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Physics 506: Solutions to Assignment #3

Problem 9.1

Note that part (a) is merely the statement that the Fourier decomposition of the multipole moments gives the same result as the multipole expansion of the Fourier decomposed charge distribution, *i.e.*, it does not matter whether you expand first and then decompose or decompose first and then expand. This is obvious since the two integrations commute.

(b) Let T be the period and ω_0 be the characteristic frequency, thus:

$$\rho(\vec{r},t) = \rho(\vec{r},t+T)$$

Therefore,

$$\rho(\vec{r},t) = \int_0^T \rho(\vec{r},t')\delta(t-t')dt' = \int_0^T \rho(\vec{r},t')\left\{\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{-in\omega_0(t-t')}\right\}dt'$$

$$= \sum_{n=-\infty}^{\infty} e^{-in\omega_0 t}\frac{1}{T}\int_0^T \rho(\vec{r},t')e^{in\omega_0 t'}dt'$$

$$= \frac{1}{T}\int_0^T \rho(\vec{r},t')dt' + \sum_{n=1}^{\infty}\left\{e^{-in\omega_0 t}\frac{1}{T}\int_0^T \rho(\vec{r},t')e^{in\omega_0 t'}dt' + e^{in\omega_0 t}\frac{1}{T}\int_0^T \rho(\vec{r},t')e^{-in\omega_0 t'}dt'\right\}$$

$$= \rho_0(\vec{r}) + \sum_{n=1}^{\infty} 2Re\left\{\rho_n(\vec{r})e^{-in\omega_0 t}\right\}$$

Here

$$\rho_n(\vec{r}) = \frac{1}{T}\int_0^T \rho(\vec{r},t')e^{in\omega_0 t'}dt'$$

Note that $\rho(\vec{r},t)$ is assumed to be real.

(c) To facilitate the calculation of multipole moments later, we can write the charge distribution in spherical coordinates as

$$\rho(\vec{r},t) = \frac{q}{R^2}\delta(r-R)\delta(\cos\theta)\delta(\phi-\omega_0 t)$$

Therefore

$$\rho_n(\vec{r}) = \frac{1}{T}\int_0^T \rho(\vec{r},t)e^{in\omega_0 t}dt = \frac{q}{R^2 T}\delta(r-R)\delta(\cos\theta)\int_0^T \delta(\phi-\omega_0 t)e^{in\omega_0 t}\frac{d(\omega_0 t)}{\omega_0} = \frac{q}{2\pi R^2}\delta(r-R)\delta(\cos\theta)e^{in\phi}$$

Thus

$$\rho(\vec{r},t) = \rho_0(\vec{r}) + \sum_{n=1}^{\infty} Re\left\{2\rho_n(\vec{r})e^{-in\omega_0 t}\right\} = \frac{q}{2\pi R^2}\delta(r-R)\delta(\cos\theta)\left\{1 + 2\sum_{n=1}^{\infty}\cos(n\phi - n\omega_0 t)\right\}$$

The $\ell = 0, 1$ multiple moments using method (a) (before the Fourier decomposition) are:

$$q_{00} = \int Y_{00}^*\rho(\vec{r},t)d\tau = \frac{1}{\sqrt{4\pi}}\int \delta(\cos\theta)\delta(\phi-\omega_0 t)d\Omega \int \frac{q}{R^2}\delta(r-R)r^2 dr = \frac{q}{\sqrt{4\pi}}$$

$$q_{10} = \int rY_{10}^*\rho(\vec{r},t)d\tau = \sqrt{\frac{3}{4\pi}} \int \cos\theta\delta(\cos\theta)\delta(\phi-\omega_0 t)d\Omega \int \frac{q}{R^2}\delta(r-R)r^3 dr = 0$$

$$q_{11} = \int rY_{11}^*\rho(\vec{r},t)d\tau = -\sqrt{\frac{3}{8\pi}} \int \sin\theta e^{i\phi}\delta(\cos\theta)\delta(\phi-\omega_0 t)d\Omega \int \frac{q}{R^2}\delta(r-R)r^3 dr = -\sqrt{\frac{3}{8\pi}}qRe^{i\omega_0 t}$$

$$q_{1-1} = \int rY_{1-1}^*\rho(\vec{r},t)d\tau = -q_{11}^* = \sqrt{\frac{3}{8\pi}}qRe^{-i\omega_0 t}$$

