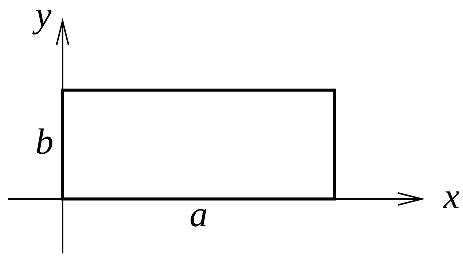
Physics 506 **Winter 2008**

Midterm — Solutions

This midterm is a two hour open book, open notes exam. Do all three problems.

[35 pts] 1. Consider the propagation of waves in a rectangular waveguide with sides of lengths a and b.



[15] *a*) Show that (for $m > 0$ and $n > 0$) a TE_{mn} and TM_{mn} mode can be superposed to make the x component of the magnetic field vanish, $H_x = 0$.

For a right-moving TE_{mn} mode defined by

$$\psi^{\text{TE}} = H_0 \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \tag{1}$$

the x component of the magnetic field is given by

$$\vec{H}_t = \frac{ik}{\gamma^2}\vec{\nabla}_t\psi^{\text{TE}} \qquad \Rightarrow \qquad H_x^{\text{TE}} = -\frac{ik}{\gamma^2}H_0\frac{m\pi}{a}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \tag{2}$$

Likewise, for a right-moving TM_{mn} mode defined by

$$\psi^{\text{TM}} = E_0 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{3}$$

the x component of the magnetic field is

$$\vec{H}_t = \frac{i\epsilon\omega}{\gamma^2}\hat{z}\times\vec{\nabla}_t\psi^{\text{TM}} \qquad \Rightarrow \qquad H_x^{\text{TM}} = -\frac{i\epsilon\omega}{\gamma^2}E_x\frac{n\pi}{b}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

We note that in both cases, the modes have identical eigenvalues

$$\gamma^2 = \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right] \tag{4}$$

and identical functional behavior for H_x. This allows us to superpose and cancel the x component of the magnetic field

$$0 = H_x = H_x^{\text{TE}} + H_x^{\text{TM}} = -\frac{i}{\gamma^2}\left(kH_0\frac{m\pi}{a} + \epsilon\omega E_0\frac{n\pi}{b}\right)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

The solution is

$$E_0 = -\frac{k}{\epsilon\omega}\frac{m/a}{n/b}H_0$$

Of course, even though $H_x = 0$, the other components of $\vec{E}$ and $\vec{H}$ do not necessarily vanish. Furthermore, the two modes that are superposed must be moving in the same direction in order to to have $H_x = 0$ everywhere along the z direction.

[15] *b*) Compute the transmitted power in this superposition mode. (Recall that TE and TM modes are orthogonal in the sense that $\frac{1}{2}\int \hat{z}\cdot(\vec{E}_{\text{TE}}\times\vec{H}^*_{\text{TM}})\,da = 0$ and similarly with TE and TM interchanged.)

The power is obtained from the Poynting vector

$$P = \int_A \hat{z}\cdot\vec{S}\,da = \tfrac{1}{2}\int_A \hat{z}\cdot(\vec{E}\times\vec{H}^*)da = \tfrac{1}{2}\int_A \hat{z}\cdot[(\vec{E}^{\text{TE}}+\vec{E}^{\text{TM}})\times(\vec{H}^{\text{TE}}+\vec{H}^{\text{TM}})^*]da$$

However, since the modes are orthogonal, the power simply decomposes into an incoherent sum $P = P^{\text{TE}} + P^{\text{TM}}$ where

$$\begin{aligned} P^{\text{TE}} &= \tfrac{1}{2}\int_A \hat{z}\cdot(\vec{E}^{\text{TE}}\times\vec{H}^{\text{TE}\,*})\,da \\ P^{\text{TM}} &= \tfrac{1}{2}\int_A \hat{z}\cdot(\vec{E}^{\text{TM}}\times\vec{H}^{\text{TM}\,*})\,da \end{aligned}$$

