Physics 506 **Winter 2006**

Practice Midterm – Solutions

The midterm will be a 120 minute open book, open notes exam. Do all three problems.

1. A resonant cavity is in the shape of a rectangular box with sides of lengths a, b and c.

   a) Assuming infinite conductivity for the walls, determine the modes of the cavity and their respective resonant frequencies.

   We assume the box is aligned with a, b and c along the x, y and z axes. The TM cavity modes are given by

$$\begin{aligned} E_z &= \psi \cos\frac{p\pi z}{c} \\ \vec{E}_t &= -\frac{p\pi}{c\gamma^2}\sin\frac{p\pi z}{c}\vec{\nabla}_t\psi \\ \vec{H}_t &= \frac{i\epsilon\omega}{\gamma^2}\cos\frac{p\pi z}{c}\hat{z}\times\vec{\nabla}_t\psi \end{aligned}$$

   Since ψ satisfies the rectangular coordinate Helmholtz equation

$$[\nabla_t^2 + k^2]\psi = 0 \qquad \psi\Big|_S = 0$$

   we have

$$\psi = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad \gamma = \pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

   The explicit modes are

$$\begin{aligned} E_z &= \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{p\pi z}{c} \\ E_x &= -\frac{p\pi}{c\gamma^2}\frac{m\pi}{a}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{p\pi z}{c} \\ E_y &= -\frac{p\pi}{c\gamma^2}\frac{n\pi}{b}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{p\pi z}{c} \\ H_x &= -\frac{i\epsilon\omega}{\gamma^2}\frac{n\pi}{b}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\cos\frac{p\pi z}{c} \\ H_y &= \frac{i\epsilon\omega}{\gamma^2}\frac{m\pi}{a}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{p\pi z}{c} \end{aligned}$$

   where $m, n \geq 1$ and $p \geq 0$. Note that here (and below) c is the length of the box along the z axis, and *not* the speed of light.

   The TE cavity modes are given by

$$\begin{aligned} H_z &= \psi \sin\frac{p\pi z}{c} \\ \vec{E}_t &= -\frac{i\mu\omega}{\gamma^2}\sin\frac{p\pi z}{c}\hat{z}\times\vec{\nabla}_t\psi \\ \vec{H}_t &= \frac{p\pi}{c\gamma^2}\cos\frac{p\pi z}{c}\vec{\nabla}_t\psi \end{aligned}$$

This time ψ satisfies

$$[\nabla_t^2 + k^2]\psi = 0 \qquad \frac{\partial\psi}{\partial n}\Big|_S = 0$$

so that

$$\psi = \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

The fields are

$$\begin{aligned}
E_x &= -\frac{i\mu\omega}{\gamma^2}\frac{n\pi}{b}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{p\pi z}{c} \\
E_y &= \frac{i\mu\omega}{\gamma^2}\frac{m\pi}{a}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{p\pi z}{c} \\
H_z &= \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{p\pi z}{c} \\
H_x &= -\frac{p\pi}{c\gamma^2}\frac{m\pi}{a}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\cos\frac{p\pi z}{c} \\
H_y &= -\frac{p\pi}{c\gamma^2}\frac{n\pi}{b}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{p\pi z}{c}
\end{aligned}$$

where $m, n \geq 0$ $(m+n \geq 1)$ and $p \geq 1$. In both TM and TE cases, the resonant frequency is

$$\omega_{mnp}^2 = \frac{\pi^2}{\mu\epsilon}\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{c}\right)^2\right]$$

Note that, in all cases, the fields are such that E_i has cos in the i-th coordinate, while H_i has sin in the i-th coordinate. This ensures that the boundary conditions $E_\parallel = H_\perp = 0$ are satisfied. The lowest modes are generally TM_{110}, TE_{101} and TE_{011}.

b) Calculate Q for each of the modes in part *a*), assuming the walls have large but finite conductivity σ.

To calculate Q, we use the expressions in the textbook whenever appropriate. For the TM modes, the stored energy and power loss are given by

$$\begin{aligned}
U &= \frac{c}{4}\epsilon\left[1 + \left(\frac{p\pi}{\gamma c}\right)^2\right]\int_A |\psi|^2 da \\
P_{\text{loss}} &= \frac{\epsilon}{\mu\sigma\delta}\left[1 + \left(\frac{p\pi}{\gamma c}\right)^2\right]\left(1 + \xi\frac{Cc}{4A}\right)\int_A |\psi|^2 da
\end{aligned}$$

