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Physics 506: Solutions to Assignment #11

Problem 14.14
(a)

$$\vec{r} = a\cos(\omega_0 t)\hat{z}, \quad \vec{\beta} = -\frac{a\omega_0}{c}\sin(\omega_0 t)\hat{z}$$

$$\hat{n}\cdot\vec{r} = a\cos\theta\cos(\omega_0 t), \quad \hat{n}\times(\hat{n}\times\vec{\beta}) = -\frac{a\omega_0}{c}\sin(\omega_0 t)\hat{n}\times(\hat{n}\times\hat{z})$$

Plugging into the result of Prob. (14.13):

$$A_m(\omega) = -\frac{a\omega_0}{c}\hat{n}\times(\hat{n}\times\hat{z})\frac{\omega_0}{2\pi}\int_0^{\frac{2\pi}{\omega_0}} dt\sin(\omega_0 t)e^{im\omega_0 t}e^{-i\frac{\omega a}{c}\cos\theta\cos(\omega_0 t)}$$

$$= -\beta_0\hat{n}\times(\hat{n}\times\hat{z})\int_0^{2\pi}\frac{d\phi}{2\pi}e^{-i\beta_0\cos\theta\cos\phi}\frac{1}{2i}\left\{e^{i(m+1)\phi} - e^{i(m-1)\phi}\right\}$$

where $\beta_0 = a\omega_0/c$.

$$A_m(m\omega_0) = i\frac{\beta_0}{2}\hat{n}\times(\hat{n}\times\hat{z})\int_0^{2\pi}\frac{d\phi}{2\pi}\left\{e^{i(-m\beta_0\cos\theta\cos\phi+(m+1)\phi)} - e^{i(-m\beta_0\cos\theta\cos\phi+(m-1)\phi)}\right\}$$

Let $\phi' = \pi - \phi$ and using the identity

$$J_m(x) = \frac{1}{i^m}\int_0^{2\pi}\frac{d\phi}{2\pi}e^{i(x\cos\phi - m\phi)}$$

we have

$$\int_0^{2\pi}\frac{d\phi}{2\pi}e^{i(-m\beta_0\cos\theta\cos\phi+(m+1)\phi)} = (-1)^{m+1}\int_0^{2\pi}\frac{d\phi'}{2\pi}e^{i(m\beta_0\cos\theta\cos\phi'-(m+1)\phi')} = (-i)^{m+1}J_{m+1}(m\beta_0\cos\theta)$$

$$\int_0^{2\pi}\frac{d\phi}{2\pi}e^{i(-m\beta_0\cos\theta\cos\phi+(m-1)\phi)} = (-1)^{m-1}\int_0^{2\pi}\frac{d\phi'}{2\pi}e^{i(m\beta_0\cos\theta\cos\phi'-(m-1)\phi')} = (-i)^{m-1}J_{m-1}(m\beta_0\cos\theta)$$

Consequently, we get

$$A_m(m\omega_0) = i\frac{\beta_0}{2}\hat{n}\times(\hat{n}\times\hat{z})\left\{(-1)^{m+1}J_{m+1}(m\beta_0\cos\theta) - (-i)^{m-1}J_{m-1}(m\beta_0\cos\theta)\right\}$$

$$= (-i)^m\frac{\beta_0}{2}\hat{n}\times(\hat{n}\times\hat{z})\left\{J_{m+1}(m\beta_0\cos\theta) + J_{m-1}(m\beta_0\cos\theta)\right\} = (-i)^m\frac{\hat{n}\times(\hat{n}\times\hat{z})}{\cos\theta}J_m(m\beta_0\cos\theta)$$

Here we have used a Bessel identity:

$$J_{m+1}(x) + J_{m-1}(x) = \frac{2m}{x}J_m(x)$$

Then

$$|A_m(m\omega_0)|^2 = \frac{|\hat{n}\times(\hat{n}\times\hat{z})|^2}{\cos^2\theta}J_m^2(m\beta_0\cos\theta) = \frac{\sin^2\theta}{\cos^2\theta}J_m^2(m\beta_0\cos\theta)$$

