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Physics 506: Solutions to Assignment #7

Problem 11.6

Let $v(t)$ be the instantaneous velocity of the rocket with respect to the earth. At a given time t, consider the rocket's motion in an inertial frame moving with (constant) velocity $v(t)$ with respect to the earth.

(a) The rocket's velocity in this frame is $u' = 0$, while its acceleration is $a'_{||} = g$ and $a'_{\perp} = 0$. Then by the Problem 11.5, we know that an observer in the earth's frame would see the rocket to have an acceleration

$$a_{||} = (1 - \frac{v(t)^2}{c^2})^{3/2} g$$

such an observer measures the acceleration by using

$$a_{||} = \frac{dv(t)}{dt} = (1 - \frac{v(t)^2}{c^2})^{3/2} g$$

Therefore $v(t)$ can be solved from the above differential equation. The initial condition for the 1st part of the journey (the five years of acceleration) is $v(0) = 0$:

$$\int_0^v \frac{dv}{(1 - \frac{v^2}{c^2})^{3/2}} = \int_0^t g dt \quad \Rightarrow \quad gt = \frac{v}{(1 - v^2/c^2)^{1/2}} \quad \Rightarrow \quad v^2 = \frac{g^2 t^2}{1 + g^2 t^2/c^2}$$

Thus

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \sqrt{1 + \frac{g^2 t^2}{c^2}}$$

The relationship between dt in the earth's frame and dt' in the instantaneous rest frame is

$$dt = \gamma(v) \left\{ dt' + \frac{\vec{v} \cdot d\vec{r'}}{c^2} \right\}$$

But in the instantaneous rest frame, $\vec{u'} = 0$, which leads to $d\vec{r'} = 0$. Therefore $dt = \gamma(v) dt'$. Thus

$$t' = \int \frac{dt}{\gamma(t)} = \int \frac{dt}{\sqrt{1 + g^2 t^2/c^2}} = \frac{c}{g} \sinh^{-1}(\frac{gt}{c}), \quad \Rightarrow \quad t = \frac{c}{g} \sinh(\frac{gt'}{c})$$

For the first leg of the journey

$$t \approx \frac{3 \cdot 10^8}{10} \sinh(\frac{3 \cdot 10^8 \times 3 \cdot 10^7}{10}) \approx 75 \times (3 \cdot 10^7) \text{ s} \approx 75 \text{ years}$$

The total journey takes four times of the first leg:

$$t_{\text{total}} \sim 4 \times 75 \sim 300 \text{ years}$$

Therefore, the year on earth is 2400 when the twin returns to visit his/her sibling's grave.

(b) The furtherest distance the rocket ship traveled

$$s = 2 \int_0^{75\text{years}} v(t) dt = \int \frac{gt}{\sqrt{1 + g^2 t^2/c^2}} = \frac{c^2}{g} \sqrt{1 + \frac{g^2 t^2}{c^2}} \Big|_0^{75 \text{ years}} \approx 148 \text{ light} - \text{years}$$

Problem 11.8

(a) In frame K' in which the fluid is at rest, the frequency and wave vector are related by $ck' = n(\omega')\omega'$ because only in this frame can we define the index of refraction. We assume the speed of light in the fluid is

$$v_p' = \frac{\omega'}{k'} = \frac{c}{n(\omega')}$$

Applying velocity addition formula for parallel velocities, we get

$$v_p \equiv u = \frac{v + v_p'}{1 + vv_p'/c^2} = \frac{c}{n(\omega')} \left\{ \frac{1 + \beta n(\omega')}{1 + \beta/n(\omega')} \right\}$$

where $\beta = v/c$. Expanding in powers of β and keeping only first order in β:

$$u \approx \frac{c}{n(\omega')} + v \left\{ 1 - \frac{1}{n^2(\omega')} \right\} + \mathcal{O}(\beta^2 c)$$

To find the correction for dispersion, we must relate ω' to the lab frequency ω. We now note that both ω'/c and k' are the time and space components of a 4-vector (because the phase of a wave is a Lorentz invariant). Thus

$$\frac{\omega}{c} = \gamma(\frac{\omega'}{c} + \beta k'), \quad \text{and} \quad k = \gamma(k' + \beta\frac{\omega'}{c})$$

Since $ck' = \omega' n(\omega')$, solving ω and ck in terms of ω', we get

$$\omega = \gamma(1 + \beta n(\omega'))\omega', \qquad ck = \gamma(n(\omega') + \beta)\omega'$$

