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Physics 506: Solutions to Assignment #6

Problem 10.15

From Prob. 8.2, the TEM fields in this case are

$$\vec{E} = \frac{V}{\ln(b/a)}\frac{\hat{\rho}}{\rho}, \qquad c\vec{B} = \frac{V}{\ln(b/a)}\frac{\hat{\phi}}{\rho}$$

In Kirchhoff approximation, the problem can be simplified as a plane wave incident on a conducting plane sheet with a ring cut out of it. Therefore, the radiated field is given by Eq. (10.109):

$$\vec{E}(\vec{r}) = \frac{ie^{ikr}}{2\pi r}\vec{k}\times\int \hat{z}\times\vec{E}(\vec{r'})e^{-i\vec{k}\cdot\vec{r'}}da' = \frac{ie^{ikr}}{2\pi r}\frac{V}{\ln(b/a)}\vec{k}\times\int\frac{e^{-i\vec{k}\cdot\vec{r'}}}{\rho'}\hat{\phi}'da'$$

Let (θ,ϕ) be the spherical angles of $\vec{k}$, ϕ' be the polar angle of $\vec{r'}$, then

$$\vec{k} = k(\sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}), \quad \vec{r'} = \rho'(\cos\phi'\hat{x} + \sin\phi'\hat{y})$$

$$\vec{k}\cdot\vec{r'} = k\rho'\sin\theta\cos(\phi'-\phi), \qquad \hat{\phi}' = -\sin\phi'\hat{x} + \cos\phi'\hat{y}$$

The electric field

$$\vec{E}(\vec{r}) = \frac{ie^{ikr}}{2\pi r}\frac{V}{\ln(b/a)}\vec{k}\times\int_a^b d\rho'\int_0^{2\pi}d\phi' e^{-ik\rho'\sin\theta\cos(\phi'-\phi)}(-\sin\phi'\hat{x} + \cos\phi'\hat{y})$$

Using the identities

$$\int_0^{2\pi}d\phi e^{i(x\cos\phi - m\phi)} = 2\pi i^m J_m(x); \quad J_{-m}(x) = (-1)^m J_m(x); \quad J_m(-x) = (-1)^m J_m(x)$$

the integrals over ϕ' can be carried out:

$$\int_0^{2\pi} e^{-ik\rho'\sin\theta\cos(\phi'-\phi)}\sin\phi' d\phi' = \int_0^{2\pi} e^{-ik\rho'\sin\theta\cos\phi'}\sin(\phi+\phi')d\phi'$$

$$= \frac{1}{2i}\int_0^{2\pi} e^{-ik\rho'\sin\theta\cos\phi'}\left\{e^{i(\phi+\phi')} - e^{-i(\phi+\phi')}\right\}$$

$$= \frac{1}{2i}\int_0^{2\pi}d\phi'\left\{e^{i\phi}e^{i(-k\rho'\sin\theta\cos\phi'+\phi')} - e^{-i\phi}e^{i(-k\rho'\sin\theta\cos\phi'-\phi')}\right\}$$

$$= \frac{1}{2i}\left\{(2\pi)i^{-1}e^{i\phi}J_{-1}(-k\rho'\sin\theta) - (2\pi)i^{+1}e^{-i\phi}J_1(-k\rho'\sin\theta)\right\}$$

$$= -2\pi i\sin\phi J_1(k\rho'\sin\theta)$$

Similarly

$$\int_0^{2\pi} e^{-ik\rho'\sin\theta\cos(\phi'-\phi)}\cos\phi' d\phi' = \int_0^{2\pi} e^{-ik\rho'\sin\theta\cos\phi'}\cos(\phi+\phi')d\phi'$$

$$= \frac{1}{2}\int_0^{2\pi} e^{-ik\rho' \sin\theta\cos\phi'} \left\{ e^{i(\phi+\phi')} + e^{-i(\phi+\phi')} \right\}$$

$$= \frac{1}{2}\int_0^{2\pi} d\phi' \left\{ e^{i\phi} e^{i(-k\rho'\sin\theta\cos\phi' + \phi')} + e^{-i\phi} e^{i(-k\rho'\sin\theta\cos\phi' - \phi')} \right\}$$

$$= \frac{1}{2}\left\{ (2\pi) i^{-1} e^{i\phi} J_{-1}(-k\rho'\sin\theta) + (2\pi) i^{+1} e^{-i\phi} J_1(-k\rho'\sin\theta) \right\}$$

$$= -2\pi i \cos\phi J_1(k\rho'\sin\theta)$$

Noting that

$$-\sin\phi\hat{x} + \cos\phi\hat{y} = \hat{\phi}, \quad \text{and} \quad \int_a^b J_1(x)dx = \frac{1}{2}(J_0(b) - J_0(a))$$

we get the electric field:

$$\vec{E}(\vec{r}) = \frac{ie^{ikr}}{2\pi r}\frac{V}{\ln(b/a)}\vec{k} \times \int_a^b d\rho'(-2\pi i)J_1(k\rho'\sin\theta)(-\sin\phi\hat{x} + \cos\phi\hat{y})$$

$$= \frac{e^{ikr}}{r}\frac{V}{\ln(b/a)}\vec{k} \times \hat{\phi}\int_a^b d\rho' J_1(k\rho'\sin\theta)$$

$$= \frac{e^{ikr}}{r}\frac{V}{\ln(b/a)}\hat{k} \times \hat{\phi}\frac{J_0(kb\sin\theta) - J_0(ka\sin\theta)}{2\sin\theta}$$