Again, we keep in mind that c is the length of the box along the z direction. The area and circumference are given by

$$A = ab, \qquad C = 2(a+b)$$

while the factor ξ is given by

$$\xi = \frac{\oint_C (1/\gamma^2)|\partial\psi/\partial n|^2 dl}{(C/A)\int_A |\psi|^2 da}$$

Since $\psi = \sin(m\pi x/a)\sin(n\pi y/b)$, we calculate

$$\int_A |\psi|^2 da = \int_0^a dx \int_0^b dy \sin^2 \frac{m\pi x}{a} \sin^2 \frac{n\pi y}{b} = \frac{A}{4}$$

and

$$\begin{aligned}\oint_C \left|\frac{\partial\psi}{\partial n}\right|^2 dl &= 2\int_0^a dx \left(\frac{n\pi}{b}\right)^2 \sin^2 \frac{m\pi x}{a} + 2\int_0^b dy \left(\frac{m\pi}{a}\right)^2 \sin^2 \frac{n\pi y}{b} \\ &= a\left(\frac{n\pi}{b}\right)^2 + b\left(\frac{n\pi}{a}\right)^2\end{aligned}$$

Hence

$$\xi = \frac{1}{\gamma^2}\frac{\pi^2[a(n/b)^2 + b(m/a)^2]}{2(a+b)/4} = \frac{2[a(n/b)^2 + b(m/a)^2]}{(a+b)[(m/a)^2 + (n/b)^2]}$$

This gives a Q value of

$$\begin{aligned}Q^{\text{TM}} &= \frac{\omega U}{P_{\text{loss}}} = \frac{\omega c\sigma\delta\mu}{4}\left[1 + \xi\frac{Cc}{4A}\right]^{-1} \\ &= \frac{c}{2\delta}\frac{\mu}{\mu_c}\left[1 + \frac{2[a(n/b)^2 + b(m/a)^2]}{(a+b)[(m/a)^2 + (n/b)^2]}\frac{c(a+b)}{2ab}\right]^{-1} \\ &= \frac{\mu}{\mu_c}\frac{c}{\delta}\frac{1}{2}\left[1 + \frac{(c/a)(m/a)^2 + (c/b)(n/b)^2}{(m/a)^2 + (n/b)^2}\right]^{-1}\end{aligned}$$

Note that we take $c \to 2c$ if $p = 0$.

The TE mode is slightly more involved to work out. In this case, the stored energy is

$$U = \frac{c}{4}\mu\left[1 + \left(\frac{p\pi}{\gamma c}\right)^2\right] int_A|\psi|^2 da$$

while the power loss may be computed from

$$P_{\text{loss}} = \frac{1}{2\sigma\delta}\left[\oint_C dl \int_0^c dz |\hat{n}\times\vec{H}|^2_{\text{sides}} + 2\int_A da |\hat{n}\times\vec{H}|^2_{\text{ends}}\right]$$

We have to consider both terms. For the ends, we have

$$|\hat{n}\times\vec{H}|^2 = |\hat{z}\times\vec{H}|^2 = \left(\frac{p\pi}{c\gamma^2}\right)^2 |\hat{z}\times\vec{\nabla}_t\psi|^2 \cos^2 \frac{p\pi z}{c}\Big|_{\text{ends}} = \left(\frac{p\pi}{c\gamma^2}\right)^2 |\vec{\nabla}_t\psi|^2$$

As a result

$$\int_A da |\hat{n}\times\vec{H}|^2_{\text{ends}} = \left(\frac{p\pi}{c\gamma^2}\right)^2 \int_A |\vec{\nabla}_t\psi|^2 da = \left(\frac{p\pi}{c\gamma}\right)^2 \int_A |\psi|^2 da$$

For the sides, we have

$$|\hat{n}\times\vec{H}|^2 = \left|\hat{n}\times\hat{z}\psi\sin\frac{p\pi z}{c} + \frac{p\pi}{c\gamma^2}\cos\frac{p\pi z}{c}\hat{n}\times\vec{\nabla}\psi\right|^2$$

Note that $\hat{n}\times\hat{z}$ lies in the transverse direction, while $\hat{n}\times\vec{\nabla}\psi$ lies in the z direction. As a result

$$|\hat{n}\times\vec{H}|^2 = \sin^2\frac{p\pi z}{c}|\psi|^2 + \left(\frac{p\pi}{c\gamma^2}\right)^2\cos^2\frac{p\pi z}{c}|\hat{n}\times\vec{\nabla}\psi|^2$$

Integrating this over the z direction gives

$$\int_0^c dz|\hat{n}\times\vec{H}|^2_{\text{sides}} = \frac{c}{2}\left[|\psi|^2 + \left(\frac{p\pi}{c\gamma^2}\right)^2|\hat{n}\times\vec{\nabla}\psi|^2\right]$$