We use the standard expressions for transmitted power in a waveguide

$$\begin{aligned} P^{\text{TE}} &= \frac{\mu\omega k}{2\gamma^2}\int_A |\psi^{\text{TE}}|^2 da = \frac{ab}{8}\frac{\mu\omega k}{\gamma^2}|H_0|^2 \\ P^{\text{TM}} &= \frac{\epsilon\omega k}{2\gamma^2}\int_A |\psi^{\text{TM}}|^2 da = \frac{ab}{8}\frac{\epsilon\omega k}{\gamma^2}|E_0|^2 \end{aligned}$$

where ψ^{TE} and ψ^{TM} are given in (1) and (3), respectively. Hence

$$P = \frac{ab}{8}\frac{\omega k}{\gamma^2}(\epsilon|E_0|^2 + \mu|H_0|^2) = \frac{ab}{8}\frac{\mu\omega k}{\gamma^2}\left[1 + \frac{k^2}{\mu\epsilon\omega^2}\left(\frac{m/a}{n/b}\right)^2\right]|H_0|^2$$

Finally, noting that $k^2 = \mu\epsilon\omega^2 - \gamma^2$ as well as the definition of γ^2 in (4), the power expression may be rewritten as

$$P = \frac{ab}{8}\frac{\mu\omega k}{(n\pi/b)^2}\left[1 - \frac{(m\pi/a)^2}{\mu\epsilon\omega^2}\right]|H_0|^2$$

This may alternatively be written in terms of E_0 as

$$P = \frac{ab}{8}\frac{\epsilon\omega k}{(m\pi/a)^2}\left[1 + \frac{(n\pi/b)^2}{k^2}\right]|E_0|^2$$

[5] *c*) Can we still have $H_x = 0$ when $m = 0$ or $n = 0$?

If either $m = 0$ or $n = 0$, then the TM mode does not exist. In this case we cannot superpose the two modes. However, the TE mode itself may have a vanishing H_x. For H_x^{TE} given in (2), this may occur when $m = 0$. In particular, the TE_{0n} mode has $H_x = 0$, while TE_{m0} cannot have $H_x = 0$. So it is possible to have $H_x = 0$ when $m = 0$ but not when $n = 0$.

[35 pts] 2. A non-conducting sphere ($\epsilon = \epsilon_0$ and $\mu = \mu_0$) of radius a carries a uniform charge density ρ_0 throughout its volume. The sphere is centered at the origin of the coordinate system and oscillates back and forth about the z-axis with an angular velocity $\vec{\omega} = \hat{z}\,\phi_0\omega\cos\omega t$.

[5] *a*) Show that the current density may be expressed as

$$\vec{J} = \Re[\hat{\phi}\,\rho_0\phi_0\omega r\sin\theta e^{-i\omega t}]$$

The velocity of a point $\vec{r}$ within the sphere is given by

$$\vec{v} = \vec{\omega}\times\vec{r} = \hat{z}\times\vec{r}\phi_0\omega\cos\omega t = \hat{\phi}\phi_0\omega r\sin\theta\cos\omega t$$

As a result, the current density is

$$\vec{J} = \rho_0\vec{v} = \hat{\phi}\rho_0\phi_0\omega r\sin\theta\cos\omega t = \Re[\hat{\phi}\rho_0\phi_0\omega r\sin\theta e^{-i\omega t}]$$

This current is only non-vanishing for $r < a$. For this time-harmonic current density, we may define the complex current as

$$\vec{J}(\vec{x}\,) = \hat{\phi}\rho_0\phi_0\omega r\sin\theta \tag{5}$$

[20] *b*) Compute the multipole radiation coefficients $a_E(l,m)$ and $a_M(l,m)$. Note that the integral $\int x^{l+2}j_l(x)\,dx = x^{l+2}j_{l+1}(x)$ may be helpful.

We start with the electric multipoles $a_E(l,m)$. Note from (5) that both $\rho = (1/i\omega)\vec{\nabla}\cdot\vec{J} = 0$ and $\vec{r}\cdot\vec{J} = 0$. Since

$$a_E(l,m) = \frac{k^2}{i\sqrt{l(l+1)}}\int Y_{lm}^*\left[c\rho\frac{\partial}{\partial r}[rj_l(kr)] + ik(\vec{r}\cdot\vec{J})j_l(kr)\right]d^3x$$

we immediately conclude that all electric multipoles vanish

$$a_E(l,m) = 0$$

For the magnetic multiples $a_M(l,m)$, we first compute

$$\vec{r}\times\vec{J} = \vec{r}\times\hat{\phi}\rho_0\phi_0\omega r\sin\theta = -\hat{\theta}\rho_0\phi_0\omega r^2\sin\theta$$

and

$$\vec{\nabla}\cdot(\vec{r}\times\vec{J}) = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta(\vec{r}\times\vec{J})_\theta = -\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\rho_0\phi_0\omega r^2\sin^2\theta = -2\rho_0\phi_0\omega r\cos\theta$$

