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Physics 506: Solutions to Assignment #10

Problem 14.4

(a) The instantaneous power radiated per unit solid angle for $\beta \ll 1$ is given by Eq. (14.20):

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c}|\hat{n} \times (\hat{n} \times \dot{\vec{\beta}})|^2$$

$$\vec{r}(t) = a\cos(\omega_0 t)\hat{z}, \quad \dot{\vec{\beta}} = \frac{1}{c}\ddot{\vec{r}} = -a\frac{\omega_0^2}{c}\cos(\omega_0 t)\hat{z}$$

Therefore

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c}\frac{a^2\omega_0^4}{c^2}\cos^2(\omega_0 t)|\hat{n} \times (\hat{n} \times \hat{z})|^2 = \frac{e^2}{4\pi c}\frac{a^2\omega_0^4}{c^2}\cos^2(\omega_0 t)\sin^2\theta$$

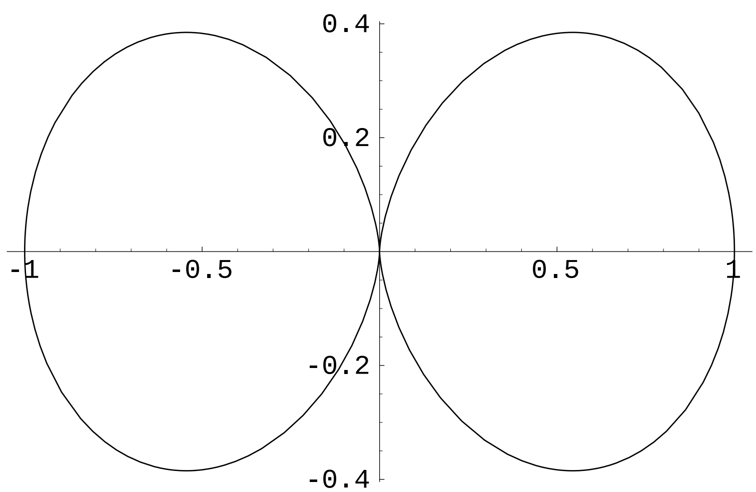
The average power per unit solid angle

$$\langle\frac{dP}{d\Omega}\rangle = \frac{e^2}{4\pi c}\frac{a^2\omega_0^4}{c^2}\sin^2\theta\langle\cos^2(\omega_0 t)\rangle = \frac{e^2}{8\pi c}\frac{a^2\omega_0^4}{c^2}\sin^2\theta$$

The total average power

$$P = \int\langle\frac{dP}{d\Omega}\rangle d\Omega = 2\pi\int_{-1}^{+1}\langle\frac{dP}{d\Omega}\rangle d(\cos\theta) = \frac{e^2}{3c^3}a^2\omega_0^4$$

The average power in the unit of $(e^2a^2\omega_0^4)/(8\pi c^3)$ is plotted below. The positive vertical axis defines $\theta = 0$.



(b)

$$\vec{r}(t) = R\cos(\omega_0 t)\hat{x} + R\sin(\omega_0 t)\hat{y}, \quad \dot{\vec{\beta}} = \frac{1}{c}\ddot{\vec{r}} = -\frac{\omega_0^2 R}{c}\{\cos(\omega_0 t)\hat{x} + \sin(\omega_0 t)\hat{y}\}$$

The problem is azimuthal symmetric and therefore, the differential power radiated is independent of ϕ. Without loosing generality, we can choose $\hat{n}$ in the $x - z$ plane. In this case, $\hat{n} = \cos\theta\hat{z} + \sin\theta\hat{x}$ and

$$|\hat{n} \times (\hat{n} \times \dot{\vec{\beta}})| = |\hat{n}(\hat{n}\cdot\dot{\vec{\beta}}) - \dot{\vec{\beta}}| = \frac{\omega_0^2 R}{c}| - \cos^2\theta\cos(\omega_0 t)\hat{x} - \sin(\omega_0 t)\hat{y} + \sin\theta\cos\theta\cos(\omega_0 t)\hat{z}|$$

$$|\hat{n}\times(\hat{n}\times\dot{\vec{\beta}})|^2 = \frac{\omega_0^4 R^2}{c^2}\left\{\cos^2\theta\cos^2(\omega_0 t)+\sin^2(\omega_0 t)\right\}$$