Summarizing the $\ell = 1$ moments, the electric dipole moment is

$$\vec{p} = \sqrt{\frac{4\pi}{3}}\left\{\frac{q_{1-1}-q_{11}}{\sqrt{2}}\hat{x} - i\frac{q_{1-1}+q_{11}}{\sqrt{2}}\hat{y} + q_{10}\hat{z}\right\} = qR\left\{\cos(\omega_0 t)\hat{x} + \sin(\omega_0 t)\hat{y}\right\} = Re\left\{qR(\hat{x}+i\hat{y})e^{-i\omega_0 t}\right\}$$

The multipole moments using method (b) (after the Fourier transformation) are:

$$q_{00} = \int Y_{00}^*\rho(\vec{r},t)d\tau = \frac{1}{\sqrt{4\pi}}\int \frac{q}{2\pi R^2}\delta(r-R)r^2 dr\delta(\cos\theta)d(\cos\theta)\left\{1+2\sum_{n=1}^{\infty}\cos(n\phi-n\omega_0 t)\right\}d\phi = \frac{q}{\sqrt{4\pi}}$$

$$\vec{p} = \int \rho(\vec{r},t)\vec{r}d\tau = \frac{q}{2\pi R^2}\int \delta(r-R)\delta(\cos\theta)\left\{1+2\sum_{n=1}^{\infty}\cos(n\phi-n\omega_0 t)\right\}(r\sin\theta\cos\phi\hat{x} + r\sin\theta\sin\phi\hat{y} + r\cos\theta\hat{z})d\tau$$

$$= \frac{qR}{2\pi}\int_0^{2\pi}\left\{1+2\sum_{n=1}^{\infty}\cos(n\phi-n\omega_0 t)\right\}(\cos\phi\hat{x}+\sin\phi\hat{y})d\phi = qR\left\{\cos(\omega_0 t)\hat{x}+\sin(\omega_0 t)\hat{y}\right\} = Re\left\{qR(\hat{x}+i\hat{y})e^{-i\omega_0 t}\right\}$$

Here we have used the following two integrals:

$$\int_0^{2\pi}\left\{1+2\sum_{n=1}^{\infty}\cos(n\phi-n\omega_0 t)\right\}\cos\phi d\phi = \frac{1}{2}\int_0^{2\pi}\sum_{n=1}^{\infty}\left\{e^{in\phi}e^{-in\omega_0 t} + e^{-in\phi}e^{in\omega_0 t}\right\}(e^{i\phi}+e^{-i\phi})d\phi$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}(2\pi)\left\{(\delta_{n,-1}+\delta_{n,1})e^{-in\omega_0 t} + (\delta_{n,1}+\delta_{n,-1})e^{in\omega_0 t}\right\} = (2\pi)\frac{1}{2}(e^{-i\omega_0 t}+e^{i\omega_0 t}) = 2\pi\cos(\omega_0 t)$$

$$\text{Similarly} \quad \int_0^{2\pi}\left\{1+2\sum_{n=1}^{\infty}\cos(n\phi-n\omega_0 t)\right\}\sin\phi d\phi = 2\pi\sin(\omega_0 t)$$

Therefore, the two calculations agree with each other. There are high order multipole moments:

$$q_{\ell m} \sim \int r^\ell Y_{\ell m}^*(\theta,\phi)\rho(\vec{r},t)d\tau \sim \frac{q}{2\pi R^2}\int \delta(r-R)r^{\ell+2}dr\int Y_{\ell m}^*\delta(\cos\theta)\left\{1+2\sum_{n=1}^{\infty}\cos(n\phi-n\omega_0 t)\right\}d\Omega$$

$$\sim qR^\ell\int P_\ell^m(\cos\theta)\delta(\cos\theta)\ d(\cos\theta)\int_0^{2\pi}\left\{1+2\sum_{n=1}^{\infty}\cos(n\phi-n\omega_0 t)\right\}e^{im\phi}d\phi$$

Now note

$$P_\ell^m(\cos\theta) \sim (\sin\theta)^{|m|}(\cos\theta)^{\ell-|m|} \quad \Rightarrow \quad \int \delta(\cos\theta)P_\ell^m(\cos\theta)d(\cos\theta) \sim \delta_{\ell,|m|}$$

$$\int_0^{2\pi}\left\{1+2\sum_{n=1}^{\infty}\cos(n\phi-n\omega_0 t)\right\}e^{im\phi}d\phi=\int_0^{2\pi}\left\{1+\sum_{n=1}^{\infty}(e^{i(n+m)\phi}e^{-in\omega_0 t}+e^{i(m-n)\phi}e^{in\omega_0 t})\right\}d\phi=2\pi e^{im\omega_0 t}$$

Thus

$$q_{\ell m}\sim qR^{\ell}\delta_{\ell,|m|}e^{im\omega_0 t}$$

Thus multipole moments $q_{\ell m}$ is nonvanishing for $m=\ell$ or $m=-\ell$ with a frequency dependence of $\omega=\ell\omega_0$.