Now that $\hat{k} = \hat{r}$ and thus $\hat{k} \times \hat{\phi} = \hat{r} \times \hat{\phi} = -\hat{\theta}$

$$\vec{E}(\vec{r}) = -\frac{e^{ikr}}{r}\frac{V}{\ln(b/a)}\frac{J_0(kb\sin\theta) - J_0(ka\sin\theta)}{2\sin\theta}\hat{\theta}$$

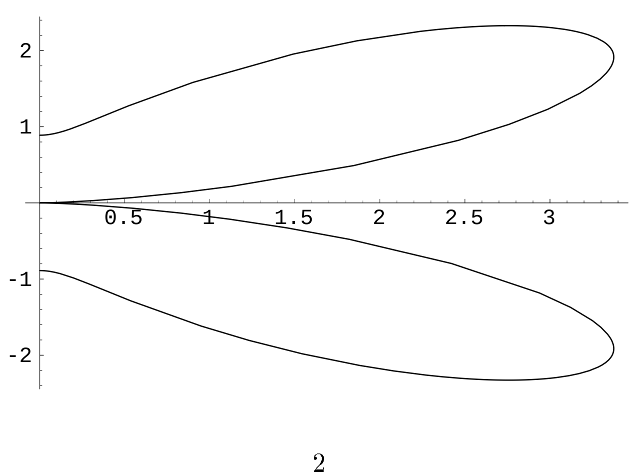
The average Poynting vector

$$\langle\vec{S}\rangle = \frac{|\vec{E}|^2}{2Z_0}\hat{k} = \frac{V^2}{8Z_0\ln^2(b/a)}\frac{\{J_0(kb\sin\theta) - J_0(ka\sin\theta)\}^2}{\sin^2\theta}\frac{\hat{k}}{r^2}$$

The average distributions of the radiation

$$\frac{dP}{d\Omega} = r^2\langle\vec{S}\rangle \cdot \hat{k} = \frac{V^2}{8Z_0\ln^2(b/a)}\frac{\{J_0(kb\sin\theta) - J_0(ka\sin\theta)\}^2}{\sin^2\theta}$$

The distribution for $kb = 4$ and $ka = 1$ in the unit of $V^2/(8Z_0\ln^2(b/a))$ is plotted below. The horizontal axis is the $z-$direction. As expected from the functional form, there is no radiation in the forward direction ($\theta = 0$). For large b/a values, the distribution has many local maxima and minima as θ is varied from 0 to $\pi/2$.



The total power

$$P = \int \frac{dP}{d\Omega} d\Omega = \frac{\pi V^2}{4Z_0 \ln^2(b/a)} \int_0^{\pi/2} \frac{\{J_0(kb\sin\theta) - J_0(ka\sin\theta)\}^2}{\sin\theta} d\theta$$

The integral does not have a simple analytical form and has to be carried out numerically.

Note to the grader: the following discussion should not be part of the grading

Long wavelength limit: $(kb \ll 1)$

$$J_0(kb\sin\theta) \sim 1 - \frac{1}{4}(kb\sin\theta)^2; \qquad J_0(ka\sin\theta) \sim 1 - \frac{1}{4}(ka\sin\theta)^2$$

Thus

$$\frac{dP}{d\Omega} \approx \frac{k^4 V^2 (b^2 - a^2)^2}{128 Z_0 \ln^2(b/a)} \sin^2\theta$$

The total radiated power is then

$$P_{\text{rad.}} = \int \frac{dP}{d\Omega} d\Omega = \frac{k^4 V^2 (b^2 - a^2)^2}{128 Z_0 \ln^2(b/a)} (2\pi) \int_0^{\pi/2} \sin^2\theta d(\cos\theta) = \frac{k^4 V^2 (b^2 - a^2)^2}{96 Z_0 \ln^2(b/a)}$$

This is to be compared with the power flow along an infinite coaxial line:

$$P_{\text{trans.}} = \frac{1}{2} \int (\vec{E} \times \vec{H}) \cdot \hat{z} da = \frac{V^2}{2Z_0 \ln^2(b/a)} \int_0^{2\pi} d\phi \int_a^b \frac{d\rho}{\rho} = \frac{\pi V^2}{Z_0 \ln(b/a)}$$

$$\frac{P_{\text{rad.}}}{P_{\text{trans.}}} = \frac{k^4 (b^2 - a^2)^2}{96\pi \ln(b/a)} \ll 1$$

Therefore, most of the power is reflected back. The fields inside the coaxial cable is very similar to those of an "open" transmission line. Note that in this case, the coaxial cable can only operate in its TEM mode. All other modes are cut off.