The differential potential

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c}\frac{\omega_0^4 R^2}{c^2}\left\{\cos^2\theta\cos^2(\omega_0 t)+\sin^2(\omega_0 t)\right\}$$

and the average

$$\langle\frac{dP}{d\Omega}\rangle = \frac{e^2}{4\pi c}\frac{\omega_0^4 R^2}{c^2}\left\{\frac{1}{2}\cos^2\theta+\frac{1}{2}\right\} = \frac{e^2}{8\pi c}\frac{\omega_0^4 R^2}{c^2}(1+\cos^2\theta)$$

The total power

$$P = \int\langle\frac{dP}{d\Omega}\rangle d\Omega = \frac{e^2}{8\pi c}\frac{\omega_0^4 R^2}{c^2}(2\pi)\int_{-1}^{+1}(1+\cos^2\theta)d(\cos\theta) = \frac{2e^2}{3c^3}\omega_0^4 R^2$$

Alternate approach

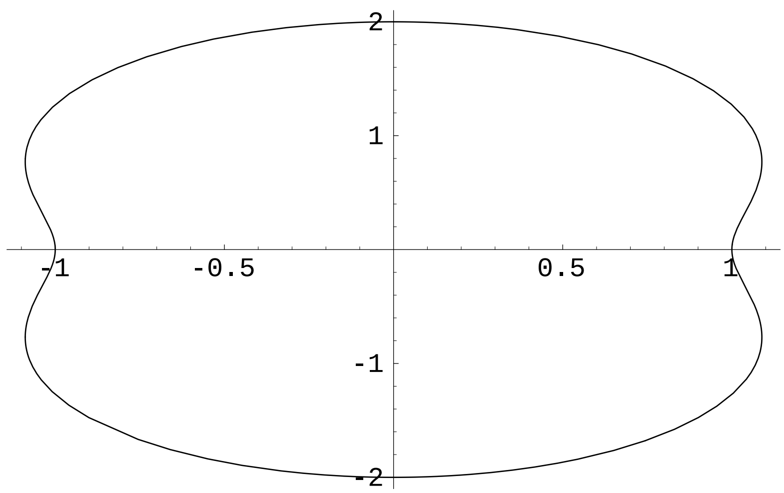
$$\vec{r}(t) = R(\hat{x}+i\hat{y})e^{-i\omega_0 t}, \qquad \dot{\vec{\beta}} = -\frac{\omega_0^2 R}{c}(\hat{x}+i\hat{y})$$

The average power

$$\langle\frac{dP}{d\Omega}\rangle = \frac{c}{8\pi}r^2 Re\left\{\vec{E}\times\vec{B}^*\right\} = \frac{e^2}{8\pi c}|\hat{n}\times(\hat{n}\times\dot{\vec{\beta}})|^2$$

$$|\hat{n}\times(\hat{n}\times\dot{\vec{\beta}})|^2 = \left\{\hat{n}(\hat{n}\cdot\dot{\vec{\beta}})-\dot{\vec{\beta}}\right\}\left\{\hat{n}(\hat{n}\cdot\dot{\vec{\beta}}^*)-\dot{\vec{\beta}}^*\right\} = \dot{\vec{\beta}}\cdot\dot{\vec{\beta}}^* - (\hat{n}\cdot\dot{\vec{\beta}})(\hat{n}\cdot\dot{\vec{\beta}}^*) = \frac{\omega_0^4 R^2}{c^2}(1+\cos^2\theta)$$

The $\langle dP/d\Omega\rangle$ (in $(e^2\omega_0^4 R^2)/(8\pi c^3)$ unit) vs θ is plotted below, again $\theta = 0$ is defined by the upper vertical axis.