In passing, we note that the lab phase velocity is

$$\frac{\omega}{k} = \frac{c}{n(\omega')} \left\{ \frac{1 + \beta n(\omega')}{1 + \beta/n(\omega')} \right\}$$

just as we found above from the velocity addition formula. And the index of refraction for the moving fluid can be defined as

$$n(\beta, \omega) = c\frac{k}{\omega} = \frac{n(\omega') + \beta}{1 + \beta n(\omega')}$$

It depends not only on the frequency, but also on the speed of the fluid.
To the first order in β, we have

$$\omega' \approx \omega(1 - \beta n(\omega)) \qquad \Rightarrow \qquad \omega' - \omega \approx -\beta\omega n(\omega)$$

Taylor expanding $n(\omega')$ at $\omega' = \omega$:

$$n(\omega') = n(\omega) + \frac{dn}{d\omega}(\omega' - \omega) + \mathcal{O}((\omega' - \omega)^2)$$

$$= n(\omega) - \beta\omega n(\omega)\frac{dn}{d\omega} + \mathcal{O}(\beta^2) \approx n(\omega) \left\{ 1 - \beta\omega\frac{dn}{d\omega} \right\}$$

Thus

$$\frac{1}{n(\omega')} \approx \frac{1}{n(\omega)} \left\{ 1 + \beta\omega\frac{dn}{d\omega} \right\}; \qquad \frac{1}{n^2(\omega')} \approx \frac{1}{n^2(\omega)} \left\{ 1 + 2\beta\omega\frac{dn}{d\omega} \right\}$$

The velocity formula therefore becomes

$$u = \frac{c}{n(\omega)} \left\{ 1 + \beta\omega\frac{dn}{d\omega} \right\} + v \left\{ 1 - \frac{1}{n^2(\omega)}(1 + 2\beta\omega\frac{dn}{d\omega}) \right\}$$

$$\approx \frac{c}{n(\omega)} + v\left\{1 - \frac{1}{n^2(\omega)} + \frac{\omega}{n(\omega)}\frac{dn(\omega)}{d\omega}\right\}$$

Similarly the velocity formula for the antiparallel case is

$$u = \frac{c}{n(\omega)} + v\left\{1 + \frac{1}{n^2(\omega)} + \frac{\omega}{n(\omega)}\frac{dn(\omega)}{d\omega}\right\}$$

Problem 11.13

(a) In the wire's rest frame K', the wire has a constant linear charge density q_0. In this frame, the electric and magnetic fields in Gaussian units are given by (in cylindrical coordinates):

$$\vec{E'} = \frac{2q_0}{r}\hat{r} \qquad \vec{B'} = 0$$

Here we used r instead of ρ to denote the polar radius to avoid confusion (see below). Lorentz-transform along the $z-$axis using the inverse of Eq. (11.149) to get the fields in the lab frame ($\vec{\beta} = \frac{v}{c}\hat{z}$):

$$\vec{E} = \gamma(\vec{E'} - \vec{\beta}\times\vec{B'}) - \frac{\gamma^2}{1+\gamma}\vec{\beta}(\vec{\beta}\cdot\vec{E'}) = \gamma\vec{E'} = \frac{2\gamma q_0}{r}\hat{r}$$

$$\vec{B} = \gamma(\vec{B'} + \vec{\beta}\times\vec{E'}) - \frac{\gamma^2}{1+\gamma}\vec{\beta}(\vec{\beta}\cdot\vec{B'}) = \gamma\vec{\beta}\times\vec{E'} = \frac{2\gamma\beta q_0}{r}\hat{\phi}$$

Note that the radial (r) and angular (ϕ) lengths (coordinates) are the same in both frames since the relative motion is in the $z-$direction.

(b) In the rest frame K', the current and charge densities are:

$$\vec{J'} = 0, \qquad \rho' = \frac{q_0}{2\pi r}\delta(r)$$

Note that it is easy to verify that the charge per unit length is q_0. Since $(c\rho', \vec{J'})$ transform as a four-vector, we have in the lab frame:

$$c\rho = \gamma(c\rho' + \beta J'_z) = \gamma c\rho' \qquad \Rightarrow \qquad \rho = \gamma\rho' = \frac{\gamma q_0}{2\pi r}\delta(r)$$

i.e., the line charge density in the lab frame is γq_0, consistent with the Lorentz contraction of the wire in $z-$direction.

$$J_z = \gamma(J'_z + \beta c\rho') = \gamma\beta c\rho' \qquad \Rightarrow \qquad \vec{J} = \frac{\gamma q_0 v}{2\pi r}\delta(r)\hat{z} = \rho v\hat{z} = \rho\vec{v}$$

This is the current density of a line current $\gamma q_0 v$.