Therefore, the average power for the mth harmonics:

$$\langle\frac{dP}{d\Omega}\rangle_{m^{th}} = \frac{m^2e^2\omega_0^2}{2\pi c}\tan^2\theta J_m^2(m\beta_0\cos\theta) = \frac{e^2c\beta_0^2}{2\pi a^2}m^2\tan^2\theta J_m^2(m\beta_0\cos\theta)$$

Here we used $\beta_0 = \omega_0 a/c$ instead of β to avoid confusions.
(b) The total power radiated in the m^{th} harmonic is

$$P_m = \int\langle\frac{dP}{d\Omega}\rangle d\Omega = \frac{e^2\beta_0^2 c}{2\pi a^2}m^2(2\pi)\int_{-1}^{+1} d(\cos\theta)\tan^2\theta J_m^2(m\beta_0\cos\theta)$$

In the non-relativistic limit, $\beta_0 \ll 1$, the contribution from large m will be negligible since

$$J_m^2(m\beta_0\cos\theta) \approx \frac{1}{(m!)^2}(\frac{m\beta_0\cos\theta}{2})^{2m}$$

The radiation is dominated by $m = 1$ harmonic:

$$P_1 = \frac{e^2\beta_0^2 c}{a^2}\int_{-1}^{+1} d(\cos\theta)\tan^2\theta\frac{1}{4}\beta_0^2\cos^2\theta = \frac{e^2\beta_0^4 c}{3a^2} = \frac{e^2a^2\omega_0^4}{3c^3} \equiv \frac{2}{3}\frac{e^2}{c^3}\omega_0^4\overline{a^2}$$

where

$$\overline{a^2} = \langle a^2\cos^2(\omega_0 t)\rangle = \frac{1}{2}a^2$$

Problem 14.26
(a) The radius of the orbit can be calculated using the numerical form Eq. (12.42):

$$\rho = \frac{p\ (\text{MeV/c})}{3.0\times 10^{-4}\ B(\text{gauss})} = \frac{10^{13}\cdot 10^{-6}}{3.0\times 10^{-4}\cdot 3\times 10^{-4}} = 1.1\times 10^{14}\ \text{cm}$$

The natural frequency of the motion

$$\omega_0 = \frac{c}{\rho} = 2.7\times 10^{-4}\ \text{s}^{-1}$$

and the critical frequency

$$\omega_c = \frac{3}{2}\omega_0\gamma^3 = \frac{3}{2}\omega_0(\frac{E}{mc^2})^3 \approx 3\cdot 10^{18}\ \text{s}^{-1}$$

$$\hbar\omega_c \approx 6.6\times 10^{-22}\ \text{MeV s}\cdot 3\times 10^{18}\ \text{s}^{-1} = 2\ \text{keV}$$

(b) The average observable power is given by

$$P(\omega, E) = \frac{1}{T}\frac{dI}{d\omega} = \frac{\omega_0}{2\pi}\frac{dI}{d\omega}$$

At low frequencies ($\omega \ll \omega_c$), $dI/d\omega$ is given by Eq. (14.89). Therefore the average power spectrum has the form

$$P(\omega, E) \sim \omega_0\frac{e^2}{c}(\frac{\omega}{\omega_0})^{1/3} \sim (\omega\omega_0^2)^{1/3}$$

At high frequencies ($\omega \gg \omega_c$), $dI/d\omega$ is given by Eq. (14.90). Thus

$$P(\omega, E) \sim \gamma\omega_0(\frac{\omega}{\omega_c})^{1/2}e^{-\omega/\omega_c} \sim \frac{1}{\gamma^2}(\omega\omega_c)^{1/2}e^{-\omega/\omega_c}$$

Note that

$$\omega_0 = \frac{c}{\rho} = \frac{ceB}{pc} \sim \frac{1}{E}, \quad \gamma = \frac{E}{mc^2} \sim E, \quad \omega_c = \frac{3}{2}\gamma^3\omega_0 \sim E^2$$

The average power

$$P(\omega, E) \sim (\frac{\omega}{E^2})^{1/3} \text{ for } \omega \ll \omega_c \qquad \text{and} \qquad P(\omega, E) \sim (\frac{\omega}{E^2})^{1/2} e^{-\omega/\omega_c} \text{ for } \omega \gg \omega_c$$