Problem 14.5
(a) For a non-relativistic particle with charge ze, the power radiated is

$$P = \frac{2}{3}\frac{(ze)^2}{c^3}\dot{v}^2$$

where $\dot{v}$ is the acceleration, given by

$$m\dot{v} = F = -\frac{dV}{dr}, \quad \Rightarrow \quad \dot{v} = -\frac{1}{m}\frac{dV}{dr}$$

Assuming the amount of energy radiated is small, we have approximately

$$\frac{1}{2}mv^2 + V(r) = E_{tot} = V(r_{min}), \quad \Rightarrow \quad v = \sqrt{\frac{2}{m}}\sqrt{V(r_{min}) - V(r)}$$

Note that the particle has zero velocity at the position of the closest approach. The total energy radiated is the time integral of power radiated

$$\Delta W = \int_{-\infty}^{\infty} P dt = 2\int_{r_{min}}^{\infty} P\frac{dr}{v} = \frac{4}{3}\frac{(ze)^2}{m^2c^3}\sqrt{\frac{m}{2}}\int_{r_{min}}^{\infty}\frac{|dV/dr|^2}{\sqrt{V(r_{min}) - V(r)}}dr$$

(b) For a Coulomb potential,

$$V(r) = \frac{zZe^2}{r},$$

the energy radiated is

$$\Delta W = \frac{4}{3}\frac{(ze)^2}{m^2c^3}\sqrt{\frac{m}{2}}(zZe^2)^{3/2}\int_{r_{min}}^{\infty}\frac{1}{\sqrt{1/r_{min} - 1/r}}\frac{dr}{r^4} = \frac{4}{3}\frac{(ze)^2}{m^2c^3}\sqrt{\frac{m}{2}}(zZe^2)^{3/2}\frac{16}{15}(\frac{1}{r_{min}})^{5/2}$$

In terms of v_0, we have

$$\frac{1}{2}mv_0^2 = V(r_{min}) = \frac{zZe^2}{r_{min}} \quad \Rightarrow \quad \frac{1}{r_{min}} = \frac{mv_0^2}{2zZe^2}$$

Therefore,

$$\Delta W = \frac{4}{3}\frac{(ze)^2}{m^2c^3}\sqrt{\frac{m}{2}}(zZe^2)^{3/2}\frac{16}{15}(\frac{mv_0^2}{2zZe^2})^{5/2} = \frac{8}{45}\frac{zmv_0^5}{Zc^3}$$

Problem 14.12

(a)

$$\vec{r}(t) = \hat{z}a\cos(\omega_0 t'), \quad \vec{\beta} = \dot{\vec{r}}(t) = -\frac{\omega_0 a}{c}\sin(\omega_0 t')\hat{z}, \quad \dot{\vec{\beta}} = -\frac{\omega_0^2 a}{c}\cos(\omega_0 t')\hat{z}$$

The differential power in particle's own time is given by Eq. (14.38):

$$\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c}\frac{|\hat{n}\times\left\{(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\right\}|^2}{(1-\hat{n}\cdot\beta)^5} = \frac{e^2}{4\pi c}\frac{|\dot{\vec{\beta}}|^2\sin^2\theta}{(1-\hat{n}\cdot\vec{\beta})^5}$$

$$= \frac{e^2a^2\omega_0^4}{4\pi c^3}\frac{\sin^2\theta\cos^2(\omega_0 t')}{\{1+\frac{\omega_0 a}{c}\cos\theta\sin(\omega_0 t')\}^2} = \frac{e^2c\beta_0^4}{4\pi a^2}\frac{\sin^2\theta\cos^2(\omega_0 t')}{(1+\beta_0\cos\theta\sin(\omega_0 t'))^5}$$

where $\beta_0 \equiv a\omega_0/c$. Here I used β_0 instead of β for the constant to avoid confusions.
(b) The average power

$$\langle\frac{dP}{d\Omega}\rangle = \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0}\frac{dP(t')}{d\Omega}dt' = \frac{e^2c\beta_0^4}{8\pi^2a^2}\sin^2\theta\int_0^{2\pi/\omega_0}\frac{\cos^2(\omega_0 t')}{(1+\beta_0\cos\theta\sin(\omega_0 t'))^5}d(\omega_0 t')$$

$$= \frac{e^2c\beta_0^4}{8\pi^2a^2}\sin^2\theta\int_0^{2\pi}\frac{\cos^2\phi}{(1+\beta_0\cos\theta\sin\phi)^5}d\phi = \frac{e^2c\beta_0^4}{8\pi^2a^2}\sin^2\theta\left\{\frac{\pi}{4}\frac{4+\beta_0^2\cos^2\theta}{(1-\beta_0^2\cos^2\theta)^{7/2}}\right\}$$

