Physics 506: Solutions to Assignment #8

Problem 11.23
(a) Let \( \mathcal{P} \) and \( \mathcal{P}' \) be 4-vectors in lab and CM frame respectively, then we have

\[
\mathcal{P}_1 = (E_1, \vec{p}_1), \quad \mathcal{P}_2 = (m_2, \vec{0}); \quad \mathcal{P}_1' = (E_1', \vec{p}_1'), \quad \mathcal{P}_2' = (E_2', -\vec{p}_2')
\]

From the energy and momentum conservation in the lab frame, we have

\[
\mathcal{P}_1 + \mathcal{P}_2 = \mathcal{P}_3 + \mathcal{P}_4
\]

The total center-of-mass energy \( W \):

\[
W^2 = (E_1 + E_2)^2 = (E_1' + E_2')^2 - (p_1' - p_2')^2 = (\mathcal{P}_1' \cdot \mathcal{P}_2')^2
\]

Now note \((\mathcal{P}_1' + \mathcal{P}_2')^2 \) is Lorentz invariant, we have

\[
W^2 = (\mathcal{P}_1' + \mathcal{P}_2')^2 = (\mathcal{P}_1 + \mathcal{P}_2)^2 = \mathcal{P}_1^2 + \mathcal{P}_2^2 + 2\mathcal{P}_1 \cdot \mathcal{P}_2 = m_1^2 + m_2^2 + 2m_2E_1
\]

To find \( \vec{p}' \), we consider \((\mathcal{P}_1 \cdot \mathcal{P}_2)^2 \) and \((\mathcal{P}_1' \cdot \mathcal{P}_2')^2 \):

\[
(\mathcal{P}_1 \cdot \mathcal{P}_2)^2 = (m_2E_1)^2 = m_2^2(p_1^2 + m_1^2) = m_2^2p^2_1 - m_1^2m_2^2
\]

\[
(\mathcal{P}_1' \cdot \mathcal{P}_2')^2 = (E_1' E_2' + p'^2)^2 = E_1'^2 E_2'^2 + 2E_1' E_2' p'^2 + p'^4
\]

\[
= (p'^2 + m_1^2)(p'^2 + m_2^2) + 2E_1' E_2' p'^2 + p'^4
\]

\[
= 2p'^4 + (m_1^2 + m_2^2)p'^2 + 2E_1' E_2' p'^2 + m_1^2m_2^2
\]

\[
= p'^2(2p'^2 + m_1^2 + m_2^2 + 2E_1' E_2') + m_1^2m_2^2
\]

\[
= p'^2(E_1'^2 + 2E_1' E_2' + E_2'^2) + m_1^2m_2^2 = p'^2W^2 + m_1^2m_2^2
\]

From Lorentz invariance, we have

\[
(\mathcal{P}_1 \cdot \mathcal{P}_2)^2 = (\mathcal{P}_1' \cdot \mathcal{P}_2')^2 \Rightarrow m_2^2p^2_1 = p'^2W^2 \Rightarrow p' = \frac{m_2}{W}p_1
\]

Since \( \vec{p}_1 \) and \( \vec{p}' \) are in the same direction (the Lorentz boost is along \( \vec{p}_1 \)), therefore we have

\[
\vec{p}' = \frac{m_2}{W}\vec{p}_1
\]

(b) We can also obtain \( \vec{p}' \) from Lorentz transformation of \( \vec{p}_1 \) (and \( -\vec{p}' \) from \( \vec{p}_2 \)):

\[
p' = \gamma_{cm}(p_1 - \beta_{cm}E_1); \quad (-\vec{p}') = \gamma_{cm}(-\beta_{cm}m_2)
\]

