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Physics 506: Solutions to Assignment #8

Problem 11.23

(a) Let $\mathcal{P}$ and $\mathcal{P}'$ be 4-vectors in lab and CM frame respectively, then we have

$$\mathcal{P}_1 = (E_1, \vec{p}_{\rm LAB}), \quad \mathcal{P}_2 = (m_2, \vec{0}); \qquad \mathcal{P}'_1 = (E'_1, \vec{p'}), \quad \mathcal{P}'_2 = (E'_2, -\vec{p'})$$

From the energy and momentum conservation in the lab frame, we have

$$\mathcal{P}_1 + \mathcal{P}_2 = \mathcal{P}_3 + \mathcal{P}_4$$

The total center-of-mass energy W:

$$W^2 = (E'_1 + E'_2)^2 = (E'_1 + E'_2)^2 - (\vec{p'}_1 + \vec{p'}_2)^2 = (\mathcal{P}'_1 + \mathcal{P}'_2)^2$$

Now note $(\mathcal{P}'_1 + \mathcal{P}'_2)^2$ is Lorentz invariant, we have

$$W^2 = (\mathcal{P}'_1 + \mathcal{P}'_2)^2 = (\mathcal{P}_1 + \mathcal{P}_2)^2 = \mathcal{P}_1^2 + \mathcal{P}_2^2 + 2\mathcal{P}_1 \cdot \mathcal{P}_2 = m_1^2 + m_2^2 + 2m_2E_1$$

To find $\vec{p'}$, we consider $(\mathcal{P}_1 \cdot \mathcal{P}_2)^2$ and $(\mathcal{P}'_1 \cdot \mathcal{P}'_2)^2$:

$$(\mathcal{P}_1 \cdot \mathcal{P}_2)^2 = (m_2E_1)^2 = m_2^2(p_1^2 + m_1^2) = m_2^2p_1^2 + m_1^2m_2^2$$

$$(\mathcal{P}'_1 \cdot \mathcal{P}'_2)^2 = (E'_1E'_2 + p'^2)^2 = E_1'^2E_2'^2 + 2E'_1E'_2p'^2 + p'^4$$

$$= (p'^2 + m_1^2)(p'^2 + m_2^2) + 2E'_1E'_2p'^2 + p'^4$$

$$= 2p'^4 + (m_1^2 + m_2^2)p'^2 + 2E'_1E'_2p'^2 + m_1^2m_2^2$$

$$= p'^2(2p'^2 + m_1^2 + m_2^2 + 2E'_1E'_2) + m_1^2m_2^2$$

$$= p'^2(E_1'^2 + 2E'_1E'_2 + E_2'^2) + m_1^2m_2^2 = p'^2W^2 + m_1^2m_2^2$$

From Lorentz invariance, we have

$$(\mathcal{P}_1 \cdot \mathcal{P}_2)^2 = (\mathcal{P}'_1 \cdot \mathcal{P}'_2)^2 \quad \Rightarrow \quad m_2^2p_1^2 = p'^2W^2 \quad \Rightarrow p' = \frac{m_2}{W}p_1$$

Since $\vec{p}_1$ and $\vec{p'}$ are in the same direction (the Lorentz boost is along $\vec{p}_1$), therefore we have

$$\vec{p'} = \frac{m_2}{W}\vec{p}_1$$

(b) We can also obtain $\vec{p'}$ from Lorentz transformation of $\vec{p}_1$ (and $-\vec{p'}$ from $\vec{p}_2$):

$$p' = \gamma_{\rm cm}(p_1 - \beta_{\rm cm}E_1); \quad (-p') = \gamma_{\rm cm}(-\beta_{\rm cm}m_2)$$

Thus

$$\beta_{\rm cm} = \frac{p_1}{m_2 + E_1}, \quad \Rightarrow \quad \vec{\beta}_{\rm cm} = \frac{\vec{p}_1}{m_2 + E_1}$$

$$\gamma_{\rm cm} = \frac{1}{\sqrt{1 - \beta_{\rm cm}^2}} = \frac{m_2 + E_1}{\sqrt{(m_2 + E_1)^2 - p_1^2}} = \frac{m_2 + E_1}{\sqrt{m_2^2 + 2m_2E_1 + E_1^2 - p_1^2}} = \frac{m_2 + E_1}{W}$$

