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Physics 506: Solutions to Assignment #4

Problem 9.12

Since β is small and the charge distribution is uniform, we can approximate the charge distribution by

$$\rho(t) \approx \frac{Q}{4\pi R^3/3} = \frac{3Q}{4\pi R^3(\theta)} = \frac{3Q}{4\pi R_0^3} \frac{1}{(1+\beta(t)P_2(\cos\theta))^3}$$

where $\beta(t) = \beta_0 \cos\omega t$. Since the problem is spherical symmetric, all mutipole moments with $m \neq 0$ vanish. Therefore, the electric multipole moments (here we have ignored any currents on the sphere) are:

$$q_{\ell,0} = \int r^\ell Y^*_{\ell,0}(\theta,\phi)\rho(t)d\tau = \frac{3Q}{4\pi R_0^3}\int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta \sqrt{\frac{2\ell+1}{4\pi}} \frac{P_\ell(\cos\theta)}{(1+\beta(t)P_2(\cos\theta))^3} \int_0^{R(\theta)} r^\ell r^2 dr$$

$$= \frac{3QR_0^\ell}{2(\ell+3)}\sqrt{\frac{2\ell+1}{4\pi}} \int_0^\pi P_\ell(\theta)(1+\beta(t)P_2(\theta))^\ell d(\cos\theta)$$

$$\approx \frac{3QR_0^\ell}{2(\ell+3)}\sqrt{\frac{2\ell+1}{4\pi}} \int_{-1}^{+1} P_\ell(x)\{1+\ell\beta(t)P_2(x)\}dx$$

$$= \frac{3QR_0^\ell}{2(\ell+3)}\sqrt{\frac{2\ell+1}{4\pi}} \frac{2}{2\ell+1}(\delta_{\ell,0}+\ell\beta(t)\delta_{\ell,2})$$

Thus, the only time-varying non-vanishing moment is the electric dipole moment ($\ell = 2$):

$$q_{2,0} = \frac{3}{5\sqrt{5\pi}}QR_0^2\beta(t) = \frac{3}{5\sqrt{5\pi}}QR_0^2 Re\{\beta_0 e^{-i\omega t}\}$$

For the long wavelength approximation,

$$a_E(\ell,m) \approx \frac{ck^{\ell+2}}{i(2\ell+1)!!}\sqrt{\frac{\ell+1}{\ell}}q_{\ell m} \quad \Rightarrow \quad a_E(2,0) = -i\frac{1}{25}\sqrt{\frac{3}{10\pi}}QR_0^2ck^4\beta_0$$

The angular distribution of the radiation

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}|a_E(2,0)|^2|\vec{X}_{2,0}|^2 = \frac{Z_0}{2k^2}\left\{\frac{3}{5^4\cdot 10\pi}Q^2R_0^4c^2k^8\beta_0^2\right\}\left\{\frac{15}{8\pi}\sin^2\theta\cos^2\theta\right\} = \frac{9c^2Z_0}{2\cdot 10^4\pi^2}Q^2R_0^4\beta_0^2k^6\sin^2\theta\cos^2\theta$$

The total power

$$P = \int d\Omega\frac{dP}{d\Omega} = \frac{9c^2Z_0}{2\cdot 10^4\pi^2}Q^2R_0^4\beta_0^2k^6\int\sin^2\theta\cos^2\theta d\Omega = \frac{3c^2Z_0}{12500\pi}Q^2R_0^4\beta_0^2k^6$$

Problem 9.16

Let the $z-$axis along the antenna so that the antenna spans between $-d/2 < z < d/2$. Therefore, the current density

$$\vec{J}(\vec{r}) = \hat{z}I\sin(kz)\delta(x)\delta(y) \quad \text{for} \quad |z| \leq \frac{d}{2}$$

where $kd = 2\pi$. The vector potential from Eq. (9.8):

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int e^{-ik\vec{n}\cdot\vec{r'}}\vec{J}(\vec{r'})d\tau' = \hat{z}\frac{\mu_0 I}{4\pi}\frac{e^{ikr}}{r}\int_{-d/2}^{d/2}\sin(kz')e^{-ikz'\cos\theta}dz' = \hat{z}\frac{\mu_0 I}{2\pi i}\frac{e^{ikr}}{kr}\frac{\sin(\pi\cos\theta)}{\sin^2\theta}$$