Thus

\[
\beta_{cm} = \frac{p_1}{m_2 + E_1}, \quad \Rightarrow \quad \beta_{cm} = \frac{\vec{p}_1}{m_2 + E_1}
\]

\[
\gamma_{cm} = \frac{1}{\sqrt{1 - \beta_{cm}^2}} = \frac{m_2 + E_1}{\sqrt{(m_2 + E_1)^2 - p_1^2}} = \frac{m_2 + E_1}{\sqrt{m_2^2 + 2m_2E_1 + E_1^2 - p_1^2}} = \frac{m_2 + E_1}{W}
\]
(c) In the non-relativistic limit,

\[ E_1 \approx m_1 + \frac{p_1^2}{2m_1} \]

therefore,

\[ W^2 \approx m_1^2 + m_2^2 + 2m_2(m_1 + \frac{p_1^2}{2m_1}) = (m_1 + m_2)^2 + \frac{m_2}{m_1}p_1^2 = (m_1 + m_2)^2 \left\{ 1 + \frac{m_2}{m_1} \frac{p_1^2}{2m_1} \right\} \]

\[ W = (m_1 + m_2) \sqrt{1 + \frac{m_2}{m_1} \frac{p_1^2}{2m_1}} \approx (m_1 + m_2) \left\{ 1 + \frac{m_2}{m_1} \frac{p_1^2}{2m_1} \right\} = m_1 + m_2 - \frac{m_2}{m_1 + m_2} \frac{p_1^2}{2m_1} \]

Similarly

\[ \vec{p}' = \frac{m_2}{W} \vec{p}_1 \approx \frac{m_2}{m_1 + m_2} \vec{p}_1 \]

\[ \vec{\beta}_{cm} = \frac{\vec{p}_1 + E_1}{m_2 + E_1} \approx \frac{\vec{p}_1}{m_1 + m_2} \]

These are the familiar Galilean relativity results.

**Problem 12.2**

(a) Let the Lagrangian \( L \) be replaced by

\[ L' = L + \frac{d}{dt} \Omega(x_\alpha), \]

with \( \Omega \) a given function of the coordinates \( x_\alpha \). The action is

\[ A = \int_{t_1}^{t_2} L \, dt, \quad \Rightarrow \quad A' = \int_{t_1}^{t_2} L \, dt + \int_{t_1}^{t_2} \frac{d\Omega}{dt} \, dt = A + \Omega(x_\alpha) \big|_{t_1}^{t_2} \]

The variation of the action

\[ \delta A' = \delta A + \delta \left\{ \Omega(x_\alpha) \big|_{t_1}^{t_2} \right\} = \delta A \]

since, under the variation of the paths \( x_\alpha(t) \), the end points remain fixed. Thus \( L \) and \( L' \) yield the same Euler-Lagrange equations.

(b)

\[ A^\alpha \rightarrow A^\alpha + \frac{\partial A}{\partial x_\alpha} \]

The Lagrangian is

\[ L = -mc^2 \sqrt{1 - \frac{u^2}{c^2}} + \frac{e}{c} \vec{u} \cdot \vec{A} - e \Phi \]

Under the gauge transformation, we have

\[ \vec{A} \rightarrow \vec{A} - \nabla \Lambda, \quad \Phi \rightarrow \Phi + \frac{1}{c} \frac{\partial A}{\partial t} \]

then

\[ L \rightarrow L' = -mc^2 \sqrt{1 - \frac{u^2}{c^2}} + \frac{e}{c} \vec{u} \cdot \vec{A} - e e \Phi - \frac{e}{c} \vec{u} \cdot \nabla \Lambda - \frac{e}{c} \frac{\partial A}{\partial t} \]

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\[
= -mc^2 \sqrt{1 - \frac{\vec{u}^2}{c^2} + \frac{e}{c} \vec{u} \cdot \vec{A} - e \Phi - \frac{e}{c} \left( \frac{\partial}{\partial t} + \vec{u} \cdot \nabla \right) \vec{A}} = L - \frac{e}{c} \frac{dA}{dt}
\]

Since \( L' \) and \( L \) differ by a total time derivative, the two Lagrangians yield the same equations of motion.