(c) In the non-relativistic limit,

$$E_1 \approx m_1 + \frac{p_1^2}{2m_1}$$

therefore,

$$W^2 \approx m_1^2 + m_2^2 + 2m_2(m_1 + \frac{p_1^2}{2m_1}) = (m_1 + m_2)^2 + \frac{m_2}{m_1} p_1^2 = (m_1 + m_2)^2 \left\{ 1 + \frac{m_2}{(m_1 + m_2)^2} \frac{p_1^2}{m_1} \right\}$$

$$W = (m_1 + m_2) \sqrt{1 + \frac{m_2}{(m_1 + m_2)^2} \frac{p_1^2}{m_1}} \approx (m_1 + m_2) \left\{ 1 + \frac{m_2}{(m_1 + m_2)^2} \frac{p_1^2}{2m_1} \right\} = m_1 + m_2 + \frac{m_2}{m_1 + m_2} \frac{p_1^2}{2m_1}$$

Similarly

$$\vec{p'} = \frac{m_2}{W} \vec{p}_1 \approx \frac{m_2}{m_1 + m_2} \vec{p}_1$$

$$\vec{\beta}_{\text{cm}} = \frac{\vec{p}_1}{m_2 + E_1} \approx \frac{\vec{p}_1}{m_1 + m_2}$$

These are the familiar Galilean relativity results.

Problem 12.2

(a) Let the Lagrangian L be replaced by

$$L' = L + \frac{d}{dt} \Omega(x_\alpha),$$

with Ω a given function of the coordinates x_α. The action is

$$A = \int_{t_1}^{t_2} L dt, \qquad \Rightarrow \qquad A' = \int_{t_1}^{t_2} L dt + \int_{t_1}^{t_2} \frac{d\Omega}{dt} dt = A + \Omega(x_\alpha)|_{t_1}^{t_2}$$

The variation of the action

$$\delta A' = \delta A + \delta \left\{ \Omega(x_\alpha)|_{t_1}^{t_2} \right\} = \delta A$$

since, under the variation of the paths $x_\alpha(t)$, the end points remain fixed. Thus L and L' yield the same Euler-Lagrange equations.

(b)

$$A^\alpha \to A^\alpha + \frac{\partial \Lambda}{\partial x_\alpha}$$

The Lagrangian is

$$L = -mc^2 \sqrt{1 - \frac{u^2}{c^2}} + \frac{e}{c} \vec{u} \cdot \vec{A} - e\Phi$$

Under the gauge transformation, we have

$$\vec{A} \to \vec{A} - \nabla \Lambda, \qquad \Phi \to \Phi + \frac{1}{c} \frac{\partial \Lambda}{\partial t}$$

then

$$L \to L' = -mc^2 \sqrt{1 - \frac{u^2}{c^2}} + \frac{e}{c} \vec{u} \cdot \vec{A} - e\Phi - \frac{e}{c} \vec{u} \cdot \nabla \Lambda - \frac{e}{c} \frac{\partial \Lambda}{\partial t}$$

$$= -mc^2\sqrt{1-\frac{u^2}{c^2}} + \frac{e}{c}\vec{u}\cdot\vec{A} - e\Phi - \frac{e}{c}\left\{\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right\}\Lambda = L - \frac{e}{c}\frac{d\Lambda}{dt}$$

Since L' and L differ by a total time derivative, the two Lagrangians yield the same equations of motion.