Thus

$$\langle\frac{dP}{d\Omega}\rangle = \frac{e^2c\beta_0^4}{32\pi a^2}\frac{4+\beta_0^2\cos^2\theta}{(1-\beta_0^2\cos^2\theta)^{7/2}}\sin^2\theta$$

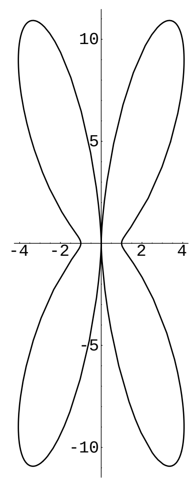
(c) For non-relativistic case, $\beta_0 \ll 1$, therefore

$$\langle \frac{dP}{d\Omega} \rangle \to \frac{e^2 c \beta_0^4}{8\pi a^2} \sin^2\theta$$

In the relativistic case, $\beta_0 \to 1$,

$$\langle \frac{dP}{d\Omega} \rangle \to \frac{e^2 c \beta_0^4}{32\pi a^2} \frac{4 + \cos^2\theta}{(1-\cos^2\theta)^{7/2}} \sin^2\theta$$

As β_0 approaches 1, $\langle dP/d\Omega \rangle$ develops peaks close to $\theta = 0, \pi$. The $\langle dP/d\Omega \rangle$ distribution for the non-relativistic case ($\beta_0 = 0$) is the same as Prob. 14.4(a). For $\beta_0 = 0.9$, $\langle dP/d\Omega \rangle$ is plotted below:



As β_0 increases, the four lobeds become narrower. The upper two are increasing clustered together, forming a strong peak in the forward direction. Similarly, the lower two lobeds form a strong peak in the backward direction.

Problem 14.13

From Eq. (14.67), we have

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} \vec{n} \times (\vec{n} \times \vec{\beta}(t)) e^{i\omega(t - \vec{n}\cdot\vec{r}(t)/c)} \; dt \right|^2$$

If the charge is in periodic motion with period T, the integrand almost repeat itself (except for a phase factor) each period. We can thus break the integral over time into a sum of terms, times a common integral over one cycle. If the charge has actually been in periodic motion always, the total radiated energy is infinite. To keep track of things and to avoid square of delta functions, we make the integral from $-NT$ to $+NT$ where N is a large integer, thus

$$A_N = \sum_{n=-N}^{N-1} \int_{nT}^{(n+1)T} dt \; \vec{n} \times (\vec{n} \times \vec{\beta}) e^{i\omega(t - \vec{n}\cdot\vec{r}(t)/c)}$$

Changing variables to $t' = t - nT$ and using the factor that $\vec{r}(t)$ and $\vec{\beta}(t)$ are periodic, we have

$$A_N = \sum_{n=-N}^{N-1} e^{in\omega T} A_0(\omega), \qquad \text{where} \qquad A_0(\omega) = \int_0^T dt' \; \vec{n} \times (\vec{n} \times \vec{\beta}(t')) e^{i\omega(t' - \vec{n}\cdot\vec{r}(t')/c)}$$

The sum of the phase factor

$$\mathcal{S}_N = \sum_{n=-N}^{N-1} e^{in\omega T} = \sum_{n=0}^{N-1} e^{in\omega T} + \sum_{n=0}^{N-1} e^{-i(n+1)\omega T} = \frac{1 - e^{iN\omega T}}{1 - e^{i\omega T}} + e^{-i\omega T}\frac{1 - e^{-iN\omega T}}{1 - e^{-i\omega T}}$$