Here we have used the following integral

$$\int_{-\pi}^{\pi} \sin z e^{-iaz} dz = \frac{2i \sin(\pi a)}{a^2 - 1}$$

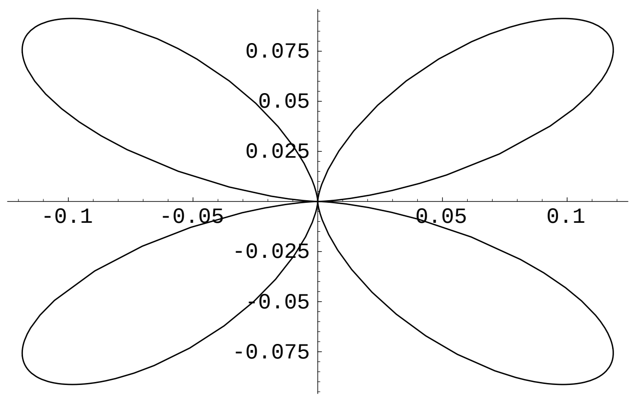
The magnetic field is given by

$$\vec{H} = \frac{1}{\mu_0} \nabla \times \vec{A} \approx \frac{ik}{\mu_0} \vec{n} \times \vec{A} = -\frac{I}{2\pi} \frac{e^{ikr}}{r} \frac{\sin(\pi \cos\theta)}{\sin\theta} \hat{\phi}$$

(a) The angular distribution of radiated power is

$$\frac{dP}{d\Omega} = \frac{r^2}{2} \vec{n} \cdot (\vec{E} \times \vec{H}^*) = \frac{Z_0 I^2}{8\pi^2} \frac{\sin^2(\pi \cos\theta)}{\sin^2\theta} = \frac{Z_0 I^2}{8} \left\{ \frac{\sin^2(\pi\cos\theta)}{\pi^2 \sin^2\theta} \right\}$$

which is plotted below (in the unit of $Z_0 I^2/8$). The $\theta = 0$ direction is vertically up.



(b) The total power of radiation

$$P = \frac{Z_0 I^2}{8\pi^2} \int \frac{\sin^2(\pi \sin\theta)}{\sin^2\theta} d\Omega = \frac{Z_0 I^2}{4\pi} \int_{-1}^{+1} \frac{\sin^2(\pi\cos\theta)}{\sin^2\theta} d(\cos\theta) \approx \frac{1.56}{4\pi} Z_0 I^2$$

Now note

$$P = \frac{1}{2} I^2 R_{\text{rad}} \quad \Rightarrow \quad R_{\text{rad}} = \frac{2P}{I^2} = \frac{1.56}{2\pi} Z_0 = \frac{1.56}{2\pi} \cdot 377 = 93.6 \text{ Ohms}$$

Problem 9.17

The charge distribution can be calculated from the continuity equation:

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0, \quad \Rightarrow \quad \rho(\vec{r}) = \frac{1}{i\omega} \nabla \cdot \vec{J} = -i \frac{I}{c} \cos(kz) \delta(x) \delta(y) \quad \text{for} \quad |z| \le d/2$$

(a) Exact calculations:

The dipole moment

$$\vec{p} = \int \vec{r} \rho(\vec{r}) d\tau = -i\frac{I}{c} \int (x\hat{x} + y\hat{y} + z\hat{z}) \delta(x)\delta(y) \cos(kz) dx dy dz = -i\frac{I}{c} \int_{-d/2}^{d/2} z \cos(kz) dz (\hat{z}) = 0$$

The magnetic dipole moment

$$\vec{m} = \frac{1}{2} \int \vec{r} \times \vec{J} d\tau = \int I(x\hat{x} + y\hat{y} + z\hat{z}) \times \hat{z} \delta(x)\delta(y) \sin(kz)\ dx dy dz = 0$$

The only non-vanishing quadrupole moments are Q_{11}, Q_{22} and Q_{33}.

$$Q_{33} = \int (3z^2 - r^2) \rho(\vec{r}) d\tau = -2i \frac{I}{c} \int_{-d/2}^{d/2} z^2 \cos(kz) dz = i\frac{Id^3}{\pi^2 c}, \quad Q_{11} = Q_{22} = -\frac{1}{2} Q_{33}$$

Long wavelength limit:
In the long wavelength limit, we have

$$\vec{J}(\vec{r}) = I\sin(kz)\delta(x)\delta(y)\ \hat{z} \approx kIz\delta(x)\delta(y)\ \hat{z}; \qquad \rho(\vec{r}) = -i\frac{I}{c}\cos(kz)\delta(x)\delta(y) \approx -i\frac{I}{c}\delta(x)\delta(y)$$

