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Physics 506: Solutions to Assignment #2

Problem 8.3

(a) Choose a rectangular coordinate system with x parallel to the strip along the side b, y perpendicular to the strip and z along the line. Let $K(\vec{z}, t) = K_0 e^{i(kz-\omega t)}\hat{z}$ be the surface current density of the top strip. Thus, the magnetic field in between the two strips is given by

$$\vec{B} = \mu K\hat{x} = \mu K_0 e^{i(kz-\omega t)}\hat{x}, \qquad \vec{H} = \frac{\vec{B}}{\mu} = K_0 e^{i(kz-\omega t)}\hat{x}$$

Therefore, $K_0 = H_0$. The electric field can be derived from the Maxwell's equation:

$$\nabla\times\vec{B} = \mu\epsilon\frac{\partial\vec{E}}{\partial t} = -i\mu\epsilon\omega\vec{E} \qquad \Rightarrow \qquad \vec{E} = -\frac{\nabla\times\vec{B}}{i\mu\epsilon\omega} = -\frac{\vec{k}\times\vec{B}}{\mu\epsilon\omega} = -\frac{kH_0}{\epsilon\omega}e^{i(kz-\omega t)}\hat{y}$$

The average Poynting vector

$$\vec{S} = \frac{1}{2}\vec{E}\times\vec{H}^* = \frac{1}{2}\left\{-\frac{kH_0}{\epsilon\omega}\right\}\hat{y}\times(H_0^*\hat{x}) = \frac{k|H_0|^2}{2\epsilon\omega}\hat{z} = \frac{\sqrt{\mu\epsilon}|H_0|^2}{2\epsilon}$$

The average power transmitted along the line

$$P = \int\vec{S}\cdot\hat{z}da = \frac{ab}{2}\sqrt{\frac{\mu}{\epsilon}}|H_0|^2$$

In terms of the power P,

$$|H_0|^2 = \frac{2P}{ab}\sqrt{\frac{\epsilon}{\mu}}$$

The power loss per unit area

$$\frac{dP}{da} = -\frac{1}{2\sigma\delta}|\vec{K}_{\text{eff}}|^2 = -\frac{1}{2\sigma\delta}|\vec{H}_{||}|^2 = -\frac{1}{2\sigma\delta}|H_0|^2$$

The power loss per unit length along the z:

$$\frac{dP}{dz} = 2b\frac{dP}{da} = -\frac{b}{\sigma\delta}|H_0|^2 = -2\left\{\frac{1}{a\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\right\}P = -2\gamma P$$

Thus

$$P(z) = P_0 e^{-2\gamma z} \qquad \text{with} \qquad \gamma = \frac{1}{a\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}$$

The potential difference between the two strips

$$V = \int\vec{E}\cdot d\vec{\ell} = \frac{kH_0}{\epsilon\omega}ae^{i(kz-\omega t)}$$

The wave impedance

$$Z = \frac{V}{I} = \frac{V}{Kb} = \frac{ka}{\epsilon\omega} = \frac{a}{b}\sqrt{\frac{\mu}{\epsilon}}$$

The series resistance per unit length

$$R = -\frac{2}{|I|^2}\frac{dP}{dz} = -\frac{2}{|H_0|^2 b^2}\left(-\frac{b}{\sigma\delta}|H_0|^2\right) = \frac{2}{\sigma\delta b}$$

The inductance per unit length

$$L = \frac{1}{|I|^2} \int \vec{B} \cdot \vec{H}^* da = \frac{1}{|H_0|^2 b^2} \left\{ ab\mu |H_0|^2 + 2 \int_{\text{conductor}} \vec{B} \cdot \vec{H}^* da \right\}$$

where the integration of the second term taking into account the magnetic energy stored inside the conductors. Note that inside the conductors

$$\vec{H}(\xi, t) = H_0 e^{-(1-i)\xi/\delta} e^{-i\omega t} \hat{x}$$

Thus

$$\int_{\text{conductor}} \vec{B} \cdot \vec{H}^* da = \mu_c \int_0^\infty |H_0|^2 e^{-2\xi/\delta} (b d\xi) = \frac{1}{2} \mu_c \delta |H_0|^2 b$$

Thus

$$L = \frac{1}{|H_0|^2 b^2} (ab\mu |H_0|^2 + 2 \cdot \frac{1}{2} \mu_c \delta |H_0|^2 b) = \frac{\mu a + \mu_c \delta}{b}$$