We now perform the circumference integrals using $\psi = \cos(m\pi x/a)\cos(n\pi y/b)$

$$\oint_C dl|\psi|^2 = 2\int_0^a dx\cos^2\frac{m\pi x}{a} + 2\int_0^b dy\cos^2\frac{n\pi y}{b} = a+b$$

and

$$\begin{aligned}\oint_C dl|\hat{n}\times\vec{\nabla}\psi|^2 &= 2\int_0^a dx\left|\frac{\partial\psi}{\partial x}\right|^2_{y=0} + 2\int_0^b dy\left|\frac{\partial\psi}{\partial y}\right|^2_{x=0}\\ &= 2\left(\frac{m\pi}{a}\right)^2\int_0^a dx\sin^2\frac{m\pi x}{a} + 2\left(\frac{n\pi}{b}\right)^2\int_0^b dy\sin^2\frac{n\pi y}{b}\\ &= a\left(\frac{m\pi}{a}\right)^2 + b\left(\frac{n\pi}{b}\right)^2\end{aligned}$$

This gives

$$\oint_C dl\int_0^c dz|\hat{n}\times\vec{H}|^2_{\text{sides}} = \frac{c}{2}\left[(a+b) + \left(\frac{p\pi}{c\gamma^2}\right)^2\left(a\left(\frac{m\pi}{a}\right)^2 + b\left(\frac{n\pi}{b}\right)^2\right)\right]$$

so that

$$P_{\text{loss}} = \frac{1}{2\sigma\delta}\left[\frac{c}{2}(a+b) + \frac{c}{2}\left(\frac{p\pi}{c\gamma}\right)^2\frac{a(m/a)^2+b(n/b)^2}{(m/a)^2+(n/b)^2} + 2\left(\frac{p\pi}{c\gamma}\right)^2\frac{ab}{4}\right]$$

The Q factor is then

$$\begin{aligned}Q^{\text{TE}} = \frac{\omega U}{P_{\text{loss}}} &= \frac{\omega c\mu 2\sigma\delta}{4}\frac{\left[1+\left(\frac{p\pi}{c\gamma}\right)^2\right]\frac{ab}{4}}{\frac{c}{2}\left[a+b+\left(\frac{p\pi}{c\gamma}\right)^2\left(\frac{a(m/a)^2+b(n/b)^2}{(m/a)^2+(n/b)^2}+\frac{ab}{c}\right)\right]}\\ &= \frac{\mu}{\mu_c}\frac{c}{\delta}\frac{\frac{ab}{2}c\left[1+\left(\frac{p\pi}{c\gamma}\right)^2\right]}{\left[a+b+\left(\frac{p\pi}{c\gamma}\right)^2\left(\frac{a(m/a)^2+b(n/b)^2}{(m/a)^2+(n/b)^2}+\frac{ab}{c}\right)\right]}\end{aligned}$$

As special cases, we take $a \to 2a$ if $m = 0$, or $b \to 2b$ if $n = 0$.

2. A current $I = \Re I_0 e^{-i\omega t}$ is flowing in a circular antenna with radius a centered on the origin and lying in the x-y plane.

   a) Compute the exact multipole radiation coefficients $a_E(l,m)$ and $a_M(l,m)$.

   In spherical coordinates, the current density takes the form

$$\vec{J} = \frac{I_0}{r}\delta(r-a)\delta(\cos\theta)\hat{\phi}$$

   Note that the factor of $1/r$ guarantees the proper normalization

$$\vec{J}d^3x = \frac{I_0}{r}\delta(r-a)\delta(\cos\theta)\hat{\phi}r^2 dr d\cos\theta d\phi = I_0\hat{\phi}r d\phi = I_0\hat{\phi}dl$$

   where $dl = r d\phi$ is the infinitesimal length along the wire. This problem does not have any intrinsic magnetization. Hence the electric multipole coefficient is

$$a_E(l,m) = \frac{k^2}{i\sqrt{l(l+1)}}\int Y_{lm}^*\left(c\rho\frac{\partial}{\partial r}[rj_l(kr)] + ik(\vec{r}\cdot\vec{J})j_l(kr)\right)d^3x$$

   Note that

$$\rho = \frac{1}{i\omega}\vec{\nabla}\cdot\vec{J} = \frac{1}{i\omega}\frac{1}{r\sin\theta}\frac{dJ_\phi}{d\phi} = 0$$

$$\vec{r}\cdot\vec{J} = rJ_r = 0$$

   Hence all electric multipole coefficients vanish

$$a_E(l,m) = 0$$

   For the magnetic multipole coefficient, we have

$$a_M(l,m) = \frac{k^2}{i\sqrt{l(l+1)}}\int Y_{lm}^*\vec{\nabla}\cdot(\vec{r}\times\vec{J})j_l(kr)d^3x$$