The dipole moment

$$\vec{p} = \int \vec{r}\rho(\vec{r})d\tau = -i\frac{I}{c}\int \vec{r}\delta(x)\delta(y)d\tau = -i\frac{I}{c}\int_{-d/2}^{d/2} z\ dz\ \hat{z} = 0$$

The magmetic dipole moment

$$\vec{m} = \frac{1}{2}\int \vec{r}\times\vec{J}d\tau = \frac{1}{2}kI\int (x\hat{x}+y\hat{y}+z\hat{z})\times\hat{z}\ \delta(x)\delta(y)\ dxdydz = 0$$

The electric quadrupole moment

$$Q_{33} = \int (3z^2 - r^2)\rho(\vec{r})d\tau = -2i\frac{I}{c}\int_{-d/2}^{d/2} z^2 dz = -i\frac{Id^3}{6c}, \quad Q_{11} = Q_{22} = -\frac{1}{2}Q_{33}$$

Not surprising, the exact calculation and the long wavelength approximation yield very different values for the electric quadrupole moment tensor in this case. With $kd = 2\pi$, the approximation does not work.
(b) The angular power of radiation (of the exact calculation of the quadrupole moments):

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^6}{1152\pi^2}\left\{|\vec{Q}(\vec{n})|^2 - |\vec{n}\cdot\vec{Q}(\vec{n})|^2\right\}$$

where

$$\vec{Q}(\vec{n}) = \sum_{i=1}^{3}\{Q_{1i}\hat{x} + Q_{2i}\hat{y} + Q_{3i}\hat{z}\}\,n_i = Q_{11}n_1\hat{x} + Q_{22}n_2\hat{y} + Q_{33}n_3\hat{z} = -\frac{1}{2}Q_{33}\{\sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} - 2\cos\theta\hat{z}\}$$

Thus

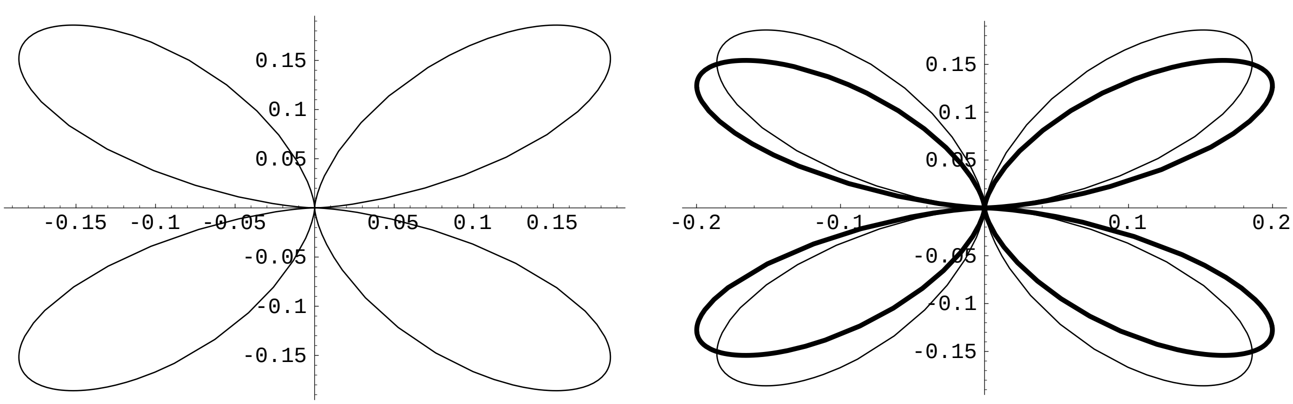
$$|\vec{Q}(\vec{n})|^2 = \frac{1}{4}|Q_{33}|^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + 4\cos^2\theta) = \frac{1}{4}|Q_{33}|^2(1 + 3\cos^2\theta)$$

$$|\vec{n}\cdot\vec{Q}(\vec{n})|^2 = \frac{1}{4}|Q_{33}|^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi - 2\cos^2\theta)^2 = \frac{1}{4}|Q_{33}|^2(1 - 3\cos^2\theta)^2$$

Therefore,

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^6}{1152\pi^2}\left\{\frac{1}{4}|Q_{33}|^2\right\}\left\{(1+3\cos^2\theta) - (1-3\cos^2\theta)^2\right\} = \frac{1}{8}Z_0 I^2\sin^2\theta\cos^2\theta$$

The left plot below shows graphically the angular distribution (in the unit of $Z_0 I^2/8$) of the quadrupole radiation. The right plot compares the shape of this distribution (thin line) with that (thick line) of the exact calculation scaled up by a factor of $157.9/93.6 = 1.69$ (see the discussion below). Evidently, apart from an overestimation of the radiation power, the angular distribution of the quadrupole agrees reasonablly well with that of the exact calculation.