**Problem 12.3**

(a) The motion of the particle is governed by Eq. (11.144):

\[
\frac{dU^\alpha}{d\tau} = \frac{e}{mc} \Gamma^{\alpha \beta U_\beta}
\]

Rewriting it in terms of familiar particle velocity \( \vec{v} \) and electric field \( \vec{E} \), we get

\[
\frac{d(\gamma \vec{c})}{d\tau} = \frac{e \vec{E}}{mc} \frac{\gamma \vec{c}}{\vec{v} \vec{v}} \quad \frac{d(\gamma \vec{v})}{d\tau} = \frac{e \vec{E}}{m} \frac{\gamma \vec{c}}{\vec{v} \vec{v}}
\]

Let \( \eta = eE/mc, v_{\parallel} \) and \( v_{\perp} \) be the parallel and perpendicular components of the velocity defined by the direction of \( \vec{E} \), we then have the following three equations:

\[
\frac{d(\gamma \vec{c})}{d\tau} = \frac{e \vec{E}}{mc} (\gamma v_{\parallel}) = \eta(\gamma v_{\parallel}); \quad \frac{d(\gamma v_{\parallel})}{d\tau} = \frac{e \vec{E}}{mc} (\gamma c) = \eta(\gamma c); \quad \frac{d(\gamma v_{\perp})}{d\tau} = 0
\]

Integrating the last equation, \( \gamma v_{\perp} = \text{constant} = \alpha \), thus \( v_{\perp} = \alpha/\gamma \). From the remaining equations, we get

\[
\frac{d^2}{d\tau^2}(\gamma v_{\parallel}) = \eta^2(\gamma v_{\parallel}) \Rightarrow \gamma v_{\parallel} = A \sinh(\eta \tau) + B \cosh(\eta \tau)
\]

\[
\frac{d^2}{d\tau^2}(\gamma c) = \eta^2(\gamma c) \Rightarrow \gamma c = A \cosh(\eta \tau) + B \sinh(\eta \tau)
\]

where \( A \) and \( B \) are the same constants due to \( d(\gamma v_{\parallel})/d\tau = \eta(\gamma c) \). The three constants \( (\alpha, A, B) \) are determined by the initial condition:

At \( \tau = 0 \): \( v_{\parallel} = 0, \ v_{\perp} = v_0 \ \Rightarrow \ \alpha = \gamma_0 v_0, \ A = \gamma_0 c, \ B = 0 \)

where \( \gamma_0 = 1/\sqrt{1 - v_0^2/c^2} \). Thus

\[
\gamma = \gamma_0 \cosh(\eta \tau); \quad v_{\parallel} = c \tanh(\eta \tau); \quad v_{\perp} = \frac{v_0}{\cosh(\eta \tau)}
\]

These results are expressed in terms of proper time. To rewrite them as functions of laboratory time, we use \( dt = \gamma d\tau \):

\[
t = \int_0^\tau \gamma d\tau = \int_0^\tau \gamma_0 \cosh(\eta \tau) d\tau = \frac{\gamma_0}{\eta} \sinh(\eta \tau)
\]

Thus

\[
\sinh(\eta \tau) = \frac{\eta t}{\gamma_0}; \quad \cosh(\eta \tau) = \sqrt{1 + \frac{\eta^2 t^2}{\gamma_0^2}}; \quad \tanh(\eta \tau) = \frac{\eta t}{\gamma_0} \sqrt{1 + \frac{\eta^2 t^2}{\gamma_0^2}}
\]

Therefore,

\[
\gamma = \gamma_0 \sqrt{1 + \frac{\eta^2 t^2}{\gamma_0^2}}; \quad v_{\parallel} = \frac{\eta ct}{\gamma_0 \sqrt{1 + \eta^2 t^2/\gamma_0^2}}, \quad v_{\perp} = \frac{v_0}{\sqrt{1 + \eta^2 t^2/\gamma_0^2}}
\]