Alternative as suggested by Mr. Ben Burrington

Taking the results of Prob. 8.2 and making the following substitutions:

$$2\pi b_1 \Rightarrow b_2, \qquad b_1 - a_1 \Rightarrow a_2$$

where a_1, b_1 and a_2, b_2 are a, b's of Prob. 8.2 and Prob. 8.3 respectively. The substitutions are justified by the geometry. With

$$b_1 = \frac{b_2}{2\pi}, \qquad a_1 = \frac{b_2}{2\pi} - a_2$$

we have

$$\ln(\frac{b_1}{a_1}) = \ln(\frac{\frac{b_2}{2\pi}}{\frac{b_2}{2\pi} - a_2}) = -\ln(1 - \frac{2\pi a_2}{b_2}) \approx \frac{2\pi a_2}{b_2}, \quad \text{and} \quad \frac{1}{a_1} + \frac{1}{b_1} = \frac{1}{b_2/(2\pi) - a_2} + \frac{1}{b_2/(2\pi)} \approx \frac{4\pi}{b_2}$$

Thus

$$P = \sqrt{\frac{\mu}{\epsilon}} \pi a_1^2 |H_0|^2 \ln(\frac{b_1}{a_1}) \Rightarrow \sqrt{\frac{\mu}{\epsilon}} \pi (\frac{b_2}{2\pi})^2 |H_0|^2 \frac{2\pi a_2}{b_2} = \frac{a_2 b_2}{2} \sqrt{\frac{\mu}{\epsilon}} |H_0|^2$$

$$\gamma = \frac{1}{2\sigma\delta} \sqrt{\frac{\epsilon}{\mu}} \frac{1/a_1 + 1/b_1}{\ln(b_1/a_1)} \Rightarrow \frac{1}{2\sigma\delta} \sqrt{\frac{\epsilon}{\mu}} \frac{4\pi}{b_2} \cdot \frac{b_2}{2\pi a_2} = \frac{1}{a_2 \sigma\delta} \sqrt{\frac{\epsilon}{\mu}}$$

$$Z_0 = \frac{1}{2\pi} \sqrt{\frac{\mu}{\epsilon}} \ln(\frac{b_1}{a_1}) \Rightarrow \frac{1}{2\pi} \sqrt{\frac{\mu}{\epsilon}} \frac{2\pi a_2}{b_2} = \frac{a_2}{b_2} \sqrt{\frac{\mu}{\epsilon}}$$

$$R = \frac{1}{2\pi\sigma\delta} (\frac{1}{a_1} + \frac{1}{b_1}) \Rightarrow \frac{1}{2\pi\sigma\delta} \frac{4\pi}{b_2} = \frac{2}{\sigma\delta b_2}$$

$$L = \frac{\mu}{2\pi} \ln(\frac{b_1}{a_1}) + \frac{\mu_c \delta}{4\pi} (\frac{1}{a_1} + \frac{1}{b_1}) \Rightarrow \frac{\mu}{2\pi} \frac{2\pi a_2}{b_2} + \frac{\mu_c \delta}{4\pi} \frac{4\pi}{b_2} = \frac{\mu a_2 + \mu_c \delta}{b_2}$$

(b) For the case $b \gg h$, the electric and magnetic fields are mostly confined in the region between the strip and the ground plane and are uniform within the region. Therefore, this case is very similar to part (a) with the slab and its mirror image. However the case $b \ll h$ is very different from (a). This case can be approximated by a wire above a grounding plane. The dielectric substrate should have little effect on the quantities calculated in (a) since both electric and magnetic fields extend mostly in the region without the substrate.

Problem 8.4

(a) The wave equation is

$$(\nabla_t^2 + \gamma^2)\psi = 0 \quad \text{with} \quad \gamma^2 = \mu\epsilon\omega^2 - k^2$$

Explicitly in polar coordinates, the equation has the form:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial\psi}{\partial\rho}) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \gamma^2\psi = 0$$

This is the Bessel's equation and has the following solutions:

$$\psi \sim A_m J_m(\gamma\rho)e^{im\phi}$$

For TM modes,

$$\psi|_{\rho=R} = 0, \quad \Rightarrow \quad J_m(\gamma R) = 0$$

Let x_{mn} be the n^{th} root of $J_m(x) = 0$, then

$$\gamma_{mn} = \frac{x_{mn}}{R}, \quad \Rightarrow \quad \omega = \sqrt{\frac{x_{mn}^2}{R^2\mu\epsilon} + \frac{k^2}{\mu\epsilon}}$$