This allows us to compute

$$\begin{aligned}
a_M(l,m) &= \frac{k^2}{i\sqrt{l(l+1)}}\int Y_{lm}^*[\vec{\nabla}\cdot(\vec{r}\times\vec{J})j_l(kr)]d^3x \\
&= \frac{k^2}{i\sqrt{l(l+1)}}(-2\rho_0\phi_0\omega)\int Y_{lm}^* r\cos\theta j_l(kr)r^2\,drd\Omega \\
&= \frac{2ik^2\rho_0\phi_0\omega}{\sqrt{l(l+1)}}\sqrt{\frac{4\pi}{3}}\int Y_{lm}^*Y_{10}r^3 j_l(kr)drd\Omega \\
&= \frac{2ik^2\rho_0\phi_0\omega}{\sqrt{l(l+1)}}\sqrt{\frac{4\pi}{3}}\delta_{l,1}\delta_{m,0}\int_0^a r^3 j_l(kr)dr \\
&= i\sqrt{\frac{8\pi}{3}}k^2\rho_0\phi_0\omega\delta_{l,1}\delta_{m,0}\int_0^a r^3 j_1(kr)dr
\end{aligned}$$

where we have used the fact that $\cos\theta = P_1(\cos\theta) = \sqrt{3/4\pi}Y_{10}(\theta,\phi)$ as well as the orthonormality of the spherical harmonics. The radial integral may be performed according to

$$\int_0^a r^3 j_1(kr)dr = \frac{1}{k^4}\int_0^{ka} x^3 j_1(x)dx = \left.\frac{x^3 j_2(x)}{k^4}\right|_0^{ka} = \frac{(ka)^3 j_2(ka)}{k^4}$$

Hence the only non-vanishing magnetic multipole is

$$a_M(1,0) = i\sqrt{\frac{8\pi}{3}}\rho_0\phi_0\omega k a^3 j_2(ka)$$

[10] *c*) Find the time-averaged power radiated per unit solid angle $dP/d\Omega$.

Since the radiation is purely magnetic dipole, the angular power distribution is given by

$$\begin{aligned}
\frac{dP}{d\Omega} &= \frac{Z_0}{2k^2}|a_M(1,0)|^2|X_{10}|^2 \\
&= \frac{Z_0}{2k^2}\frac{8\pi}{3}\rho_0^2\phi_0^2\omega^2k^2a^6j_2^2(ka)\left[\frac{3}{8\pi}\sin^2\theta\right] \\
&= \frac{Z_0}{2}\rho_0^2\phi_0^2\omega^2a^6j_2^2(ka)\sin^2\theta
\end{aligned}$$

[30 pts] 3. Three *small* uniform and non-permeable ($\mu = \mu_0$) dielectric spheres of radii b and relative dielectric constant $\epsilon_r \approx 1$ are located along the z axis, centered at positions $z = -a$, $z = 0$ and $z = a$.

[15] *a*) Using the Born approximation, compute the unpolarized differential scattering cross section $d\sigma/d\Omega$. Assume $kb \ll 1$, but allow a to be arbitrarily small or large.

The Born approximation is given by

$$\begin{aligned}\vec{\epsilon}^* \cdot \vec{f} &= \frac{k^2}{4\pi}\int d^3x e^{i\vec{q}\cdot\vec{x}}\left[\vec{\epsilon}^* \cdot \vec{\epsilon}_0 \frac{\delta\epsilon}{\epsilon_0} + (\hat{n}\times\vec{\epsilon}^*)\cdot(\hat{n}\times\vec{\epsilon}^*)\frac{\delta\mu}{\mu_0}\right] \\ &= \frac{k^2}{4\pi}\int d^3x e^{i\vec{q}\cdot\vec{x}}\vec{\epsilon}^* \cdot \vec{\epsilon}_0 \frac{\delta\epsilon}{\epsilon_0}\end{aligned}$$

where $\vec{q} = \vec{k}_0 - \vec{k}$ and where we have used the fact that the dielectric is non-permeable. Since $\delta\epsilon$ is only non-vanishing in the interior of the dielectric spheres, the volume integral may be restricted to the three small spheres located at $\vec{x} \approx 0$, $\vec{x} \approx a\hat{z}$ and $\vec{x} \approx -a\hat{z}$. Since each sphere has a volume of $(4/3)\pi b^3$, we obtain

