constants are plotted as follows



where

$$\overline{\beta} = \frac{1}{\sigma\delta_{mn}} \sqrt{\frac{\epsilon}{\mu}} \frac{1}{R}$$