In the coordinate system defined by \( \vec{E} = E \hat{z} \) and \( v_0 = v_0 \hat{x} \), and assuming the particle is at the origin initially, the the position of the particle is given by

\[
x = \int_0^t v_{\perp} dt = \int_0^t \frac{v_0 dt}{\sqrt{1 + \eta^2 t^2/\gamma_0^2}} = \frac{v_0 \gamma_0}{\eta} \sinh^{-1} \left( \frac{\eta t}{\gamma_0} \right)
\]
\[ z = \int_0^t \eta \, dt = \int_0^t \frac{\eta \, dt}{\gamma_0 \sqrt{1 + \frac{\eta^2 t^2}{\gamma_0^2}}} = \frac{c \gamma_0}{\eta} \left\{ \sqrt{1 + \frac{\eta^2 t^2}{\gamma_0^2}} - 1 \right\} \]

We could also get this result by starting from the Lorentz force equations (11.124):

\[ \frac{d\vec{p}}{dt} = e \vec{E} = e E \hat{z}, \quad \Rightarrow \quad \vec{p} = \vec{p}_0 + e E t \hat{z} \]

In perpendicular and parallel components:

\[ p_\parallel = \gamma m v_\parallel = e E t, \quad p_\perp = \gamma m v_\perp = \gamma_0 m v_0 \]

Then

\[ \gamma v_\parallel = \frac{e E}{m} t = \eta c t, \quad \gamma v_\perp = \gamma_0 v_0 \]

so

\[ \gamma^2 v^2 = \gamma_0^2 v_0^2 + \eta^2 c^2 t^2 \quad \Rightarrow \quad v = \frac{\gamma_0 v_0 + \eta c^2 t^2}{\gamma_0^2 - \eta^2 c^2 t^2} \quad \text{or} \quad \gamma = \sqrt{\gamma_0^2 + \eta^2 c^2 t^2} \]

(b) To determine the trajectory, we need to eliminate the time-dependence. From the equation for \( x \), we get

\[ \frac{\eta^2}{\gamma_0} = \frac{\gamma_0}{\eta} \sinh\left( \frac{\eta \tau}{\gamma_0 v_0} \right) \]

Plugging it into the equation for \( z \):

\[ z = \frac{c \gamma_0}{\eta} \left\{ \sqrt{1 + \sinh^2 \left( \frac{\eta \tau}{\gamma_0 v_0} \right)} - 1 \right\} = \frac{c \gamma_0}{\eta} \left\{ \cosh \left( \frac{\eta \tau}{\gamma_0 v_0} \right) - 1 \right\} \]

For \( t \ll \gamma_0/\eta \) \( \text{i.e.} \ x \ll \eta/(\gamma_0 v_0) \):

\[ \cosh \left( \frac{\eta \tau}{\gamma_0 v_0} \right) \approx 1 + \frac{1}{2} \left( \frac{\eta \tau}{\gamma_0 v_0} \right)^2 \]

\[ z \approx \frac{c \gamma_0}{\eta} \left( \frac{\eta^2 x^2}{2 \gamma_0 v_0} \right) = \frac{1}{2} \frac{\eta c}{\gamma_0 v_0} x^2 \]

It is a parabola. In terms of \( t \), we have

\[ x \approx v_0 t, \quad z \approx \frac{c \eta^2}{2 \gamma_0} t^2 = \frac{e E}{2m \gamma_0^2} t^2 \]

For \( t \gg \gamma_0/\eta \):

\[ x \approx \frac{\gamma_0 v_0}{\eta} \ln \left( \frac{2 \eta t}{\gamma_0} \right), \quad z \approx c t \]

Eliminating \( t \):

\[ z \approx \frac{c \gamma_0}{2 \eta} e^{\eta \gamma_0 v_0} \]

The particle moves along the \( z \)-direction with a speed close to \( c \) with a gradual motion in \( x \)-direction.

