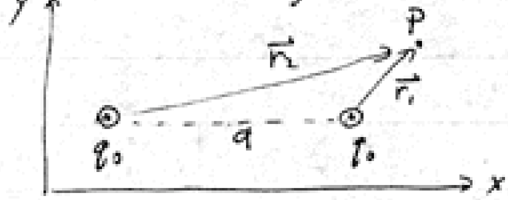
1 a) The magnetic field vanishes in the rest frame $\boxed{\vec{B}'=0}$

the electric field may be obtained by Gauss' law

for a single wire $E' = \dfrac{2q_0}{r}$ $\Rightarrow$ $\boxed{\vec{E}' = \dfrac{2q_0\hat{r}_1}{r_1} + \dfrac{2q_0\hat{r}_2}{r_2}}$



The force on wire 2 is $\vec{F}_{21}' = q_0\,\vec{E}_1' = \boxed{\dfrac{2q_0^2}{a}\,\hat{r}_{21}}$ (repulsive)

(per unit length)

b) Now boost along the $\hat{z}$ direction (parallel to wires)

$$\vec{\beta} = \frac{v}{c}\,\hat{z}$$

$$\Rightarrow \vec{E} = \gamma(\vec{E}' - \vec{\beta}\times\vec{B}') + (1-\gamma)\,\hat{\beta}(\hat{\beta}\cdot\vec{E}') = \gamma\vec{E}'$$

$$\vec{B} = \gamma(\vec{B}' + \vec{\beta}\times\vec{E}') + (1-\gamma)\,\hat{\beta}(\hat{\beta}\cdot\vec{B}') = \gamma\vec{\beta}\times\vec{E}'$$

ie
$$\boxed{\begin{aligned}\vec{E} &= 2\gamma q_0\left(\frac{\hat{r}_1}{r_1} + \frac{\hat{r}_2}{r_2}\right)\\ \vec{B} &= 2\gamma\frac{v}{c}q_0\,\hat{z}\times\left(\frac{\hat{r}_1}{r_1} + \frac{\hat{r}_2}{r_2}\right)\end{aligned}}$$

c) Note that the above fields correspond to a charge per unit length

$q = \gamma q_0$ and a current $I = qv = \gamma q_0 v$

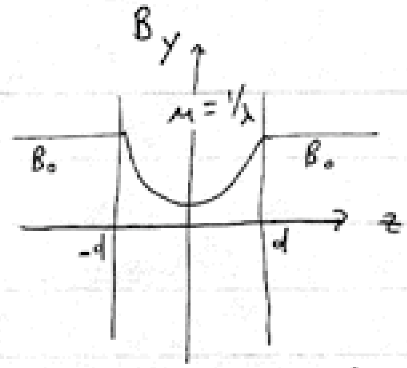
(this can also be obtained by Lorentz boost)

Then the force per length on wire 2 is

$$\begin{aligned}\vec{F}_{21} &= q\,\vec{E}_1 + \frac{1}{c}I\,\hat{z}\times\vec{B}_1\\ &= 2\gamma^2 q_0^2\,\frac{\hat{r}_{21}}{a} + 2\gamma^2\left(\frac{v}{c}\right)^2 q_0^2\,\hat{z}\times\left(\hat{z}\times\frac{\hat{r}_{21}}{a}\right)\\ &= 2\gamma^2 q_0^2\left(1-\frac{v^2}{c^2}\right)\frac{\hat{r}_{21}}{a}\\ &= \boxed{\frac{2q_0^2}{a}\,\hat{r}_{21}}\end{aligned}$$

This is the same as the force computed in part (a)

2a)



Since $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{A}$ satisfies the Proca equation

$(\Box + \mu^2)\vec{A} = 0$ inside the superconductor

we must also have $(\Box + \mu^2)\vec{B} = 0$

Because of uniformity, we take $\vec{B} = B(z)\hat{y}$

$$\Rightarrow \quad -B'' + \mu^2 B = 0$$

this is solved by $B = A e^{\mu z} + C e^{-\mu z}$ constants A & C

Matching at $z = \pm d$: $B_0 = A e^{\mu d} + C e^{-\mu d} = A e^{-\mu d} + C e^{\mu d}$