Problem 9.2

Let the rotational axis be the $z-$axis and the coordinate origin at the center of the square, properly choose $t=0$ such that $\phi=\omega t$ for one of the $+q$ charge. In this case, the charge distribution is given by

$$\rho(\vec{r},t)=q\delta(z)\left\{\delta(x-\frac{a}{\sqrt{2}}\cos\omega t)\delta(y-\frac{a}{\sqrt{2}}\sin\omega t)-\delta(x+\frac{a}{\sqrt{2}}\cos\omega t)\delta(y-\frac{a}{\sqrt{2}}\sin\omega t)\right.$$

$$\left.+\delta(x+\frac{a}{\sqrt{2}}\cos\omega t)\delta(y+\frac{a}{\sqrt{2}}\sin\omega t)-\delta(x-\frac{a}{\sqrt{2}}\cos\omega t)\delta(y+\frac{a}{\sqrt{2}}\sin\omega t)\right\}$$

The quadrupole moments

$$Q_{ij}=\int(3x_ix_j-r^2\delta_{ij})\rho(\vec{r},t)d\tau$$

Now that

$$\int x^2\rho(\vec{r},t)d\tau=qa^2\cos(2\omega t),\quad \int y^2\rho(\vec{r},t)d\tau=-qa^2\cos(2\omega t),\quad \int z^2\rho(\vec{r},t)d\tau=0$$

$$\int xy\rho(\vec{r},t)d\tau=qa^2\sin(2\omega t),\quad \int xz\rho(\vec{r},t)d\tau=0,\quad \int yz\rho(\vec{r},t)d\tau=0$$

Therefore,

$$Q_{11}=\int(2x^2-y^2-z^2)\rho d\tau=3qa^2\cos(2\omega t)=3qa^2 Re\left\{e^{-i(2\omega t)}\right\}$$

$$Q_{22}=\int(2y^2-x^2-z^2)\rho d\tau=-3qa^2\cos(2\omega t)=-3qa^2 Re\left\{e^{-i(2\omega t)}\right\}$$

$$Q_{12}=Q_{21}=\int 3xy\rho d\tau=3qa^2\sin(2\omega t)=3qa^2 Re\left\{ie^{-i(2\omega t)}\right\}$$

All other Q_{ij}'s vanish. Therefore, the quadrupole moment tensor is (with the understanding of $e^{-i(2\omega t)}$):

$$Q_{ij}=\ 3qa^2\left\{\begin{array}{ccc}1 & i & 0\\ i & -1 & 0\\ 0 & 0 & 0\end{array}\right\}$$

Evidently, the frequency of the radiation is 2ω as expected from the periodicity of the charge distribution $\rho(\vec{r},t+T/2)=\rho(\vec{r},t)$, here T is the rotation period. Thus $k=2\omega/c$. The electric dipole moment vanishes for this configuration (two equal dipoles antiparallel to each other). The magnetic dipole moment also vanishes since the rotating square with net zero charge has zero net current flowing. Thus the radiation is dominated by the electric quadrupole. In the long wavelength limit, the radiation magnetic field is given by Eq. (9.44):

$$\vec{H}=-\frac{ick^3}{24\pi}\frac{e^{ikr}}{r}\vec{n}\times\vec{Q}(\vec{n})$$

where $\vec{n} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$ and

$$\vec{Q}(\vec{n}) = \sum_{i=1}^{3}(\hat{x}Q_{1i}n_i + \hat{y}Q_{2i}n_i + \hat{z}Q_{3i}n_i) = 3qa^2\sin\theta(\cos\phi + i\sin\phi)\hat{x} + 3qa^2\sin\theta(i\cos\phi - \sin\phi)\hat{y} = 3qa^2\sin\theta e^{i\phi}(\hat{x} + i\hat{y})$$