Therefore,

$$\frac{d^2I}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}|\mathcal{S}_N(\omega)|^2|A_0(\omega)|^2$$

Multiplying both sides of $\mathcal{S}_N$ by $e^{i\omega T/2}$, we get

$$\mathcal{S}_N e^{i\omega T/2} = e^{i\omega T/2}\frac{1-e^{iN\omega T}}{1-e^{i\omega T}} + c.c.$$

where $c.c.$ is a short-hand for complex conjugate. This is a standard diffraction pattern function that peaks up strongly at $\omega = (2\pi/T)m$ if N is large. Here m is an integer. Let $\omega T = 2\pi m + x$ and assume $x \ll 1$, then

$$\mathcal{S}_N e^{i\omega T/2} = e^{im\pi}e^{ix/2}\frac{1-e^{i2\pi Nm}e^{iNx}}{1-e^{i2\pi m}e^{ix}} + c.c. \approx (-1)^m\frac{1-e^{iNx}}{-ix} + c.c.$$

$$= (-1)^m e^{iNx/2}\frac{e^{iNx/2}-e^{-iNx/2}}{ix} + c.c. = 2(-1)^m e^{iNx/2}\frac{\sin(Nx/2)}{x} + cc.$$

$$= (-1)^m\frac{\sin(Nx/2)}{x}\times 2(e^{iNx/2}+e^{-iNx/2}) = 2(-1)^m\frac{\sin(Nx)}{x}$$

Thus for frequencies near $\omega = m(2\pi/T) \equiv m\omega_0$, the frequency spectrum is sharply peaked. Evidently as $N \to \infty$ the frequency spectrum becomes a series of lines at $\omega = m\omega_0$. The integral over frequency of $|\mathcal{S}_N|^2$ near $\omega = m\omega_0$ is

$$\int_{\omega\sim m\omega_0} d\omega|\mathcal{S}_N|^2|A_0(\omega)|^2 \approx \frac{4}{T}|A_0(m\omega_0)|^2\int_{-\infty}^{\infty}\frac{\sin^2(Nx)}{x^2}dx = \frac{8N}{T}\int_0^{\infty}\frac{\sin^2 t}{t^2}dt = \frac{4\pi}{T}N = 2\omega_0 N$$

The radiated energy in each line is proportional to N. Since the total time interval is $2NT = 4\pi N/\omega_0$, the power radiated in each harmonic is

$$\frac{dP_m}{d\Omega} = \frac{e^2\omega_0^2 m^2}{4\pi^2 c}|A_0(m\omega_0)|^2 \times 2\omega_0 N \times \frac{\omega_0}{4\pi N} = \frac{e^2\omega_0^4}{8\pi^3 c}m^2|A_0(m\omega_0)|^2$$

or

$$\frac{dP_m}{d\Omega} = \frac{e^2\omega_0^4 m^2}{(2\pi c)^3}|\int_0^{2\pi/\omega_0} dt\ \vec{n}\times(\vec{n}\times\vec{v}(t))e^{im\omega_0(t-\vec{n}\cdot\vec{r}(t)/c)}|^2$$

$$= \frac{e^2\omega_0^4 m^2}{(2\pi c)^3}|\int_0^{2\pi/\omega_0} dt\ \vec{n}\times\vec{v}(t)\ e^{im\omega_0(t-\vec{n}\cdot\vec{r}(t)/c)}|^2$$

<u>*Alternate Approach*</u>
The energy distribution is given by Eq. (14.70):

$$\frac{d^2I}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}|\int_{-\infty}^{\infty}\vec{n}\times(\vec{n}\times\vec{\beta})e^{i\omega(t-\vec{n}\cdot\vec{r}(t)/c)}\ dt|^2$$

Expanding the integrand in Fourier series

$$\vec{n}\times(\vec{n}\times\vec{\beta})e^{-i\vec{n}\cdot\vec{r}(t)/c} = \sum_{m=-\infty}^{\infty}A_m(\omega)e^{-im\omega_0 t}$$