(c) The total power of the exact calculation

$$P = \int \frac{dP}{d\Omega} d\Omega = \frac{1}{8} Z_0 I^2 \int \sin^2\theta \cos^2\theta d\Omega = \frac{\pi}{15} Z_0 I^2$$

The corresponding radiation resistance:

$$R_{\text{rad}} = \frac{2\pi}{15} Z_0 = \frac{2\pi}{15} \cdot 377 = 157.9 \text{ Ohms}$$

The total power of the long wavelength approximation:

$$P_{\text{LW}} = \frac{c^2 Z_0 k^6}{1440\pi} \sum_{ij} |Q_{ij}|^2 = \frac{c^2 Z_0 k^6}{1440\pi} \cdot \frac{3}{2} |Q_{33}|^2 = \frac{\pi^5}{540} I^2 Z_0$$

The corresponding radiation resistance

$$R_{\text{rad}}^{\text{LW}} = \frac{\pi^5}{270} Z_0 = 427.3 \text{ Ohms}$$

Obviously the long wavelength approximation does not work in this case. In Problem 9.16, we have $R_{\text{rad}} = 93.4$ Ohms from the exact calculation without expansion. There is a puzzle here that the radiation by the electric dipole mode is greater than the sum of all modes. This is because the leading term in the expansion (electric $\ell = 2$ term or E2) is not a good approximation whenever the dimensions of the source are comparable to or larger than a wavelength.

<u>Problem 9.22</u> (*Only TE modes are worked out*)
The general solutions to the Maxwell equations are given by Eq. (9.122):

$$\vec{H} = \sum_{\ell,m} \left\{ a_E(\ell,m) f_\ell(kr) \vec{X}_{\ell m} - \frac{i}{k} a_M(\ell,m) \nabla \times g_\ell(kr) \vec{X}_{\ell m} \right\}$$

$$\vec{E} = Z_0 \sum_{\ell,m} \left\{ \frac{i}{k} a_E(\ell,m) \nabla \times f_\ell(kr) \vec{X}_{\ell m} + a_M(\ell,m) g_\ell(kr) \vec{X}_{\ell m} \right\}$$

where $a_E(\ell,m)$ and $a_M(\ell,m)$ are the electric and magnetic multipoles respectively. $f_\ell(kr)$ and $g_\ell(kr)$ are linear combinations of spherical Bessel functions $j_\ell(kr)$ and $n_\ell(kr)$. Furthermore, the fields must be finite at $r = 0$. Thus, we have $f_\ell(kr) = j_\ell(kr)$ and $g_\ell(kr) = j_\ell(kr)$.

The fields of the TE modes are given by Eq. (9.116):

$$\vec{E}_{\ell m} = Z_0 j_\ell(kr) \vec{L} Y_{\ell m}(\theta,\phi); \qquad \vec{H}_{\ell m} = -\frac{i}{kZ_0} \nabla \times \vec{E}_{\ell m} \quad (\ell \neq 0)$$

The $\ell = 0$ case leads to null fields everywhere inside the cavity. The corresponding components are

$$\begin{aligned} (E_{\ell m})_r &= 0 & (H_{\ell m})_r &= \tfrac{1}{kr}\, \ell(\ell+1) Y_{\ell m}\, j_\ell(kr) \\ (E_{\ell m})_\theta &= i Z_0 \tfrac{1}{\sin\theta} \tfrac{\partial}{\partial\phi} Y_{\ell m}\, j_\ell(kr) & (H_{\ell m})_\theta &= \tfrac{1}{kr} \tfrac{\partial}{\partial\theta} Y_{\ell m}\, \tfrac{\partial}{\partial r}(r j_\ell(kr)) \\ (E_{\ell m})_\phi &= -i Z_0 \tfrac{\partial}{\partial\theta} Y_{\ell m}\, j_\ell(kr) & (H_{\ell m})_\phi &= \tfrac{1}{kr} \tfrac{1}{\sin\theta} \tfrac{\partial}{\partial\phi} Y_{\ell m}\, \tfrac{\partial}{\partial r}(r j_\ell(kr)) \end{aligned}$$