$$\Rightarrow \quad A = C = B_0 / (e^{\mu d} + e^{-\mu d})$$

This gives
$$\boxed{\vec{B} = \begin{cases} B_0 \hat{y} & |z| > d \\ B_0 \dfrac{\cosh z/\lambda}{\cosh d/\lambda}\,\hat{y} & |z| < d \end{cases}}$$

b) Compute $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J}$ (static case)

or $\vec{J} = \frac{c}{4\pi}\,\vec{\nabla} \times \left(B_0 \frac{\cosh z/\lambda}{\cosh d/\lambda}\,\hat{y} \right)$

$$= \boxed{-\frac{c}{4\pi}\,\frac{B_0}{\lambda}\,\frac{\sinh z/\lambda}{\cosh d/\lambda}\,\hat{x}}$$

the current is concentrated near the surface of the superconductor

3 Take two charges:

1) $-e$ at $\vec r_1 = \vec r(t')$

2) e at $\vec r_2 = \vec r(t') + \vec d(t')$

Then $R_1 = R$, $R_2 = |\vec x - \vec r - \vec d| = |\vec R - \vec d| \simeq R - \hat n\cdot\vec d = R(1 - \hat n\cdot\vec d/R)$

$\hat n_1 = \hat n$, $\hat n_2 = \frac{1}{R_2}(\vec x - \vec r - \vec d) = \frac{1}{R_2}(\vec R - \vec d) \simeq \frac{1}{R}(1 + \hat n\cdot\vec d/R)(\vec R - \vec d)$

$\simeq \frac{1}{R}(\vec R - \vec d + \hat n(\hat n\cdot\vec d)) = \hat n - \frac{1}{R}(\vec d - \hat n(\hat n\cdot\vec d))$

$\vec\beta_1 = \vec\beta$, $\vec\beta_2 = \vec\beta + \frac{1}{c}\dot{\vec d}$

This gives us enough to evaluate Φ and $\vec A$

$$\Phi = \frac{-e}{R_1(1-\vec\beta_1\cdot\hat n_1)} + \frac{e}{R_2(1-\vec\beta_2\cdot\hat n_2)}$$

$$= \frac{-e}{R(1-\vec\beta\cdot\hat n)} + \frac{e}{R(1-\frac{1}{R}\hat n\cdot\vec d)(1-(\vec\beta+\frac{1}{c}\dot{\vec d})\cdot(\hat n - \frac{1}{R}(\vec d - \hat n(\hat n\cdot\vec d))))}$$

$$= \frac{-e}{R(1-\vec\beta\cdot\hat n)} + \frac{e}{R(1-\frac{1}{R}\hat n\cdot\vec d)(1-\vec\beta\cdot\hat n - \frac{1}{c}\hat n\cdot\dot{\vec d} + \frac{1}{R}\vec\beta\cdot(\vec d - \hat n(\hat n\cdot\vec d)))}$$

$$= \frac{-e}{R(1-\vec\beta\cdot\hat n)} + \frac{e}{R(1-\vec\beta\cdot\hat n - \frac{1}{c}\hat n\cdot\dot{\vec d} - \frac{1}{R}(\hat n - \vec\beta)\cdot\vec d)}$$

$$= \frac{-e}{R(1-\vec\beta\cdot\hat n)} + \frac{e}{R(1-\vec\beta\cdot\hat n)}\left(1 + \frac{\frac{1}{c}\hat n\cdot\dot{\vec d} + \frac{1}{R}(\hat n - \vec\beta)\cdot\vec d}{1-\vec\beta\cdot\hat n}\right)$$

$$\boxed{\Phi = \left[\frac{(\hat n - \vec\beta)\cdot\vec p}{R^2(1-\vec\beta\cdot\hat n)^2}\right]_{ret} + \frac{1}{c}\left[\frac{\hat n\cdot\dot{\vec p}}{R(1-\vec\beta\cdot\hat n)^2}\right]_{ret}}$$

we have restored the ret notation

For $\vec A$ we have an extra factor of $\vec\beta_1$ or $\vec\beta_2$ in the numerator

$$\vec A = \frac{-e\vec\beta}{R(1-\vec\beta\cdot\hat n)} + \frac{e}{R(1-\vec\beta\cdot\hat n)}(\vec\beta + \frac{1}{c}\dot{\vec d})\left(1 + \frac{\frac{1}{c}\hat n\cdot\dot{\vec d} + \frac{1}{R}(\hat n - \vec\beta)\cdot\vec d}{1-\vec\beta\cdot\hat n}\right)$$