Problem 12.3

(a) The motion of the particle is governed by Eq. (11.144):

$$\frac{dU^\alpha}{d\tau} = \frac{e}{mc}F^{\alpha\beta}U_\beta$$

Rewriting it in terms of familiar particle velocity $\vec{v}$ and electric field $\vec{E}$, we get

$$\frac{d(\gamma c)}{d\tau} = \frac{\gamma e}{mc}\vec{E}\cdot\vec{v}; \qquad \frac{d(\gamma\vec{v})}{d\tau} = \frac{\gamma e}{m}\vec{E}$$

Let $\eta \equiv eE/mc$, $v_{||}$ and $v_\perp$ be the parallel and perpendicular components of the velocity defined by the direction of $\vec{E}$, we then have the following three equations:

$$\frac{d(\gamma c)}{d\tau} = \frac{eE}{mc}(\gamma v_{||}) = \eta(\gamma v_{||}); \qquad \frac{d(\gamma v_{||})}{d\tau} = \frac{eE}{mc}(\gamma c) = \eta(\gamma c); \qquad \frac{d(\gamma v_\perp)}{d\tau} = 0$$

Integrating the last equation, $\gamma v_\perp = \text{constant} \equiv \alpha$, thus $v_\perp = \alpha/\gamma$. From the remaining equations, we get

$$\frac{d^2}{d\tau^2}(\gamma v_{||}) = \eta^2(\gamma v_{||}) \qquad \Rightarrow \qquad \gamma v_{||} = A\sinh(\eta\tau) + B\cosh(\eta\tau)$$

$$\frac{d^2}{d\tau^2}(\gamma c) = \eta^2(\gamma c) \qquad \Rightarrow \qquad \gamma c = A\cosh(\eta\tau) + B\sinh(\eta\tau)$$

where A and B are the same constants due to $d(\gamma v_{||})/d\tau = \eta(\gamma c)$. The three constants $(\alpha,\ A,\ B)$ are determined by the initial condition:

$$\text{At } \tau = 0: \ v_{||} = 0, \ \ v_\perp = v_0 \qquad \Rightarrow \qquad \alpha = \gamma_0 v_0, \ A = \gamma_0 c, \ B = 0$$

where $\gamma_0 = 1/\sqrt{1 - v_0^2/c^2}$. Thus

$$\gamma = \gamma_0\cosh(\eta\tau); \qquad v_{||} = c\tanh(\eta\tau); \qquad v_\perp = \frac{v_0}{\cosh(\eta\tau)}$$

These results are expressed in terms of proper time. To rewrite them as functions of laboratory time, we use $dt = \gamma d\tau$:

$$t = \int_0^\tau \gamma d\tau = \int_0^\tau \gamma_0\cosh(\eta\tau)d\tau = \frac{\gamma_0}{\eta}\sinh(\eta\tau)$$

Thus

$$\sinh(\eta\tau) = \frac{\eta t}{\gamma_0}; \qquad \cosh(\eta\tau) = \sqrt{1 + \eta^2 t^2/\gamma_0^2}; \qquad \tanh(\eta\tau) = \frac{\eta t}{\gamma_0}\frac{1}{\sqrt{1+\eta^2 t^2/\gamma_0^2}}$$

Therefore,

$$\gamma = \gamma_0\sqrt{1 + \frac{\eta^2 t^2}{\gamma_0^2}}; \qquad v_{||} = \frac{\eta c t}{\gamma_0\sqrt{1+\eta^2t^2/\gamma_0^2}}, \qquad v_\perp = \frac{v_0}{\sqrt{1+\eta^2 t^2/\gamma_0^2}}$$

In the coordinate system defined by $\vec{E} = E\hat{z}$ and $\vec{v_0} = v_0\hat{x}$, and assuming the particle is at the origin initially, the the position of the particle is given by

$$x = \int_0^t v_\perp dt = \int_0^t \frac{v_0 dt}{\sqrt{1+\eta^2t^2/\gamma_0^2}} = \frac{v_0\gamma_0}{\eta}\sinh^{-1}(\frac{\eta t}{\gamma_0})$$

$$z = \int_0^t v_{||} dt = \int_0^t \frac{\eta c t dt}{\gamma_0 \sqrt{1 + \eta^2 t^2/\gamma_0^2}} = \frac{c\gamma_0}{\eta} \left\{ \sqrt{1 + \frac{\eta^2 t^2}{\gamma_0^2}} - 1 \right\}$$