It can be written in the form

$$P(\omega, E) = \text{const } (\frac{\omega}{E^2})^{1/3} f(\frac{\omega}{\omega_c})$$

where $f(x) = 1$ for $x \ll 1$ and $f(x) \sim x^{1/6} e^{-x}$ for $x \gg 1$.
(c) Now with

$$N(E)dE \sim E^{-n} dE$$

we have

$$P(\omega) = \frac{\int_0^\infty P(E,\omega) N(E) dE}{\int_0^\infty N(E) dE} = \text{const} \int P(E,\omega) N(E) dE = \text{const} \int \frac{\omega^{1/3}}{E^{2/3}} f(\frac{\omega}{\omega_c}) E^{-n} dE$$

Approximate f as a step function such that $f(x) = 1$ for $x < 1$ and $f(x) = 0$ for $x > 1$. Given that $\omega_c \sim E^2 = \delta E^2$ (here δ is a constant, see below), the non-vanishing contribution to the integral is therefore from $E > \sqrt{\omega/\delta}$.

$$P(\omega) = \text{const } \omega^{1/3} \int_{\sqrt{\omega/\delta}}^\infty E^{-(n+2/3)} dE = \text{const } \omega^{1/3} \left\{ \sqrt{\frac{\omega}{\delta}} \right\}^{-(n-1/3)}$$

$$= \text{const } \frac{\omega^{1/3}}{\omega^{n/2-1/6}} = \text{const } \omega^{-(n-1)/2} = \text{const } \omega^{-\alpha}$$

where $\alpha = (n-1)/2$.
(d) The critical frequency ω_c of the radiation is related to electron energies through

$$\omega_c = \frac{3}{2}\gamma^3 \omega_0 = \frac{3}{2}\gamma^3 \frac{c}{\rho} = \frac{3}{2}\gamma^3 \frac{ceB}{cp} \approx \frac{3}{2}\gamma^3 \frac{ceB}{E} = \frac{3}{2}\gamma^2 \frac{eB}{mc} = \frac{3}{2}\frac{eB}{m^3c^5} E^2 (= \delta E^2)$$

Thus

$$E = \gamma mc^2 = mc^2 \sqrt{\frac{2}{3}\omega_c \frac{mc}{e}\frac{1}{B}}$$

Taking the cutoff frequency 10^{18} Hz as the critical frequency and note that $e/mc = 1.76 \times 10^7$ s^{-1} gauss^{-1}, we have

$$E = mc^2 \sqrt{\frac{2}{3} \cdot 10^{18} \cdot \frac{1}{1.76 \times 10^7} \cdot \frac{1}{3 \times 10^{-4}}} \approx 1.1 \times 10^7 mc^2 \approx 5.7 \times 10^{12} \text{ eV}$$

consistent with the electron energy (and therefore all other numbers) of part (a). In this frequency region, we have $n = 2\alpha + 1 = 1.70$.
(e) The half-life is given by

$$t_{1/2} = \frac{3m^3c^5}{2e^4B^2}\frac{1}{\gamma} = \frac{3}{2}(\frac{mc}{e})^2 \frac{1}{B^2}\frac{mc^2}{e^2}\frac{mc^2}{E} c$$

Again note that

$$\frac{e}{mc} = 1.76 \times 10^7 \text{ s}^{-1} \text{ gauss}^{-1}; \qquad \frac{e^2}{mc^2} = 2.82 \times 10^{-13} \text{ cm}$$

Therefore, the half-life can be expressed in terms of B in milli-gauss and E in GeV as

$$t_{1/2} = \frac{3}{2} \cdot \frac{1}{(1.76 \times 10^7)^2} \cdot \frac{10^6}{B^2} \cdot \frac{1}{2.82 \times 10^{-13}} \frac{0.511 \times 10^{-3}}{E} \cdot 3 \times 10^{10} = \frac{2.63 \times 10^{11}}{EB^2} \text{ s}$$

For the numbers in part (a),

$$t_{1/2} = \frac{2.63 \times 10^{11}}{10^4 \cdot 0.3^2} = 2.92 \times 10^8 \text{ s} \sim 9.3 \text{ years}$$

The Crab nebula was observed in year 1054, more than 900 years ago. Therefore, initial energetic electrons are probably long gone. However, my astrophysics colleagues told me that there is not much trouble making energetic electrons from the pulsar at the center. Electrons and positrons can be pair produced from the energetic photons from the pulsar and they are accelerated by the rapidly rotating magnetic field associated with the neutron star. It is interesting to note that earlier editions of Jackson had $E = 10^{12}$ eV and $B = 10^{-4}$ gauss, which results a half-life about 834 years. Presumably the change is due to recent progresses made in this area.