   In this case

$$\vec{r}\times\vec{J} = r\hat{r}\times\vec{J} = -I_0\delta(r-a)\delta(\cos\theta)\hat{\theta}$$

   Hence

$$\begin{aligned}\vec{\nabla}\cdot(\vec{r}\times\vec{J}) &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}[\sin\theta(-I_0 r)\delta(r-a)\delta(\cos\theta)] \\ &= -\frac{I_0}{r\sin\theta}\delta(r-a)\frac{\partial}{\partial\theta}[\sin\theta\delta(\cos\theta)] \\ &= -\frac{I_0}{r\sin\theta}\delta(r-a)[\cos\theta\delta(\cos\theta) + \sin\theta\delta'(\cos\theta)(-\sin\theta)] \\ &= \frac{I_0}{r}\delta(r-a)\delta'(\cos\theta)\end{aligned}$$

Note that we have substituted $\sin\theta = 1$ and $\cos\theta = 0$ to obtain the last line, since the delta functions vanish except when $\theta = \pi/2$. Substituting this into the expression for $a_M(l,m)$ gives

$$a_M(l,m) = \frac{k^2}{i\sqrt{l(l+1)}} \int Y^*_{lm} j_l(kr) \frac{I_0}{r} \delta(r-a)\delta'(\cos\theta) r^2 dr d\cos\theta d\phi$$

The r and ϕ integrals are trivial to perform. The result is

$$\begin{aligned} a_M(l,m) &= \frac{I_0 k^2}{i\sqrt{l(l+1)}} 2\pi a \delta_{m,0} j_l(ka) \int_{-1}^{1} Y^*_{l0}(\theta)\delta'(\cos\theta) d(\cos\theta) \\ &= \frac{I_0 k^2}{i\sqrt{l(l+1)}} 2\pi a \delta_{m,0} j_l(ka) \sqrt{\frac{2l+1}{4\pi}} \int_{-1}^{1} P_l(\cos\theta)\delta'(\cos\theta) d(\cos\theta) \end{aligned}$$

Note that the $\delta'(\cos\theta)$ can be removed by integration by parts. The result is

$$a_M(l,m) = iI_0 k^2 a j_l(ka) \sqrt{\frac{\pi(2l+1)}{l(l+1)}} P'_l(0)\delta_{m,0}$$

b) What is the dominant radiation mode (electric dipole, magnetic dipole, electric quadrupole, etc.) in the limit $ka \ll 1$? Compute the total radiated power for this mode (in the limit $ka \ll 1$).

Since the electric multipole coefficients vanish, the dominant mode is the magnetic dipole. Using $P_1(x) = x$, we see that $P'_1(x) = 1$. Hence

$$a_M(1,0) = iI_0 k^2 a j_1(ka) \sqrt{\frac{3\pi}{2}}$$

In the long wavelength limit ($ka \ll 1$) the spherical Bessel function behaves as $j_1(ka) \approx \frac{1}{3}(ka)$. Hence

$$a_M(1,0) \approx iI_0 k^3 a^2 \sqrt{\frac{\pi}{6}}$$

The total radiated power is then

$$P = \frac{Z_0}{2k^2} \sum_{l,m} [|a_E(l,m)|^2 + |a_M(l,m)|^2] \approx \frac{Z_0}{2k^2} |I_0|^2 k^6 a^2 \frac{\pi}{6} = \frac{Z_0 k^4 a^2 \pi}{12} |I_0|^2$$

3. *a*) Using the Born approximation, compute the unpolarized differential cross section for the scattering of electromagnetic radiation off of a uniform dielectric sphere of relative dielectric constant $\epsilon_r \approx 1$.

In the Born approximation, the differential cross section is

$$\frac{d\sigma}{d\Omega} = |\hat{\epsilon}^* \cdot \vec{f}|^2$$

where

$$\hat{\epsilon}^* \cdot \vec{f} = \frac{k^2}{4\pi} \int d^3x e^{\vec{q}\cdot\vec{x}} \left[(\hat{\epsilon}^* \cdot \hat{\epsilon}_0) \frac{\delta\epsilon}{\epsilon_0} + (\hat{n} \times \hat{\epsilon}^*) \cdot (\hat{n}_0 \times \hat{\epsilon}_0) \frac{\delta\mu}{\mu_0} \right]$$

and where $\vec{q} = \vec{k}_0 - \vec{k}$. We assume the sphere is non-permeable. Then

$$\frac{\delta\epsilon}{\epsilon_0} = \epsilon_r - 1, \qquad \frac{\delta\mu}{\mu_0} = 0$$