We could also get this result by starging from the Lorentz force equations (11.124):

$$\frac{d\vec{p}}{dt} = e\vec{E} = eE\hat{z}, \quad \Rightarrow \quad \vec{p} = \vec{p}_0 + eEt\hat{z}$$

In perpendicular and parallel components:

$$p_{||} = \gamma m v_{||} = eEt, \quad p_{\perp} = \gamma m v_{\perp} = \gamma_0 m v_0$$

Then

$$\gamma v_{||} = \frac{eE}{m} t = \eta c t, \quad \gamma v_{\perp} = \gamma_0 v_0$$

so

$$\gamma^2 v^2 = \gamma_0^2 v_0^2 + \eta^2 c^2 t^2 \quad \Rightarrow \quad v^2 = \frac{\gamma_0^2 v_0^2 + \eta^2 c^2 t^2}{\gamma_0^2 + \eta^2 t^2} \text{ or } \gamma = \sqrt{\gamma_0^2 + \eta^2 t^2}$$

(b) To determine the trajectory, we need to eliminate the time-dependence. From the equation for x, we get

$$\frac{\eta t}{\gamma_0} = \frac{\gamma_0}{\eta} \sinh\left(\frac{\eta x}{\gamma_0 v_0}\right)$$

Pluggint it into the equation for z:

$$z = \frac{c\gamma_0}{\eta} \left\{ \sqrt{1 + \sinh^2\left(\frac{\eta x}{\gamma_0 v_0}\right)} - 1 \right\} = \frac{c\gamma_0}{\eta} \left\{ \cosh\left(\frac{\eta x}{\gamma_0 v_0}\right) - 1 \right\}$$

For $t \ll \gamma_0/\eta$ (*i.e.* $x \ll \eta/(\gamma_0 v_0)$):

$$\cosh\left(\frac{\eta x}{\gamma_0 v_0}\right) \approx 1 + \frac{1}{2}\left(\frac{\eta x}{\gamma_0 v_0}\right)^2$$

$$z \approx \frac{c\gamma_0}{\eta}\left(\frac{\eta^2 x^2}{2\gamma_0^2 v_0^2}\right) = \frac{1}{2}\frac{\eta c}{\gamma_0 v_0^2} x^2$$

It is a parabola. In terms of t, we have

$$x \approx v_0 t, \quad z \approx \frac{c\eta}{2\gamma_0} t^2 = \frac{eE}{2m\gamma_0} t^2$$

For $t \gg \gamma_0/\eta$:

$$x \approx \frac{\gamma_0 v_0}{\eta} \ln\left(\frac{2\eta t}{\gamma_0}\right), \quad z \approx ct$$

Eliminating t:

$$z \approx \frac{c\gamma_0}{2\eta} e^{\eta x} \gamma_0 v_0$$

The particle moves along the $z-$direction with a speed close to c with a gradual motion in $x-$direction.

Problem 12.6(b)

Choose the $z-$axis along the $\vec{E}$ and $\vec{B}$ direction, we have

$$F^{03} = -E, \; F^{12} = -B, \; F^{21} = B, \; F^{30} = E, \text{ and the rest } F^{\alpha\beta} = 0$$

The equation:

$$\frac{dU^\alpha}{d\tau} = \frac{e}{mc} F^{\alpha\beta} U_\beta$$

becomes

$$\frac{dU^0}{d\tau} = -\frac{eE}{mc}U_3, \quad \frac{dU^1}{d\tau} = -\frac{eB}{mc}U_2, \quad \frac{dU^2}{d\tau} = \frac{eB}{mc}U_1, \quad \frac{dU^3}{d\tau} = \frac{eE}{mc}U_0$$

Use $U^\alpha = dx^\alpha/d\tau$, the above four equations become to:

$$\frac{d^2(ct)}{d\tau^2} = \frac{eE}{mc}\frac{dz}{d\tau}, \quad \frac{d^2x}{d\tau^2} = \frac{eB}{mc}\frac{dy}{d\tau}, \quad \frac{d^2y}{d\tau^2} = -\frac{eB}{mc}\frac{dx}{d\tau}, \quad \frac{d^2z}{d\tau^2} = \frac{eE}{mc}\frac{d(ct)}{d\tau}$$