Problem 13.1

(a) Let $\vec{v} = v\hat{x}$ be the velocity of the incident particle (of mass M). Since electron is much light ($m \ll M$), $\vec{v}$ is also the velocity of the center-of-mass frame. In this frame, the electron moves at a velocity $-\vec{v}$ before the scattering and therefore its 4-momentum is given by $\mathcal{P}^i_{CM} = (\gamma mc; -\gamma mv, 0, 0)$. After the scattering, the electron energy remains the same, but the momentum is deflected by a scattering angle θ. Thus, the 4-momentum after the scattering is $\mathcal{P}^f_{CM} = (\gamma mc; -\gamma mv\cos\theta, \gamma mv\sin\theta, 0)$, here we have chosen the $x - y$ plane as the scattering plane. The invariant 4-momentum transfer squared is

$$(\delta\mathcal{P})^2 = (\mathcal{P}^f_{CM} - \mathcal{P}^i_{CM})^2 = -(-\gamma mv\cos\theta + \gamma mv)^2 - (\gamma mv\sin\theta)^2 = -2(\gamma mv)^2(1-\cos\theta)$$

$(\delta\mathcal{P})^2$ can also be calculated in the laboratory frame. In this regard, the electron 4-momenta before and after the scattering are given respectively by

$$\mathcal{P}^i_{LAB} = (mc; \vec{0}); \qquad \mathcal{P}^f_{LAB} = (\frac{E}{c}; \vec{p})$$

where E and $\vec{p}$ are electron's energy and momentum after the scattering. The 4-momentum transfer squared calculated using the laboratory variables is

$$(\delta\mathcal{P})^2 = (\mathcal{P}^f_{LAB} - \mathcal{P}^i_{LAB})^2 = (\frac{E}{c} - mc)^2 - p^2 = -2m(E - mc^2)$$

Equating the two 4-momentum transfer squared, we get the energy transfer

$$T(b) \equiv E - mc^2 = -\frac{(\delta\mathcal{P})^2}{2m} = \gamma^2 mv^2(1-\cos\theta) = 2\gamma^2 mv^2 \sin^2\frac{\theta}{2}$$

The angle factor can be calculated from the relationship between b and θ,

$$b = \frac{ze^2}{\gamma mv^2}\cot\frac{\theta}{2} \quad \Rightarrow \quad \sin^2\frac{\theta}{2} = \frac{1}{1 + (\gamma mv^2 b)^2/(ze^2)^2} = (\frac{ze^2}{\gamma mv^2})^2 \frac{1}{b^2 + b^2_{min}}$$

where $b_{min} = ze^2/\gamma mv^2$. Thus the energy transfer is

$$T(b) = 2\gamma^2 mv^2\sin^2\frac{\theta}{2} = \frac{2z^2e^4}{mv^2}\frac{1}{b^2 + b^2_{min}}$$

(b) The transverse electric field is

$$E_\perp = E_2 = \frac{\gamma zeb}{(b^2 + \gamma^2v^2t^2)^{3/2}}$$

The transverse momentum impulse

$$\Delta p = \int F_\perp dt = e\int E_\perp dt = \gamma ze^2 b\int_{-\infty}^{\infty}\frac{dt}{(b^2+\gamma^2v^2t^2)^{3/2}} = \frac{2ze^2}{vb}$$

The energy transfer

$$T \approx \frac{(\Delta p)^2}{2m} = \frac{2z^2e^4}{mv^2}\frac{1}{b^2}$$

Withe the exception of the cutoff b_{min} in the exact classical calculation, the two results are the same. Note that the energy transfer diverges without the cutoff b_{min}. This is because the two particles can get infinitely close to each other with the assumption we made in (b). In practice, this cannot be the case.