$$= \frac{-e\vec\beta}{R(1-\vec\beta\cdot\hat n)} + \frac{e}{R(1-\vec\beta\cdot\hat n)}\left(\vec\beta + \frac{\frac{1}{R}((\hat n - \vec\beta)\cdot\vec d)\vec\beta + \frac{1}{c}(\hat n\cdot\dot{\vec d})\vec\beta + \dot{\vec d}(1-\vec\beta\cdot\hat n)}{1-\vec\beta\cdot\hat n}\right)$$

$$\boxed{\vec A = \left[\frac{((\hat n - \vec\beta)\cdot\vec p)\vec\beta}{R^2(1-\vec\beta\cdot\hat n)^2}\right]_{ret} + \frac{1}{c}\left[\frac{\dot{\vec p}}{R(1-\vec\beta\cdot\hat n)} + \frac{(\hat n\cdot\dot{\vec p})\vec\beta}{R(1-\vec\beta\cdot\hat n)^2}\right]_{ret}}$$

4a) For a single <u>non-relativistic</u> charge, the acceleration field is

$$\vec{E} = \frac{q}{c^2} \frac{\hat{n} \times (\hat{n} \times \dot{\vec{v}})}{R}$$

Working in the radiation zone ($R \to \infty$), we may ignore the different locations of the $+q$ and $-q$ charges and use linear superposition to write

$$\frac{dP(t)}{d\Omega} = \frac{c}{4\pi} \left| R \sum_i \vec{E}_i \right|^2 = \frac{1}{4\pi c^3} \left| \hat{n} \times \hat{n} \times \left(\sum_i q_i \dot{\vec{v}}_i\right) \right|^2$$

$$= \frac{1}{4\pi c^3} \left| \hat{n} \times \sum_i q_i \dot{\vec{v}}_i \right|^2$$

Take the charges to be located at

$+q \qquad \vec{r}_1 = \frac{d}{2}(\hat{x}\cos\omega t + \hat{y}\sin\omega t) \qquad \dot{\vec{v}}_1 = \frac{-d\omega^2}{2}(\hat{x}\cos\omega t + \hat{y}\sin\omega t)$

$-q \qquad \vec{r}_2 = \frac{-d}{2}(\hat{x}\cos\omega t + \hat{y}\sin\omega t) \qquad \dot{\vec{v}}_2 = \frac{d\omega^2}{2}(\hat{x}\cos\omega t + \hat{y}\sin\omega t)$

this gives

$$q_1\dot{\vec{v}}_1 + q_2\dot{\vec{v}}_2 = -q d\omega^2 (\hat{x}\cos\omega t + \hat{y}\sin\omega t)$$

using spherical coordinates, we take the normal vector to be

$$\hat{n} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta \qquad \left(+\text{ higher order in } \frac{d}{R}\right)$$

This yields

$$\frac{dP(t)}{d\Omega} = \frac{q^2 d^2 \omega^4}{4\pi c^3} \left| (\hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta) \times (\hat{x}\cos\omega t + \hat{y}\sin\omega t) \right|^2$$

$$= \frac{q^2 d^2 \omega^4}{4\pi c^3} \left| -\hat{x}\cos\theta\sin\omega t + \hat{y}\cos\theta\cos\omega t + \hat{z}\sin\theta(\cos\phi\sin\omega t - \sin\phi\cos\omega t) \right|^2$$

$$\boxed{\frac{dP(t)}{d\Omega} = \frac{q^2 d^2 \omega^4}{4\pi c^3} \left(\cos^2\theta + \sin^2\theta\sin^2(\phi - \omega t)\right)}$$

b) Integrate over the solid angle

$$P(t) = \frac{q^2 d^2 \omega^4}{4\pi c^3} \int \left[\cos^2\theta + \sin^2\theta\sin^2(\phi - \omega t)\right] d\phi \, d\cos\theta$$

$$= \frac{q^2 d^2 \omega^4}{2 c^3} \int_{-1}^{1} \left[\cos^2\theta + \tfrac{1}{2}\sin^2\theta\right] d\cos\theta$$

$$= \frac{q^2 d^2 \omega^4}{4 c^3} \int_{-1}^{1} \left[1 + \cos^2\theta\right] d\cos\theta$$

$$\boxed{P(t) = \frac{2 q^2 d^2 \omega^4}{3 c^3}} \qquad \text{independent of } t$$