Integrating over proper time,

$$\frac{d(ct)}{d\tau} = \frac{eE}{mc}z, \quad \frac{dx}{d\tau} = \frac{eB}{mc}y, \quad \frac{dy}{d\tau} = -\frac{eB}{mc}x, \quad \frac{dz}{d\tau} = \frac{eE}{mc}(ct)$$

Let $\omega \equiv eB/mc$ and $\eta \equiv eE/mc$, the second and the third equations are coupled and can be solved

$$\frac{d^2x}{d\tau^2} = -\omega^2 x, \ \frac{d^2y}{d\tau^2} = -\omega^2 y \quad \Rightarrow \quad x \sim \sin(\omega\tau), \ \ y \sim \cos(\omega\tau) \text{ (by an appropriate choice of axes)}$$

Note that

$$x\frac{dx}{d\tau} + y\frac{dy}{d\tau} = 0 \quad \Rightarrow \quad x^2 + y^2 = \text{constant} \equiv \mathcal{A}^2 R^2$$

Therefore,

$$x = \mathcal{A}R\sin\phi, \quad y = \mathcal{A}R\cos\phi \quad \text{with } \phi = \omega\tau$$

Also

$$\frac{d^2z}{d\tau^2} = \eta^2 z, \ \ \frac{d^2(ct)}{d\tau^2} = \eta^2(ct) \quad \Rightarrow \quad z \sim \cosh(\eta\tau), \ \ ct \sim \sinh(\eta\tau)$$

Note that

$$ct\frac{d}{d\tau}(ct) - z\frac{d}{d\tau}z = 0 \quad \Rightarrow \quad z^2 - c^2t^2 = \text{constant} \equiv \mathcal{B}^2$$

Therefore,

$$z = \mathcal{B}\cosh(\rho\phi), \quad ct = \mathcal{B}\sinh(\rho\phi) \quad \text{with} \ \ \rho\phi = \eta\tau \ \ (\rho = \frac{\eta}{\omega} = \frac{E}{B})$$

Thus the position and velocity 4-vectors are

$$x^\alpha = (ct, x, y, z) = (\mathcal{B}\sinh(\rho\phi), \mathcal{A}R\sin\phi, \mathcal{A}R\cos\phi, \mathcal{B}\cosh(\rho\phi))$$

$$U^\alpha = (\eta z, \omega y, -\omega x, \eta(ct)) = (\mathcal{B}\eta\cosh(\rho\phi), \mathcal{A}\omega R\cos\phi, -\mathcal{A}\omega R\sin\phi, \mathcal{B}\eta\sinh(\rho\phi))$$

From $U^\alpha U_\alpha = c^2$, we get

$$\mathcal{B}^2\eta^2\cosh^2(\rho\phi) - \mathcal{A}^2\omega^2R^2\cos^2\phi - \mathcal{A}^2\omega^2R^2\sin^2\phi - \mathcal{B}^2\eta^2\sinh^2(\rho\phi) = c^2 \quad \Rightarrow \quad \mathcal{B}^2\eta^2 - \omega^2\mathcal{A}^2R^2 = c^2$$

which leads to

$$\mathcal{B} = \frac{1}{\eta}\sqrt{c^2 + \omega^2\mathcal{A}^2R^2} = \frac{\omega R}{\eta}\sqrt{\mathcal{A}^2 + \frac{c^2}{\omega^2R^2}} = \frac{R}{\rho}\sqrt{1 + \mathcal{A}^2}$$

Therefore we have

$$x = \mathcal{A}R\sin\phi, \quad y = \mathcal{A}R\cos\phi, \quad z = \frac{R}{\rho}\sqrt{1+\mathcal{A}^2}\cosh(\rho\phi), \quad ct = \frac{R}{\rho}\sqrt{1+\mathcal{A}^2}\sinh(\rho\phi)$$