**Problem 12.6(b)**

Choose the \( z \)-axis along the \( \vec{E} \) and \( \vec{B} \) direction, we have

\[ F^{03} = -E, \quad F^{12} = -B, \quad F^{21} = B, \quad F^{30} = E, \quad \text{and the rest} \quad F^{\alpha \beta} = 0 \]
The equation:
\[ \frac{dU^\alpha}{d\tau} = \frac{e}{mc} F^{\alpha \beta} U^\beta \]
becomes
\[ \frac{dU^0}{d\tau} = -\frac{eE}{mc} U_3, \quad \frac{dU^1}{d\tau} = -\frac{eB}{mc} U_2, \quad \frac{dU^2}{d\tau} = \frac{eB}{mc} U_1, \quad \frac{dU^3}{d\tau} = \frac{eE}{mc} U_0 \]
Use \( U^\alpha = dx^\alpha / d\tau \), the above four equations become to:
\[ \frac{d^2 x}{dt^2} = -\omega^2 x, \quad \frac{d^2 y}{dt^2} = -\omega^2 y \quad \Rightarrow \quad x \sim \sin(\omega \tau), \quad y \sim \cos(\omega \tau) \text{ (by an appropriate choice of axes)} \]
Note that
\[ x \frac{dx}{d\tau} + y \frac{dy}{d\tau} = 0 \quad \Rightarrow \quad x^2 + y^2 = \text{constant} \equiv A^2 R^2 \]
Therefore,
\[ x = AR \sin \phi, \quad y = AR \cos \phi \quad \text{with} \quad \phi = \omega \tau \]
Also
\[ \frac{d^2 z}{dt^2} = \eta^2 z, \quad \frac{d^2 (ct)}{dt^2} = \eta^2 (ct) \quad \Rightarrow \quad z \sim \cosh(\eta \tau), \quad ct \sim \sinh(\eta \tau) \]
Note that
\[ ct \frac{d}{d\tau}(ct) - z \frac{dz}{d\tau} = 0 \quad \Rightarrow \quad z^2 - c^2 t^2 = \text{constant} \equiv B^2 \]
Therefore,
\[ z = B \cosh(\rho \phi), \quad ct = B \sinh(\rho \phi) \quad \text{with} \quad \rho \phi = \eta \tau (\rho = \frac{\eta}{\omega} = \frac{E}{B}) \]
Thus the position and velocity 4-vectors are
\[ x^\alpha = (ct, x, y, z) = (B \cosh(\rho \phi), AR \sin \phi, AR \cos \phi, B \sinh(\rho \phi)) \]
\[ U^\alpha = (\eta z, \omega y, -\omega x, \eta (ct)) = (Bq \cosh(\rho \phi), A \omega R \cos \phi, -A \omega R \sin \phi, B \eta \sinh(\rho \phi)) \]
From \( U^\alpha U_\alpha = c^2 \), we get
\[ B^2 \eta^2 \cosh^2(\rho \phi) - A^2 \omega^2 R^2 \cos^2 \phi - A^2 \omega^2 R^2 \sin^2 \phi - B^2 \eta^2 \sinh^2(\rho \phi) = c^2 \quad \Rightarrow \quad B^2 \eta^2 - \omega^2 A^2 R^2 = c^2 \]
which leads to
\[ B = \frac{1}{\eta} \sqrt{c^2 + \omega^2 A^2 R^2} = \frac{\omega R}{\eta} \sqrt{A^2 + \frac{c^2}{\omega^2 R^2}} = \frac{R}{\rho} \sqrt{1 + A^2} \]
Therefore we have
\[ x = AR \sin \phi, \quad y = AR \cos \phi, \quad z = \frac{R}{\rho} \sqrt{1 + A^2} \cosh(\rho \phi), \quad ct = \frac{R}{\rho} \sqrt{1 + A^2} \sinh(\rho \phi) \]