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Physics 506: Solutions to Assignment #9

Problem 12.5

(a) For $|\vec{E}| < |\vec{B}|$, we can also find a frame K' in which $\vec{E}' = 0$. In this frame, the particle is moving in a uniform magnetic field $\vec{B}'$. Let $\vec{E}$ pointing to $+x$ and $\vec{B}$ pointing to $+y$ direction, the velocity of frame K' in frame K can be obtained from Eq. 12.43 to be

$$\vec{u} = c\frac{\vec{E}\times\vec{B}}{B^2}.$$

Thus,

$$\vec{E}' = 0, \qquad \vec{B}' = \frac{1}{\gamma}\vec{B} = \sqrt{B^2 - E^2}\frac{\vec{B}}{B}$$

In frame K' with a Cartesian coordinate system, the motion will be helix, *i.e.*, an uniform motion along $\vec{B}'$ and gyration in the transverse plane. With a properly chosen origin, the position of the particle can be written as

$$x' = a\cos(\omega_B t'); \qquad y' = v_{||}t'; \qquad z' = a\sin(\omega_B t')$$

where a is the gyration radius determined by particle's transverse momentum ($cp_\perp = eB'a$) and $v'_{||}$ is the velocity component along the $\vec{B}'$ in frame K', $\omega_B = eB'/(\gamma' mc)$ and γ' is the Lorentz boost factor of the particle in frame K'. Translating back to frame K:

$$\begin{aligned} x =& \; x' = a\cos(\omega_B t') \\ y =& \; y' = v_{||}t' \\ z =& \; \gamma(z' + ut') = \gamma\{a\sin(\omega_B t') + ut'\} = \tfrac{B}{\sqrt{B^2-E^2}}\{a\sin(\omega_B t') + \tfrac{E}{B}(ct')\} \end{aligned}$$

These are explicit parametric equations for the particle's trajectory in terms of parameter t'. (b) For the case of $|\vec{E}| > |\vec{B}|$, the magnetic field $\vec{B}'$ vanishes in the frame K' moving with a velocity

$$\vec{u} = c\frac{\vec{E}\times\vec{B}}{E^2}.$$

In this frame, the particle moves in a uniform electric field $\vec{E}'$:

$$\vec{E}' = \frac{\vec{E}}{\gamma'} = \sqrt{E^2 - B^2}\frac{\vec{E}}{E}$$

Thus

$$\frac{d\vec{p'}}{dt'} = q\vec{E}'$$

which leads to

$$\frac{d}{dt'}(\gamma' m v'_x) = qE', \quad \frac{d}{dt'}(\gamma' m v'_y) = 0, \quad \frac{d}{dt'}(\gamma' m v'_z) = 0$$

The differential equations for the most general case of initial velocities are difficult to integrate. Assuming the particle is at rest for simplicity, integrating the above equations

$$v'_x = \frac{dx'}{dt'} = c\frac{\alpha t'}{\sqrt{1+\alpha^2 t'^2}}, \quad v'_y = 0, \quad v'_z = 0$$

Here $\alpha = qE'/mc$. Integrating the above equations (with a properly chosen origin):

$$x'(t') = \frac{c}{\alpha}\left\{\sqrt{1+\alpha^2 t'^2} - 1\right\}; \qquad y'(t') = 0; \qquad z'(t') = 0$$

Translating back to frame K, the parametric equations for the particle's trajectory are

$$\begin{aligned} x = &\ x' = \tfrac{c}{\alpha}\left\{\sqrt{1+\alpha^2 t'^2} - 1\right\} \\ y = &\ y' = 0 \\ z = &\ \gamma(z' + ut') = \gamma u t' = \tfrac{B}{\sqrt{E^2 - B^2}}(ct') \end{aligned}$$

Problem 12.9

(a) Let $z-$axis points from south to north, in this case, $\vec{M} = -M\ \hat{z}$. The vector potential of the earth's magnetic dipole moment

$$\vec{A}(\vec{r}) = \frac{\vec{M}\times\vec{r}}{r^3} = -\frac{M\sin\theta}{r^2}\hat{\phi} \equiv A_\phi\hat{\phi}$$

Thus the magnetic field $\vec{B}$:

$$\vec{B}(\vec{r}) = \nabla\times\vec{A} = -\frac{2M\cos\theta}{r^3}\ \hat{r} - \frac{M\sin\theta}{r^3}\ \hat{\theta}$$