Short wavelength limit: $(ka \gg 1)$
The radiation can be appreciable and higher modes are excited. The fields in the plane at $z = 0$ are far from the simple TEM fields. The Smythe-Kirchhoff approximation has only qualitative validity.

Problem 11.3
Let the frame K' be moving with velocity $v_1\hat{z}$ with respect to K, and let K'' be moving with velocity $v_2\hat{z}$ with respect to K'. Then,

$$ct' = \gamma_1(ct - \beta_1 z), \quad z' = \gamma_1(z - \beta_1 ct), \quad x' = x, \quad y' = y$$

$$ct'' = \gamma_2(ct' - \beta_2 z'), \quad z'' = \gamma_2(z' - \beta_2 ct'), \quad x'' = x', \quad y'' = y'$$

with

$$\beta_1 = \frac{v_1}{c}, \quad \gamma_1 = \frac{1}{\sqrt{1-\beta_1^2}}; \qquad \beta_2 = \frac{v_2}{c}, \quad \gamma_2 = \frac{1}{\sqrt{1-\beta_2^2}}$$

Then $x'' = x$, $y'' = y$ and

$$z'' = \gamma_1\gamma_2 \{(z - \beta_1 ct) - \beta_2(ct - \beta_1 z)\} = \gamma_1\gamma_2 \{(1 + \beta_1\beta_2)z - (\beta_1 + \beta_2)ct\}$$

$$ct'' = \gamma_1\gamma_2 \{(ct - \beta_1 z) - \beta_2(z - \beta_1 ct)\} = \gamma_1\gamma_2 \{(1 + \beta_1\beta_2)ct - (\beta_1 + \beta_2)z\}$$

Thus

$$z'' = \gamma(z - \beta ct), \qquad ct'' = \gamma(ct - \beta z)$$

with

$$\gamma = \gamma_1\gamma_2(1+\beta_1\beta_2), \qquad \beta = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}$$

It is easy to show that $\gamma = 1/\sqrt{1-\beta^2}$:

$$\gamma^2 = \gamma_1^2\gamma_2^2(1+\beta_1\beta_2)^2 = \frac{(1+\beta_1\beta_2)^2}{(1+\beta_1\beta_2)^2-(\beta_1+\beta_2)^2} = \left\{1 - \frac{(\beta_1+\beta_2)^2}{(1+\beta_1\beta_2)^2}\right\}^{-1} = \frac{1}{1-\beta^2}$$

Thus two successive Lorentz transformations in the same direction are equivalent to a single Lorentz transformation in that direction with velocity

$$v = c\beta = \frac{v_1+v_2}{1+v_1v_2/c^2}$$

Problem 11.5

Begins with Eq. (11.31):

$$u_{||} = \frac{u'_{||}+v}{1+\frac{v}{c^2}u'_{||}}, \qquad \vec{u}_\perp = \frac{\vec{u}'_\perp}{\gamma_v(1+\frac{v}{c^2}u'_{||})}$$

Note that

$$\frac{d}{dt} = \frac{1}{\gamma_v(1+\frac{v}{c^2}u'_{||})}\frac{d}{dt'}$$

Therefore

$$a_{||} = \frac{1}{\gamma_v(1+\frac{v}{c^2}u'_{||})}\left\{\frac{a'_{||}}{(1+\frac{v}{c^2}u'_{||})} - \frac{u'_{||}+v}{c^2}\frac{va'_{||}}{(1+\frac{v}{c^2}u'_{||})^2}\right\} = \frac{1}{\gamma_v^3}\frac{1}{(1+\vec{v}\cdot\vec{u}'/c^2)^3}a'_{||}$$

Similarly

$$\vec{a}_\perp = \frac{1}{\gamma_v^2(1+\vec{v}\cdot\vec{u}'/c^2)}\left\{\frac{\vec{a'}_\perp}{(1+\vec{v}\cdot\vec{u}'/c^2)} - \frac{u'_\perp v}{c^2}\frac{a'_{||}}{(1+\vec{v}\cdot\vec{u}'/c^2)^2}\right\}$$

$$= \frac{1}{\gamma_v^2(1+\vec{v}\cdot\vec{u}'/c^2)^3}\left\{\vec{a'}_\perp(1+\frac{\vec{v}\cdot\vec{u'}}{c^2}) - \vec{u'}_\perp a'_{||}\frac{v}{c^2}\right\}$$

$$= \frac{1}{\gamma_v^2(1+\vec{v}\cdot\vec{u}'/c^2)^3}\left\{\vec{a'}_\perp + \frac{1}{c^2}\{(\vec{v}\cdot\vec{u'})\vec{a'}_\perp - (\vec{v}\cdot\vec{a'})\vec{u'}_\perp\}\right\}$$

$$= \frac{1}{\gamma_v^2(1+\vec{v}\cdot\vec{u}'/c^2)^3}\left\{\vec{a'}_\perp + \frac{\vec{v}}{c^2}\times(\vec{a'}\times\vec{u'})\right\}$$