Then we have

$$\frac{d^2I}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\sum_m\sum_{m'}\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}dt'\ A_m(\omega)A^*_{m'}(\omega)e^{i\omega t}e^{-im\omega_0 t}e^{-i\omega t'}e^{im'\omega_0 t'}$$

$$= \frac{e^2\omega^2}{4\pi^2 c}\sum_m\sum_{m'}\int_{-\infty}^{\infty} dt\ A_m(\omega)A^*_{m'}(\omega)e^{i(\omega-m\omega_0)t}\ \{2\pi\delta(\omega-m'\omega_0)\}$$

The above equation shows that the frequency spectrum is discrete. Integrating over ω, the total energy radiated per unit solid angle is

$$\frac{dI}{d\Omega} = \int \frac{d^2I}{d\omega d\Omega}d\omega = \frac{e^2}{2\pi c}\sum_m\sum_{m'}(m'\omega_0)^2 A_m(m'\omega_0)A^*_{m'}(m'\omega_0)\ \int_{-\infty}^{\infty} e^{i(m'-m)\omega_0 t}\ dt$$

To facilitate the power calculation, we replace the time interval $(-\infty,\infty)$ with $(-NT, NT)$ where N is a large integer and $T = 2\pi/\omega_0$ is the period. In this case, the energy radiated per unit solid angle in the time interval $2NT$ is

$$\frac{dI}{d\Omega}(-NT,NT) = \frac{e^2}{2\pi c}\sum_m\sum_{m'}(m'\omega_0)^2 A_m(m'\omega_0)A^*_{m'}(m'\omega_0)\ \int_{-NT}^{NT} e^{i(m'-m)\omega_0 t}\ dt$$

The average power per unit solid angle is therefore

$$\frac{dP}{d\Omega} = \frac{1}{2NT}\frac{dI}{d\Omega}(-NT,NT) = \frac{e^2}{2\pi c}\sum_m\sum_{m'}(m'\omega_0)^2 A_m(m'\omega_0)A^*_{m'}(m'\omega_0)\ \left\{\frac{1}{2NT}\int_{-NT}^{NT} e^{i(m'-m)\omega_0 t}\ dt\right\}$$

Note that

$$\frac{1}{2NT}\int_{-NT}^{NT} e^{i(m'-m)\omega_0 t}\ dt = \frac{1}{T}\int_0^T e^{i(m'-m)\omega_0 t}dt = \delta_{m'm}$$

Thus

$$\frac{dP}{d\Omega} = \frac{e^2}{2\pi c}\sum_m\sum_{m'}(m'\omega_0)^2 A_m(m'\omega_0)A^*_{m'}(m'\omega_0)\ \{\delta_{mm'}\} = \frac{e^2}{2\pi c}\sum_m (m\omega_0)^2|A_m(m\omega_0)|^2$$

This is the total power for all harmonics. For m^{th} harmonics, we have

$$\langle\frac{dP}{d\Omega}\rangle_{m^{th}} = \frac{m^2e^2\omega_0^2}{2\pi c}|A_m(m\omega_0)|^2$$

$A_m(\omega)$ given by reverse Fourier integration:

$$A_m(\omega) = \frac{\omega_0}{2\pi}\int_0^{\frac{2\pi}{\omega_0}} \vec{n}\times(\vec{n}\times\vec{\beta})e^{-i\omega\hat{n}\cdot\vec{r}/c}e^{im\omega_0 t}\ dt$$

Therefore

$$\langle\frac{dP}{d\Omega}\rangle_{m^{th}} = \frac{m^2e^2\omega_0^4}{(2\pi c)^3}|\int_0^{2\pi/\omega_0} \vec{n}\times(\vec{n}\times\vec{v})e^{im\omega_0(t-\hat{n}\cdot\vec{r}/c)}\ dt|^2$$

$$= \frac{m^2e^2\omega_0^4}{(2\pi c)^3}|\int_0^{2\pi/\omega_0} \vec{n}\times\vec{v}\ e^{im\omega_0(t-\hat{n}\cdot\vec{r}/c)}\ dt|^2$$