Thus, the cutoff frequencies are

$$\omega_{mn}^{TM} = \frac{x_{mn}}{R\sqrt{\mu\epsilon}} = \frac{v}{R}x_{mn}$$

Here $v \equiv 1/\sqrt{\mu\epsilon}$. The four lowest cutoff frequencies are

$$\omega_1^{TM} = \frac{v}{R}x_{01} = 2.405\frac{v}{R}, \quad \omega_2^{TM} = \frac{v}{R}x_{11} = 3.832\frac{v}{R}, \quad \omega_3^{TM} = \frac{v}{R}x_{21} = 5.136\frac{v}{R}, \quad \omega_4^{TM} = \frac{v}{R}x_{02} = 5.520\frac{v}{R}$$

For TE modes,

$$\frac{\partial\phi}{\partial\rho}|_{\rho=R} = 0, \quad \Rightarrow \quad J_m'(\gamma R) = 0$$

Let y_{mn} be the n^{th} root of $J_m'(y) = 0$, then

$$\gamma_{mn} = \frac{y_{mn}}{R}, \quad \Rightarrow \quad \omega = \sqrt{\frac{y_{mn}^2}{R^2\mu\epsilon} + \frac{k^2}{\mu\epsilon}}$$

Thus the lowest four cutoff frequencies are

$$\omega_1^{TE} = \frac{v}{R}y_{11} = 1.841\frac{v}{R}, \quad \omega_2^{TE} = \frac{v}{R}y_{21} = 3.054\frac{v}{R}, \quad \omega_3^{TE} = \frac{v}{R}y_{01} = 3.832\frac{v}{R}, \quad \omega_4^{TE} = \frac{v}{R}y_{31} = 4.201\frac{v}{R}$$

Combining TM and TE modes, the mode with the lowest cutoff frequency is TE_{11}:

$$\omega_0 = 1.841\frac{v}{R}$$

The other four modes with the lowest frequencies are TM_{01}, TE_{21}, TE_{01} and TM_{11} with the ratios of frequencies given by

$$\frac{\omega_{01}^{TM}}{\omega_0} = \frac{2.405}{1.841} = 1.3, \quad \frac{\omega_{21}^{TE}}{\omega_0} = \frac{3.504}{1.841} = 1.9, \quad \frac{\omega_{01}^{TE}}{\omega_0} = \frac{3.832}{1.841} = 2.1, \quad \frac{\omega_{11}^{TM}}{\omega_0} = \frac{3.832}{1.841} = 2.1$$

Note that the modes TE_{01} and TM_{11} are degenerate.

(b) The lowest mode is TE_{11}. The longitudinal magnetic field of the mode has the form

$$\psi(\rho,\phi) = AJ_1(y_{11}\frac{\rho}{R})e^{i\phi} \quad \text{with the cutoff frequency} \quad \omega_{11} = y_{11}\frac{v}{R}$$

Here A is a constant describing the strength of the field and $y_{11} = 1.841$ is the first root of $J'(y) = 0$. The average power

$$P = \frac{\mu}{2\sqrt{\mu\epsilon}}(\frac{\omega}{\omega_{11}})^2\sqrt{1-\frac{\omega_{11}^2}{\omega^2}}\int|\psi|^2 da = \frac{\pi\mu|A|^2}{\sqrt{\mu\epsilon}}(\frac{\omega}{\omega_{11}})^2\sqrt{1-\frac{\omega_{11}^2}{\omega^2}}\int_0^R |J_1(y_{11}\frac{\rho}{R})|^2\rho d\rho$$

Note the identity

$$\int_0^R |J_m(y_{mn}\frac{\rho}{R})|^2\rho d\rho = \frac{R^2}{2}(1-\frac{m^2}{y_{mn}^2})|J_m(y_{mn})|^2$$

Thus

$$P = (1-\frac{1}{y_{11}^2})\frac{\pi\mu|A|^2R^2}{2\sqrt{\mu\epsilon}}(\frac{\omega}{\omega_{11}})^2\sqrt{1-\frac{\omega_{11}^2}{\omega^2}}|J_1(y_{11})|^2$$

To calculate the power loss, we note

$$|\vec{n}\times\nabla_t\psi| = |(-\hat{\rho})\times\left\{\hat{\rho}\frac{\partial\psi}{\partial\rho}+\hat{\phi}\frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\right\}| = \frac{1}{\rho}|\frac{\partial\psi}{\partial\phi}| = \frac{1}{\rho}|AJ_1(y_{11}\frac{\rho}{R})|$$