Similarly the TM fields are given by Eq. (9.118):

$$\vec{H}_{\ell m} = j_\ell(kr) \vec{L} Y_{\ell m}(\theta,\phi); \qquad \vec{E}_{\ell m} = \frac{iZ_0}{k} \nabla \times \vec{H}_{\ell m} \quad (\ell \neq 0)$$

with the following components

$$\begin{aligned} (H_{\ell m})_r &= 0 & (E_{\ell m})_r &= -\tfrac{Z_0}{kr}\, \ell(\ell+1) Y_{\ell m}\, j_\ell(kr) \\ (H_{\ell m})_\theta &= i \tfrac{1}{\sin\theta} \tfrac{\partial}{\partial\phi} Y_{\ell m}\, j_\ell(kr) & (E_{\ell m})_\theta &= -\tfrac{Z_0}{kr} \tfrac{\partial}{\partial\theta} Y_{\ell m}\, \tfrac{\partial}{\partial r}(r j_\ell(kr)) \\ (H_{\ell m})_\phi &= -i \tfrac{\partial}{\partial\theta} Y_{\ell m}\, j_\ell(kr) & (E_{\ell m})_\phi &= -\tfrac{Z_0}{kr} \tfrac{1}{\sin\theta} \tfrac{\partial}{\partial\phi} Y_{\ell m}\, \tfrac{\partial}{\partial r}(r j_\ell(kr)) \end{aligned}$$

(a) At $r = a$, the electric field must be perpendicular to the conducting surface and there must be no normal component of the magnetic field. Applying this boundary condition to the TE modes leads to $j_\ell(ka) = 0$. Let $x_{\ell n}$ be the n^{th} root of $j_\ell(x)$, the characteristic frequencies $\omega_{\ell n}^{TE}$ are therefore given by

$$\frac{\omega_{\ell n}^{TE}}{c} a = x_{\ell n} \qquad \Rightarrow \qquad \omega_{\ell n}^{TE} = \frac{c}{a} x_{\ell n}$$

For the TM modes, we have

$$\frac{\partial}{\partial r}\{r j_\ell(kr)\}|_{r=a} = 0$$

Let $y_{\ell n}$ be the n^{th} root of $\partial(x j_\ell(x))/\partial x$, the characteristic frequencies are then given by

$$\frac{\omega_{\ell n}^{TM}}{c} a = y_{\ell n} \qquad \Rightarrow \qquad \omega_{\ell n}^{TM} = \frac{c}{a} y_{\ell n}$$

In both cases, the characteristic frequencies are independent of m (degenerate in m), as result of the $\phi-$symmetry. We proceed with TE modes only.
(b) The four lowest roots of $j_\ell(x) = 0$ $(\ell \neq 0)$ are $x_{1,1} = 4.5, x_{2,1} = 5.8, x_{3,1} = 6.85$ and $x_{1,2} = 7.64$. Thus the corresponding wavelengths

$$\lambda_{\ell n} = \frac{2\pi}{x_{\ell n}} a \quad \Rightarrow \lambda_{1,1} = 1.4a,\ \lambda_{2,1} = 1.1a,\ \lambda_{3,1} = 0.9a,\ \lambda_{1,2} = 0.8a$$

(c) The lowest TE mode corresponds to $\ell = 1, n = 1$, independent of m. In this case, $k_{1,1} = \frac{2\pi}{\lambda_{1,1}} = 4.5a$. The fields for $m = 0$ are (apart from normalization constants):

$$(E_{1,0})_r = 0 \qquad (H_{1,0})_r = \sqrt{\tfrac{3}{\pi}} \tfrac{1}{(4.5r/a)} j_1(4.5r/a) \cos\theta$$

$$(E_{1,0})_\theta = 0 \qquad (H_{1,0})_\theta = -\sqrt{\tfrac{3}{4\pi}} \tfrac{1}{(4.5r/a)} \tfrac{\partial}{\partial r}(r j_1(4.5r/a)) \sin\theta$$

$$(E_{1,0})_\phi = i\sqrt{\tfrac{3}{4\pi}} Z_0 j_1(4.5r/a) \sin\theta \qquad (H_{1,0})_\phi = 0$$