Therefore, the magnetic field

$$\vec{H} = -\frac{ick^3qa^2}{8\pi}\frac{e^{ikr}}{r}e^{i\phi}\sin\theta(-i\hat{x}\cos\theta + \hat{y}\cos\theta + \hat{z}i\sin\theta e^{i\phi})$$

and the electric field is given by

$$\vec{E} = \frac{iZ_0}{k}\nabla\times\vec{E} = Z_0\vec{H}\times\vec{n}$$

The angular distribution of the radiation is given by Eq. (9.45):

$$\frac{dP}{d\Omega} = \frac{c^2Z_0}{1152\pi^2}k^6|(\vec{n}\times\vec{Q}(\vec{n}))\times\vec{n}|^2 = \frac{c^2Z_0}{1152\pi^2}k^6\left\{|\vec{Q}(\vec{n})|^2 - |\vec{Q}(\vec{n})\cdot\vec{n}|^2\right\}$$

$$= \frac{c^2Z_0}{1152\pi^2}k^6\left\{(3qa^2\sin\theta)^2\times 2 - (3qa^2\sin\theta)^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi)\right\}$$

$$= \frac{c^2Z_0}{1152\pi^2}k^6(3qa^2)^2\sin^2\theta(2 - \sin^2\theta) = \frac{c^2Z_0k^6}{128\pi^2}(q^2a^4)(1 - \cos^4\theta)$$

$$= \frac{Z_0\omega^6}{2\pi^2c^4}(q^2a^4)(1 - \cos^4\theta)$$

Total power of radiation

$$P = \int d\Omega\frac{dP}{d\Omega} = \frac{Z_0\omega^6}{2\pi^2c^4}(q^2a^4)\int d\Omega(1 - \cos^4\theta) = \frac{8Z_0\omega^6}{5\pi c^4}q^2a^4$$

Problem 9.3

In the long wavelength limit, we can calculate the multipole moments from the static problem and keep only the lowest non-vanishing multipoles. From Eq. (3.36), the corresponding potential outside the shell is

$$\Phi(r,\theta) = V\left\{\frac{3}{2}(\frac{R}{r})^2P_1(\cos\theta) - \frac{7}{8}(\frac{R}{r})^4P_3(\cos\theta) + ...\right\}$$

The potential is dominated by the dipole term. Compared with the potential of an electric dipole p

$$\Phi(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2}$$

we infer the dipole moment of the sphere to be:

$$\vec{p} = 6\pi\epsilon_0VR^2\hat{z}$$

Thus, the radiation fields are given by Eq. (9.19):

$$\vec{H} = \frac{ck^2}{4\pi}(\vec{n}\times\vec{p})\frac{e^{ikr}}{r} = -\frac{3}{2}(\frac{\omega R}{c})^2\frac{V}{Z_0}\sin\theta\frac{e^{i(\omega/c)r}}{r}\hat{\phi}$$

$$\vec{E} = Z_0\vec{H}\times\vec{n} = -\frac{3}{2}V(\frac{\omega R}{c})^2\sin\theta\frac{e^{i(\omega/c)r}}{r}\hat{\theta}$$

The radiation power per unit solid angle

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 |(\vec{n} \times \vec{P}) \times \vec{n}|^2 = \frac{c^2 Z_0}{32\pi^2} k^4 (|\vec{p}|^2 - |\vec{p} \cdot \vec{n}|^2) = \frac{9}{8} (\frac{\omega R}{c})^4 \frac{V^2}{Z_0} \sin^2\theta$$

The total power

$$P = \int \frac{dP}{d\Omega} d\Omega = 3\pi (\frac{\omega R}{c})^4 \frac{V^2}{Z_0}$$

Added note

There are charges and currents on the sphere. But the magnetic dipole moment vanishes. From the scalar potential, we can calculate the surface charge distribution

$$\sigma(\theta) = \epsilon_0 E_n|_{r=R} = -\epsilon_0 \frac{\partial \Phi}{\partial n}|_{r=R} = \frac{3\epsilon_0 V}{R} \cos\theta$$

Therefore, the surface current density $\vec{K} = K\hat{\theta}$ (by symmetry, the current only flows in the θ direction) can be calculated from the continuity equation:

$$\nabla \cdot \vec{K} + \frac{\partial \sigma}{\partial t} = 0 \quad \Rightarrow \quad \frac{1}{R \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta K) = i\omega \frac{3\epsilon_0 V}{R} \cos\theta \quad \Rightarrow \quad K = \frac{3i}{2} \epsilon_0 \omega V \sin\theta$$

The magnetic dipole moment

$$\vec{m} = \frac{1}{2} \int \vec{r} \times \vec{K} da = \frac{3i}{4} \epsilon_0 \omega R^3 V \int_0^{\pi} \sin^2\theta d\theta \int_0^{2\pi} \hat{\phi} d\phi = 0$$