Thus the power loss per unit length

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}(\frac{\omega}{\omega_{11}})^2\left\{\frac{1}{\mu\epsilon\omega_{11}^2}(1-\frac{\omega_{11}^2}{\omega^2})\oint|\vec{n}\times\nabla_t\psi|^2 d\ell + (\frac{\omega_{11}}{\omega})^2\oint|\psi|^2 d\ell\right\}$$

$$= \frac{\pi R|A|^2}{\sigma\delta}(\frac{\omega}{\omega_{11}})^2\left\{\frac{1}{\mu\epsilon\omega_{11}^2}(1-\frac{\omega^2}{\omega_{11}^2})\frac{1}{R^2}+(\frac{\omega_{11}}{\omega})^2\right\}|J_1(y_{11})|^2$$

Thus the attenuation constant

$$\beta_{11}^{TE} = \frac{1}{2P}(-\frac{dP}{dz}) = \frac{y_{11}^2}{y_{11}^2-1}\frac{\sqrt{\mu\epsilon}}{\mu R}\frac{1}{\sqrt{1-\omega_{11}^2/\omega^2}}\frac{1}{\sigma\delta}\left\{\frac{1}{\mu\epsilon\omega_{11}^2}(1-\frac{\omega_{11}^2}{\omega^2})\frac{1}{R^2}+(\frac{\omega_{11}}{\omega})^2\right\}$$

Note that

$$\delta = \sqrt{\frac{2}{\mu\omega\sigma}} \qquad \Rightarrow \qquad \sigma = \frac{2}{\mu\omega\delta^2}$$

Thus

$$\beta_{11}^{TE} = \frac{y_{11}^2}{y_{11}^2-1}\frac{\sqrt{\mu\epsilon}\delta}{2R}\frac{\omega}{\sqrt{1-\omega_{11}^2/\omega^2}}\left\{\frac{1}{\mu\epsilon\omega_{11}^2}(1-\frac{\omega_{11}^2}{\omega^2})\frac{1}{R^2}+(\frac{\omega_{11}}{\omega})^2\right\}$$

$$= \frac{\delta}{2Rv}\frac{\omega^2}{\sqrt{\omega^2-\omega_{11}^2}}\left\{\frac{1}{y_{11}^2-1}+(\frac{\omega_{11}}{\omega})^2\right\} = \frac{1}{R}\sqrt{\frac{\epsilon}{2\sigma}}\sqrt{\frac{\omega^3}{\omega^2-\omega_{11}^2}}\left\{\frac{1}{y_{11}^2-1}+(\frac{\omega_{11}}{\omega})^2\right\}$$

The second lowest mode is TM_{01}. In this case, the longitudinal component of the electric field is given by

$$\psi(\rho,\phi) = AJ_0(x_{01}\frac{\rho}{R}) \quad \text{with the cutoff frequency} \quad \omega_{01} = x_{01}\frac{v}{R}$$

where A is a constant and $x_{01} = 2.405$ is the first root of $J_0(x)$. The average power and the power loss per unit length

$$P = \frac{\epsilon}{2\sqrt{\mu\epsilon}}(\frac{\omega}{\omega_{01}})^2\sqrt{1-\frac{\omega_{01}^2}{\omega^2}}\int|\psi|^2 da = \frac{\pi\epsilon|A|^2}{\sqrt{\mu\epsilon}}(\frac{\omega}{\omega_{01}})^2\sqrt{1-\frac{\omega_{01}^2}{\omega^2}}\int_0^R|J_0(x_{01}\frac{\rho}{R})|^2\rho d\rho$$

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}(\frac{\omega}{\omega_{01}})^2\frac{1}{\mu^2\omega_{01}^2}\oint|\frac{\partial\psi}{\partial n}|^2 d\ell = \frac{\pi|A|^2R}{\mu^2\sigma\delta\omega_{01}^2}(\frac{\omega}{\omega_{01}})^2\left\{|\frac{\partial}{\partial\rho}J_0(x_{01}\frac{\rho}{R})|^2\right\}_{\rho=R}$$