Let $d\vec{s}$ be the small displacement

$$d\vec{s} = \hat{r}dr + \hat{\theta}rd\theta + \hat{\phi}r\sin\theta d\phi$$

and let $d\vec{s}$ point to the direction of the magnetic field, we get

$$\frac{dr}{B_r} = \frac{rd\theta}{B_\theta} \quad\Rightarrow\quad \frac{dr}{r} = \frac{2\cos\theta}{\sin\theta}d\theta$$

Integrating the above equation yields the equation for a line magnetic force to be:

$$r = r_0\sin^2\theta$$

The magnetic field as a function of θ

$$B = \sqrt{B_r^2 + B_\theta^2} = \frac{M}{r^3}\sqrt{4\cos^2\theta + \sin^2\theta} = \frac{M}{r_0^3}\frac{\sqrt{1+3\cos^2\theta}}{\sin^6\theta}$$

(b) The gradient drift velocity is given by Eq. (12.55):

$$\vec{V}_G = \omega_B a\cdot\frac{a}{2B^2}\vec{B}\times\nabla_\perp B$$

where $\vec{B}$ is the field at the equator:

$$\vec{B} = -\frac{M}{r^3}|_{r=R}\hat{\theta} = -\frac{M}{R^3}\hat{\theta}$$

and B is its magnitude. Since the problem is azimuthal symmetric, we have

$$\nabla_\perp B = \frac{\partial B}{\partial r}|_{r=R}\hat{r} = -\frac{3M}{R^4}\hat{r}$$

Thus

$$\vec{V}_G = \frac{\omega_B a^2}{2B^2}(-B\hat{\theta})\times\left\{-\frac{3M}{R^4}\hat{r}\right\} = -\frac{3\omega_B a^2}{2R}\hat{\phi}$$

Now note that

$$\vec{V}_G = R\dot{\phi}\hat{\phi}, \quad \Rightarrow \quad R\dot{\phi} = -\frac{3a^2}{2R}\omega_B$$

Integrating the above equation of motion,

$$\phi(t) = \phi_0 - \frac{3a^2}{2R^2}\omega_0(t - t_0)$$

(c) Let $\theta = \pi/2 + \alpha$, the magnetic field along the line of the force is then given by

$$B(\alpha) = \frac{M}{R^3}\frac{\sqrt{1 + 3\sin^2\alpha}}{\cos^6\alpha}$$

For small α values,

$$B(\alpha) \approx \frac{M}{R^3}\frac{\sqrt{1+3\alpha^2}}{(1-\alpha^2/2)^6} \approx \frac{M}{R^3}(1 + \frac{9}{2}\alpha^2 + ...)$$

Note that from Eq. (12.72), we have

$$v_{||}^2(\alpha) = v^2(0) - v_\perp^2(0)\frac{B(z)}{B_0} = v^2(0) - v_\perp^2(0)(1 + \frac{9}{2}\alpha^2) = v_{||}^2(0) - \frac{9}{2}v_\perp^2(0)\alpha^2$$

Note that $v_\perp(0) = \omega_B a$ and $v_{||}^2(\alpha) = (R\dot{\alpha})^2$. Plugging these into the above equation, we get

$$\dot{\alpha}^2 + \frac{9\omega_B^2 a^2}{2R^2}\alpha^2 = \frac{v_{||}^2(0)}{R^2}$$

This is the "energy equation" of a harmonic oscillator with the corresponding frequency given by

$$\Omega = \frac{3}{\sqrt{2}}\frac{a}{R}\omega_B$$

The change in azimuth in one period of oscillation is

$$\Delta\phi = \frac{3}{2}(\frac{a}{R})^2\omega_B \times \frac{2\pi\sqrt{2}}{3\omega_B}(\frac{R}{a}) = \sqrt{2}\pi\frac{a}{R}$$

independent of M.
(d) For $R = 3 \times 10^9$ cm ($\sim 5R_{\mathrm{earth}}$), $M = 8.1 \times 10^{25}$ gauss-cm^3, we have

$$B = B_\theta = \frac{8.1 \times 10^{25}}{27 \times 10^{27}} = 3 \times 10^{-3} \text{ gauss}$$

$$\omega_B = \frac{eB}{\gamma mc} = \frac{e}{mc}\frac{B}{\gamma} = 1.76 \times 10^7 \text{ s}^{-1}\text{gauss}^{-1}\ \frac{3 \times 10^{-3} \text{ gauss}}{\gamma} = \frac{5.3 \times 10^4}{\gamma} \text{ s}^{-1}$$

and $a = v/\omega_B$. The time to drift once around the earth (in azimuth) is

$$T_\phi = 2\pi\frac{2R^2}{3\omega_B a^2}$$

and the time for one oscillation in latitude is

$$T_\theta = \frac{2\pi}{\Omega} = \frac{2\pi\sqrt{2}R}{3\omega_B a} = \frac{2\pi\sqrt{2}}{3}\frac{R}{v}$$