For the charge distribution of a dipole potential, all elements of the electric quadrupole moment tensors are zero.

$$Q_{11} = Q_{22} = \int (3x^2 - R^2) \sigma R^2 d\cos\theta d\phi \sim \int \sin^2\theta \cos\theta d\cos\theta \int \cos^2\phi d\phi = 0 \quad \Rightarrow \quad Q_{33} = 0$$

$$Q_{12} = Q_{21} = \int (3xy) \sigma R^2 d\cos\theta d\phi \sim \int \sin^2\theta \cos\theta d\cos\theta \int \sin\phi \cos\phi d\phi = 0$$

$$Q_{13} = Q_{31} = Q_{23} = Q_{32} = \int (3xz) \sigma R^2 d\cos\theta d\phi \sim \int \sin\theta \cos^2\theta d\cos\theta \int \cos\phi d\phi = 0$$

Problem 9.10

(b) The charge distribution given is not the one in usual sense since the total charge is nonvanishing and oscillating with time, against the teaching that the radiation does not having a monopole term. I guess that is why Jackson called it "transitional charge". Note that in a spherical coordinate system

$$z = r\cos\theta = \sqrt{\frac{4\pi}{3}} r Y_{10}$$

Therefore, the charge distribution can be written

$$\rho(\vec{r}, t) = \frac{2e}{\sqrt{6} a_0^4} r e^{-3r/2a_0} Y_{00} Y_{10} e^{-i\omega_0 t} = \frac{\sqrt{2} e}{4\pi a_0^4} z e^{-3r/2a_0} e^{-i\omega_0 t}$$

Therefore, the electric dipole moment only has non-vanishing contribution in the $z-$ direction.

$$p_z = \int z\rho(\vec{r}, t) d\tau = \frac{\sqrt{2} e}{3a_0^4} \int r^4 e^{-3r/2a_0} dr \int d\Omega Y_{10} Y_{10} = \frac{\sqrt{2} e}{3a_0^4} \frac{4!}{(3/2a_0)^5} = \frac{2^8 \sqrt{2}}{3^5} e a_0$$

Here we have used the orthogonality condition of the spherical harmonics and

$$\int_0^\infty x^n e^{-\alpha x} dx = \frac{n!}{\alpha^{n+1}}$$

The average power is given by Eq. (9.24):

$$P_{\text{quan.}} = \frac{c^2 Z_0 k^4}{12\pi} |\vec{p}|^2 = \frac{2^{15}}{3^{11}\pi} \frac{Z_0 \omega_0^4}{c^2} e^2 a_0^2 = (\frac{2}{3})^8 (\hbar\omega_0)(\frac{\alpha^4 c}{a_0})$$

(c) Let $\mathcal{N}(2p \to 1s)$ be the transition probability (this Jackson's probability is not the probability in usual sense, it is really the number of atoms making the transition from $2p \to 2s$ per unit time to yield the calculated power):

$$P_{\text{quan.}} = \hbar\omega_0 \mathcal{N}(2p \to 1s) \qquad \Rightarrow \qquad \mathcal{N}(2p \to 1s) = \frac{P_{\text{quan.}}}{\hbar\omega_0} = (\frac{2}{3})^8 \frac{\alpha^4 c}{a_0} \approx 6 \cdot 10^8 \text{ s}^{-1}$$

(d) Let the orbit plane be the $x - y$ plane, the electric dipole moment of the electron can be written as

$$\vec{p} = e(x\hat{x} + y\hat{y}) = 2ea_0 \left\{\cos(\omega_0 t)\hat{x} + \sin(\omega_0 t)\hat{y}\right\} = 2ea_0 Re\left\{(\hat{x} + i\hat{y})e^{-i\omega_0 t}\right\}$$

Or in complex form

$$\vec{p} = 2ea_0(\hat{x} + i\hat{y})e^{-i\omega_0 t} \qquad \Rightarrow \qquad |\vec{p}|^2 = 8e^2 a_0^2$$

The average power

$$P_{\text{class.}} = \frac{c^2 Z_0 k^4}{12\pi} |\vec{p}|^2 = \frac{9}{2^6} (\hbar\omega_0)(\frac{\alpha^4 c}{a_0})$$

The ratio of two powers

$$\frac{P_{\text{quan.}}}{P_{\text{class.}}} = \frac{(2/3)^8}{(9/2^6)} \approx 0.28$$