In this case

$$\hat{\epsilon}^* \cdot \vec{f} = \frac{k^2}{4\pi} (\epsilon_r - 1)(\hat{\epsilon}^* \cdot \hat{\epsilon}_0) \int_{r<a} d^3x e^{i\vec{q}\cdot\vec{x}}$$

The integral may be performed in spherical coordinates. Lining up the coordinate system with the $\vec{q}$ direction gives $\vec{q} \cdot \vec{x} = qr\cos\theta$. In this case

$$\begin{aligned}\int_{r<a} d^3x e^{i\vec{q}\cdot\vec{x}} &= \int_0^a r^2 dr \int_{-1}^1 d\cos\theta \int_0^{2\pi} d\phi e^{iqr\cos\theta} \\ &= 2\pi \int_0^a r^2 dr \frac{e^{iqr} - e^{-iqr}}{iqr} = \frac{4\pi}{q} \int_0^a dr\, r \sin qr\end{aligned}$$

This can be integrated by parts to give

$$\int d^3x e^{i\vec{q}\cdot\vec{x}} = \frac{4\pi a^3}{(qa)^3} (\sin qa - qa\cos qa)$$

As a result, the cross section becomes

$$\frac{d\sigma}{d\Omega} = k^4 a^6 |\epsilon_r - 1|^2 |\hat{\epsilon}^* \cdot \hat{\epsilon}_0|^2 \left(\frac{\sin qa - qa\cos qa}{(qa)^3} \right)^2$$

Note that

$$q = |\vec{k}_0 - \vec{k}| = k|\hat{n}_0 - \hat{n}| = 2k\sin\frac{\theta}{2}$$

where θ is the scattering angle. Finally, for the unpolarized cross section, we average over $\hat{\epsilon}_0$ and sum over $\hat{\epsilon}$. This gives a factor

$$|\hat{\epsilon}^* \cdot \hat{\epsilon}_0|^2 \qquad \Rightarrow \qquad \tfrac{1}{2}(1 + \cos^2\theta)$$

so that

$$\frac{d\sigma}{d\Omega} = \tfrac{1}{2} k^4 a^6 |\epsilon_r - 1|^2 (1 + \cos^2\theta) \left(\frac{\sin qa - qa\cos qa}{(qa)^3} \right)^2$$

b) Show that in the limit $ka \ll 1$ the differential cross section reduces to the small sphere result

$$\frac{d\sigma}{d\Omega} = \tfrac{1}{18}k^4a^6|1-\epsilon_r|^2(1+\cos^2\theta)$$

For $ka \ll 1$ we expand the trig function

$$\frac{\sin x - x\cos x}{x^3} = \frac{\left(x - \frac{1}{6}x^3 + \cdots\right) - x\left(1 - \frac{1}{2}x^2 + \cdots\right)}{x^3} = \tfrac{1}{3} + \cdots$$

This gives

$$\frac{d\sigma}{d\Omega} = \tfrac{1}{18}k^4a^6|\epsilon_r - 1|^2(1+\cos^2\theta)$$

c) Show that for $ka \gg 1$ the differential cross section is highly peaked in the forward direction.

For $ka \gg 1$ note that $qa = 2ka\sin(\theta/2) \gg 1$ unless $\theta \approx 0$. In this limit, the trig function can be approximated

$$\frac{\sin x - x\cos x}{x^3} \approx -\frac{\cos x}{x^2} \qquad \rightarrow \qquad \text{bounded by } \frac{1}{x^2}$$

This demonstrates that

$$\left(\frac{\sin qa - qa\cos qa}{(qa)^3}\right)^2 \approx \frac{\cos^2 qa}{(qa)^4} \sim \frac{1}{2(qa)^4} = \frac{1}{32(ka)^4}\frac{1}{\sin^4(\theta/2)}$$

where we have averaged $\cos^2(qa)$ to $1/2$. This gives

$$\frac{d\sigma}{d\Omega} \approx \frac{a^2}{64}|\epsilon_r - 1|^2(1+\cos^2\theta)\frac{1}{\sin^4(\theta/2)}$$

unless $|\theta| \lesssim 1/ka$, in which case the trig function becomes of order one. The cross section in this forward region gets considerably larger than away from the forward region. Note that, since $ka \gg 1$, the restriction of $|\theta| \lesssim 1/ka$ gives rise to a highly peaked behavior in the forward direction.