For a 10 MeV electron, we have

$$\gamma = \frac{E}{m} = \frac{10 + 0.511}{0.511} \approx 20.5, \quad v \approx c, \quad \omega_B = 2.57 \times 10^3 \text{ s}^{-1}, \quad a = 117 \text{ km}, \quad T_\phi = 107 \text{ s}, \quad T_\theta = 0.30 \text{ s}$$

For a 10 keV electron, we have

$$\gamma = \frac{E}{m} = \frac{0.511 + 0.010}{0.511} \approx 1.02, \quad v \approx 0.2c, \quad \omega_B = 5.2 \times 10^4 \text{ s}^{-1}, \quad a = 1.2 \text{ km}, \quad T_\phi = 5.5 \times 10^4 \text{ s}, \quad T_\theta = 1.5 \text{ s}$$

<u>Problem 12.11</u>

(a) The Thomas precession formula is

$$(\frac{d\vec{s}}{dt})_{\text{lab}} = \frac{1}{\gamma}(\frac{d\vec{s}}{d\tau})_{\text{rest}} + \vec{\omega}_T \times \vec{s}$$

$\vec{\omega}_T$ given by Eq. (11.119):

$$\vec{\omega}_T = \frac{\gamma^2}{1+\gamma}\frac{\vec{a} \times \vec{v}}{c^2}$$

From the Lorentz force and Newton's second law, we have

$$\frac{d\vec{p}}{dt} = \frac{e}{c}\vec{v} \times \vec{B}$$

where $\vec{p}$ is muon momentum and e is the muon charge. Since the magnetic field does not do any work, γ is a constant of the motion. Therefore, the above equation can be written as

$$\frac{d\vec{v}}{dt} = \frac{e}{\gamma mc}\vec{v} \times \vec{B} = \left\{ -\frac{e}{\gamma mc}\vec{B} \right\} \times \vec{v} = \vec{\omega}_B \times \vec{v}$$

Here $\vec{\omega}_B = -(e\vec{B})/(\gamma mc)$ is the orbital gyration frequency. Therefore the acceleration

$$\vec{a} = \frac{d\vec{v}}{dt} = \vec{\omega}_B \times \vec{v} = \frac{e}{\gamma mc}\vec{v} \times \vec{B}$$

Therefore

$$\vec{\omega}_T = \frac{\gamma^2}{1+\gamma}\frac{1}{c^2}\frac{e}{\gamma mc}(\vec{v} \times \vec{B}) \times \vec{v} = \frac{\gamma^2}{1+\gamma}\frac{e}{\gamma mc}\frac{v^2}{c^2}\vec{B} = \frac{\gamma - 1}{\gamma}\frac{e\vec{B}}{mc}$$

The precession in the rest frame is given by Eq. (11.101):

$$(\frac{d\vec{s}}{d\tau})_{\text{rest}} = \vec{\mu} \times \vec{B'} = \frac{ge}{2mc}\vec{s} \times \vec{B'}$$

where

$$\vec{B'} = \gamma(\vec{B} - \frac{\vec{v}}{c} \times \vec{E}) - \frac{\gamma^2}{1+\gamma}\frac{\vec{v}}{c}(\frac{\vec{v} \cdot \vec{B}}{c}) = \gamma\vec{B}$$

Then

$$(\frac{d\vec{s}}{dt})_{\text{lab}} = \frac{1}{\gamma}(\frac{ge}{2mc}\vec{s}) \times (\gamma\vec{B}) + (\frac{\gamma - 1}{\gamma}\frac{e\vec{B}}{mc}) \times \vec{s} = \frac{e}{mc}\left\{ \frac{\gamma - 1}{\gamma} - \frac{g}{2} \right\} \vec{B} \times \vec{s} = \vec{W} \times \vec{s}$$

where the spin precession frequency is

$$\vec{W} = (1 - \frac{g}{2} - \frac{1}{\gamma})\frac{e}{mc}\vec{B}$$

The difference between the spin precession and the orbital gyration frequencies is

$$\vec{\Omega} = \vec{W} - \vec{\omega}_B = (1 - \frac{g}{2} - \frac{1}{\gamma})\frac{e\vec{B}}{mc} + \frac{e\vec{B}}{\gamma mc} = \frac{e\vec{B}}{mc}\frac{2-g}{2} = \Omega\frac{\vec{B}}{B} \quad \Rightarrow \quad \Omega = \frac{eBa}{mc}$$