Problem 13.11

Fields of a magnetic monopole g are the same as for a charge q, with the exchanges $\vec{E} \to \vec{B}$, $\vec{B} \to -\vec{E}$ and $q \to g$. For a magnetic particle moving in the x direction, there is only an electric field in the z direction (if the observation point is on the $y-$axis). Following Eqs. (11.152) we have,

$$B_1 = -\frac{g\gamma vt}{(b^2+\gamma^2v^2t^2)^{3/2}}; \quad B_2 = \frac{\gamma gb}{(b^2+\gamma^2v^2t)^{3/2}}; \quad E_3 = -\beta B_2 = -\frac{\gamma(\beta g)b}{(b^2+\gamma^2v^2t^2)^{3/2}}$$

Since a magnetic field does no work and therefore does not cause energy transfer, the energy loss is mainly caused by the action of the electric field of the passing particle on the atomic electrons. The momentum transfer can be calculated in exactly the same way as in Prob. 13.1(b) with the following replacement for the electric field:

$$E_\perp = \frac{\gamma zeb}{(b^2+\gamma^2v^2t^2)^{3/2}} \quad \Rightarrow \quad E_\perp = \frac{\gamma(\beta g)b}{(b^2+\gamma^2v^2t^2)^{3/2}}$$

Therefore, the momentum and energy transfer can be obtained with the replacement $ze \to \beta g$:

$$\Delta p_g = \frac{2e(\beta g)}{bv} = \frac{2eg}{bc}; \qquad T_g(b) = \frac{2e^2(\beta g)^2}{mv^2} = \frac{2g^2e^2}{mc^2}\frac{1}{b^2}$$

Since the limits on $b_{\max}$ and $b_{\min}$ are essentially the same, having to do with the electrons binding frequency and the electron's Compton wavelength, the whole calculation proceeds as before. The Bethe formula thus has the following analog for energy loss by a magnetic monopole:

$$\frac{dE}{dx} \approx 4\pi NZ\frac{g^2e^2}{mc^2}\ln\left\{\frac{2\gamma^2mv^2}{\hbar\langle\omega\rangle}\right\} = 4\pi NZ\frac{g^2e^2}{mc^2}\ln\left\{\frac{2\gamma^2\beta^2mc^2}{\hbar\langle\omega\rangle}\right\} = 4\pi NZ\frac{g^2e^2}{mc^2}\left\{2\ln(\gamma\beta)+\ln\{\frac{2mc^2}{\hbar\langle\omega\rangle}\}\right\}$$

We have omitted the v^2/c^2 term, because its presence in the monopole situation is not clear. It comes in part from close collisions of the electrons with nuclei and involves the electron's spin. Evidently, the loss is linear in $\ln(\gamma\beta)$. At high energies, the dE/dx energy loss by a monopole is identical to that of a charged particle. The difference is at low energies where dE/dx is more or less flat for monoples. However it should be noted that the formula above is not valid for an extremely slow monopole.

(b) Dirac quantization condition is

$$\frac{ge}{\hbar c} = \frac{n}{2} \quad \Rightarrow \quad g = \frac{n}{2}e\,\frac{\hbar c}{e^2} = \frac{137}{2}ne$$

Thus the losses in the two cases can be written as

$$(\frac{dE}{dx})_{ze} = 4\pi NZ\frac{z^2e^4}{mc^2\beta^2}\ln\left\{\frac{2\gamma^2\beta^2mc^2}{\hbar\langle\omega\rangle}\right\} = 4\pi NZ\frac{e^4}{mc^2}\left\{\frac{z^2}{\beta^2}\right\}\ln\left\{\frac{2\gamma^2\beta^2mc^2}{\hbar\langle\omega\rangle}\right\}$$

$$(\frac{dE}{dx})_g = 4\pi NZ\frac{g^2e^2}{mc^2\beta^2}\ln\left\{\frac{2\gamma^2\beta^2mc^2}{\hbar\langle\omega\rangle}\right\} = 4\pi NZ\frac{e^4}{mc^2}\left\{\frac{137n}{2}\right\}^2\ln\left\{\frac{2\gamma^2\beta^2mc^2}{\hbar\langle\omega\rangle}\right\}$$

For $\beta \sim 1$, the charged particle will lose energy at the same rate as a monopole provided $z = 137n/2$. For $n = 1$, $z = 68.5$. For $n = 2$, $z = 137$. A Dirac monopole is thus expected to ionize and lose energy like a relativistic heavy nucleus. At low energies, the log-term $\ln(2mc^2/\hbar\langle\omega\rangle)$ dominates and therefore the loss is more or less constant.