$$\begin{aligned}\vec{\epsilon}^* \cdot \vec{f} &\approx \frac{k^2}{4\pi}\left(\frac{4}{3}\pi b^3\right)(\epsilon_r - 1)(\vec{\epsilon}^* \cdot \vec{\epsilon}_0)\left(1 + e^{iaq_z} + e^{-iaq_z}\right) \\ &= \tfrac{1}{3}k^2 b^3(\epsilon_r - 1)(\vec{\epsilon}^* \cdot \vec{\epsilon}_0)(1 + 2\cos(aq_z))\end{aligned}$$

where

$$q_z = k_{0\,z} - k_z = k(\cos\theta_0 - \cos\theta)$$

Here θ_0 is the polar angle of the incident wave, and θ is the polar angle of the scattered wave. The differential scattering cross section is then

$$\frac{d\sigma}{d\Omega} = |\vec{\epsilon}^* \cdot \vec{f}|^2 \approx k^4 b^6\left(\frac{\epsilon_r - 1}{3}\right)^2 |\vec{\epsilon}^* \cdot \vec{\epsilon}_0|^2\left[1 + 2\cos(ka(\cos\theta_0 - \cos\theta))\right]^2$$

For unpolarized scattering, the polarization average is given by

$$|\vec{\epsilon}^* \cdot \vec{\epsilon}| \qquad \longrightarrow \qquad \tfrac{1}{2}(1 + \cos^2\gamma)$$

where γ is the angle between $\vec{k}_0$ and $\vec{k}$ (ie the incident and scattered wave). Hence

$$\frac{d\sigma}{d\Omega} \approx k^4 b^6\left(\frac{\epsilon_r - 1}{3}\right)^2 \frac{1 + \cos^2\gamma}{2}\left[1 + 2\cos(ka(\cos\theta_0 - \cos\theta))\right]^2 \qquad (6)$$

where $\cos\gamma$ is given in spherical coordinates by

$$\cos\gamma = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi - \phi_0)$$

[5] *b*) Show that, for $ka \ll 1$, the cross section is nine times as large as that for a single sphere.

For $ka \ll 1$, we may approximate

$$1 + 2\cos(ka(\cos\theta_0 - \cos\theta)) \approx 3$$

Substituting this into (6) gives

$$\frac{d\sigma}{d\Omega} \approx 9k^4b^6\left(\frac{\epsilon_r - 1}{3}\right)^2 \frac{1+\cos^2\gamma}{2}$$

which is nine times as large as the single sphere result

$$\frac{d\sigma(\text{one sphere})}{d\Omega} = k^4b^6\left(\frac{\epsilon_r - 1}{3}\right)^2 \frac{1+\cos^2\gamma}{2}$$

Note that for N small spheres, the long wavelength scattering cross section would be N^2 as large. At long wavelengths, the scattering is coherent, and the amplitude scales as total dielectric volume (so the cross section scales as the square of the volume).

[10] *c*) Now suppose the incident plane wave is traveling in the $+x$ direction. For what values of a does the scattered power vanish along the z axis? Give your answer in terms of the wavelength $\lambda = 2\pi/k$.

If the incident wave travels along the $+x$, this corresponds to taking $\theta_0 = \pi/2$ and $\phi_0 = 0$. Inserting this into (6) gives

$$\frac{d\sigma}{d\Omega} \approx k^4b^6\left(\frac{\epsilon_r - 1}{3}\right)^2 \frac{1+\sin^2\theta\cos^2\phi}{2}[1+2\cos(ka\cos\theta)]^2$$

In order to examine the scattered power along the z axis, we take $\theta = 0$ or π. In either case, the result is

$$\frac{d\sigma}{d\Omega} \approx k^4b^6\left(\frac{\epsilon_r - 1}{3}\right)^2 \tfrac{1}{2}[1+2\cos ka]^2$$

This vanishes when

$$1+2\cos ka = 0 \qquad \Rightarrow \qquad \cos ka = -\frac{1}{2} \qquad \Rightarrow \qquad ka = \pm\frac{2\pi}{3} + 2n\pi$$

Using $k = 2\pi/\lambda$ gives

$$a = (n \pm \tfrac{1}{3})\lambda$$

This result can also be obtained by simply demanding destructive interference of the scattered waves from the centers of the three (small) spheres.