(b) Newton's first law on the centripetal motion,

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{e}{c}\vec{v} \times \vec{B} \quad \Rightarrow \quad \gamma m \frac{v^2}{R} = \frac{evB}{c}$$

Thus, the muon momentum

$$p = \gamma m v = \frac{eRB}{c} = 1.28 \cdot 10^3 \text{ MeV/c}$$

The Lorentz boost factor

$$\gamma = \frac{E}{mc^2} = \frac{\sqrt{p^2c^2 + m^2c^4}}{mc^2} = 12.1$$

The number of periods of precession per observed laboratory mean lifetime is

$$\frac{\gamma\tau_0}{T} = \frac{\gamma\tau_0\Omega}{2\pi} = \frac{eBa\gamma\tau_0}{2\pi mc} = \frac{eB\alpha\gamma\tau_0}{(2\pi)^2 mc} = 7.12$$

(c)

$$\Omega = \frac{eBa}{mc}, \text{ and } \omega_B = \frac{eB}{\gamma mc}, \quad \Rightarrow \quad \Omega = a\gamma\omega_B = \frac{\alpha\gamma}{2\pi}\omega_B$$

(i) E=300 MeV, $m = m_\mu = 106$ MeV,

$$\gamma = \frac{E}{mc^2} = 2.83, \quad \Omega = (\frac{\alpha\gamma}{2\pi})\omega_B = 0.0033\omega_B$$

(ii) E=300 MeV, $m = m_e = 0.511$ MeV,

$$\gamma = \frac{E}{mc^2} = 587, \quad \Omega = (\frac{\alpha\gamma}{2\pi})\omega_B = 0.682\omega_B$$

(iii) E=5 GeV, $m = m_e = 0.511$ GeV,

$$\gamma = \frac{E}{mc^2} = 9.78 \cdot 10^3, \quad \Omega = (\frac{\alpha\gamma}{2\pi})\omega_B = 11.4\omega_B$$

Problem 12.14

$$\mathcal{L} = -\frac{1}{8\pi}\partial_\alpha A_\beta \partial^\alpha A^\beta - \frac{1}{c}J_\alpha A^\alpha$$

(a)

$$\frac{\partial \mathcal{L}}{\partial A^\beta} = -\frac{1}{c}J_\beta$$

$$\partial^\alpha \frac{\partial \mathcal{L}}{\partial(\partial^\alpha A^\beta)} = \partial^\alpha(-\frac{1}{8\pi})(2\partial_\alpha A_\beta) = -\frac{1}{4\pi}\partial_\alpha\partial^\alpha A_\beta$$

Thus, the Euler-Lagrange equations are

$$\partial_\alpha\partial^\alpha A^\beta = \frac{4\pi}{c}J^\beta$$

These are Maxwell's equations in the Lorentz gauge:

$$\partial_\alpha A^\alpha = 0, \quad i.e. \quad \nabla \cdot \vec{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0$$

(b) From Eq. (12.85), we have

$$\mathcal{L}' = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{c} J_\alpha A^\alpha$$

then

$$\mathcal{L}' - \mathcal{L} = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} + \frac{1}{8\pi} \partial_\alpha A_\beta \partial^\alpha A^\beta = -\frac{1}{16\pi} (\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial^\alpha A^\beta - \partial^\beta A^\alpha) + \frac{1}{8\pi} \partial_\alpha A_\beta \partial^\alpha A^\beta$$

$$= \frac{1}{8\pi} \partial_\alpha A_\beta \partial^\beta A^\alpha = \frac{1}{8\pi} \left\{ \partial_\alpha (A_\beta \partial^\beta A^\alpha) - A_\beta \partial_\alpha \partial^\beta A^\alpha \right\}$$

The second term vanishes in the Lorentz gauge, and the 1st term is the divergence of a four-vector:

$$\mathcal{L}' - \mathcal{L} = \partial_\alpha \Lambda^\alpha, \quad \text{with} \quad \Lambda^\alpha = \frac{1}{8\pi} A_\beta \partial^\beta A^\alpha$$

The two actions differ by

$$A' - A = \int (\mathcal{L}' - \mathcal{L}) d^4x = \int \partial_\alpha \Lambda^\alpha d^4x = \int_S \Lambda^\alpha d^3x$$

where the surface integral is over the surface in four-dimension. Now note that since Λ is not varied on the surface, we have

$$\delta(A' - A) = \delta \int_S \Lambda^\alpha d^3x = 0$$

Thus the equations of motion are unchanged.