(c) An observer in the laboratory frame sees a line charge of density γq_0 and a line current $\gamma q_0 v$. Therefore, the electric and magnetic fields can be readily calculated from Gauss's and Ampere's laws to be:

$$\vec{E} = \frac{2\gamma q_0}{r}\hat{r}; \qquad \vec{B} = \frac{4\pi}{c}\,(\gamma q_0 v)\,\frac{1}{2\pi r}\hat{\phi} = \frac{2\gamma\beta q_0}{r}\hat{\phi}$$

in agreement with those of (a).

Problem 11.16

(a) Since the equation

$$J^\alpha - \frac{1}{c^2}(U_\beta J^\beta)U^\alpha = \frac{\sigma}{c}F^{\alpha\beta}U_\beta$$

is a covariant equation, it is valid in all inertial frames if it is valid in one of them. In the rest frame of the conducting medium, $U^\alpha = (c, \vec{0})$, so that in this frame we have

$$\alpha = 0: \quad c\rho - \frac{1}{c^2}(c\cdot c\rho)c = \frac{\sigma}{c}F^{00}\cdot c \qquad \Rightarrow \qquad c\rho - c\rho = 0$$

$$\alpha = i: \quad J^i - \frac{1}{c^2}(c \cdot c\rho) \cdot 0 = \frac{\sigma}{c} F^{i0} \cdot c \quad \Rightarrow \quad J^i = \sigma E^i$$

The equation gives Ohm's law in the rest frame and therefore valid in all frames.
(b) If the medium has a velocity $\vec{v} = c\vec{\beta}$, then $U^\alpha = \gamma c(1, \vec{\beta})$ and the equation becomes to:

$$\alpha = 0: \quad c\rho - \frac{(\gamma c)^2}{c^2}(c\rho - \vec{\beta} \cdot \vec{J}) = \frac{\sigma}{c} F^{0i} U_i = \gamma\sigma\vec{\beta} \cdot \vec{E} \quad \Rightarrow \quad \gamma^2(c\rho - \vec{\beta} \cdot \vec{J}) = c\rho - \gamma\sigma\vec{\beta} \cdot \vec{E}$$

$$\alpha = i: \quad J^i - \frac{(\gamma c)^2}{c^2}(c\rho - \vec{\beta} \cdot \vec{J})\beta^i = \frac{\sigma}{c}(F^{i0}\gamma c - \sum_j F^{ij}\gamma c \beta^j) \quad \Rightarrow \quad \vec{J} - \gamma^2(c\rho - \vec{\beta} \cdot \vec{J})\vec{\beta} = \gamma\sigma(\vec{E} + \vec{\beta} \times \vec{B})$$

Here we have used the following identity:

$$\sum_j F^{1j}\beta^j = F^{12}\beta^2 + F^{13}\beta^3 = -B_z\beta_y + B_y\beta_z = -(\vec{\beta} \times \vec{B})_x$$

and similar ones for $\sum_j F^{2j}\beta^j$ and $\sum_j F^{3j}\beta^j$. Therefore

$$\vec{J} = \gamma\sigma(\vec{E} + \vec{\beta} \times \vec{B}) + \gamma^2(c\rho - \vec{\beta} \cdot \vec{J})\vec{\beta} = \gamma\sigma(\vec{E} + \vec{\beta} \times \vec{B}) + (c\rho - \gamma\sigma\vec{\beta} \cdot \vec{E})\vec{\beta}$$

$$= \gamma\sigma\left\{\vec{E} + \vec{\beta} \times \vec{B} - \vec{\beta}(\vec{\beta} \cdot \vec{E})\right\} + \rho\vec{v}$$

(c) Since $(c\rho, \vec{J})$ is a four-vector,

$$0 = c\rho' = \gamma(c\rho - \vec{\beta} \cdot \vec{J}) \quad \Rightarrow \quad c\rho = \vec{\beta} \cdot \vec{J} \quad \Rightarrow \quad \rho\vec{v} = \vec{\beta}(\vec{\beta} \cdot \vec{J})$$

Thus Ohm's law generalizes to:

$$\vec{J} = \gamma\sigma\left\{\vec{E} + \vec{\beta} \times \vec{B} - \vec{\beta}(\vec{\beta} \cdot \vec{E})\right\} + \vec{\beta}(\vec{\beta} \cdot \vec{J}) \quad \Rightarrow \quad \vec{J} - \vec{\beta}(\vec{\beta} \cdot \vec{J}) = \gamma\sigma\left\{\vec{E} - \vec{\beta}(\vec{\beta} \cdot \vec{E}) + \vec{\beta} \times \vec{B}\right\}$$