Applying the following identity equations for Bessel's functions

$$\int_0^R |J_m(x_{mn}\frac{\rho}{R})|^2 \rho d\rho = \frac{R^2}{2}|J_{m+1}(x_{mn})|^2 \quad \text{and} \quad \frac{d}{dx}\{x^m J_m(x)\} = x^m J_{m-1}(x)$$

we get

$$\int_0^R |J_0(x_{01}\frac{\rho}{R})|^2 \rho d\rho = \frac{R^2}{2}|J_1(x_{01})|^2$$

$$\left\{|\frac{\partial}{\partial\rho} J_0(x_{01}\frac{\rho}{R})|^2\right\}_{\rho=R} = \frac{x_{01}^2}{R^2}|J_1(x_{01})|^2$$

Thus the attenuation constant

$$\beta_{01}^{TM} = \frac{1}{2P}(-\frac{dP}{dz}) = \frac{\sqrt{\mu\epsilon}}{\mu^2\epsilon}\frac{x_{01}^2}{\sigma\delta R^3}\frac{1}{\omega_{01}^2\sqrt{1-\omega_{01}^2/\omega^2}} = \frac{x_{01}^2}{2\sqrt{\mu\epsilon}}\frac{\delta}{R^3}(\frac{\omega}{\omega_{01}})^2\frac{1}{\sqrt{\omega^2-\omega_{01}^2}}$$

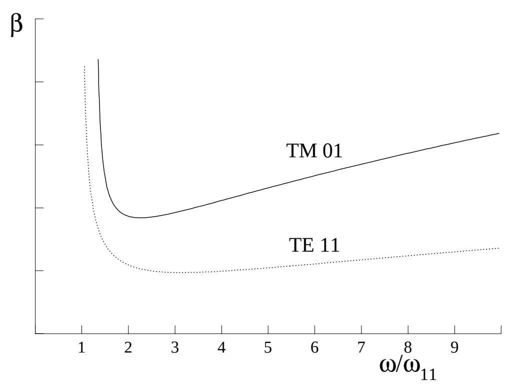
$$= \frac{\delta}{2Rv}\frac{\omega^2}{\sqrt{\omega^2-\omega_{01}^2}} = \frac{1}{R}\sqrt{\frac{\epsilon}{2\sigma}}\sqrt{\frac{\omega^3}{\omega^2-\omega_{01}^2}}$$

Note that for TM modes, β is always minimum at $\omega = \sqrt{3}\omega_0$, where ω_0 is the cutoff frequency. To facilitate the comparision, we rewrite the two constants as

$$\beta_{11}^{TE} = \frac{1}{R}\sqrt{\frac{\epsilon\omega_{11}}{2\sigma}}\sqrt{\frac{(\omega/\omega_{11})^3}{(\omega/\omega_{11})^2-1}}\left\{\frac{1}{y_{11}^2-1}+\frac{1}{(\omega/\omega_{11})^2}\right\}$$

$$\beta_{01}^{TM} = \frac{1}{R}\sqrt{\frac{\epsilon}{2\sigma}}\sqrt{\frac{\omega^3}{\omega^2-\omega_{01}^2}} = \frac{1}{R}\sqrt{\frac{\epsilon\omega_{11}}{2\sigma}}\sqrt{\frac{(\omega/\omega_{11})^3}{(\omega/\omega_{11})^2-(x_{01}/y_{11})^2}}$$

The two constants in the unit of $(1/R)\sqrt{(\epsilon\omega_{11})/(2\sigma)}$ are plotted in the figure below as functions of ω/ω_{11}.



Problem 8.16(a)

The eigenangle θ_p of the p^{th} mode is the solution of Eq. (8.123):

$$\tan(ka\sin\theta_p - \frac{p\pi}{2}) = \sqrt{\frac{2\Delta}{\sin^2\theta_p} - 1}$$

Note that

$$k_z = k\cos\theta_p \qquad \Rightarrow \qquad \sin\theta_p = \sqrt{1-\cos^2\theta_p} = \sqrt{1-\frac{k_z^2}{k^2}} = \frac{1}{k}\sqrt{k^2-k_z^2}$$

Thus

$$\tan\left\{a\sqrt{k^2-k_z^2}-\frac{p\pi}{2}\right\} = \sqrt{\frac{2\Delta k^2}{k^2-k_z^2}-1}$$

Differentiating the above equation with respect to k:

$$\frac{1}{\cos^2\left\{a\sqrt{k^2-k_z^2}-p\pi/2\right\}}\frac{a}{2}\frac{2k-2k_z(dk_z/dk)}{\sqrt{k^2-k_z^2}} = \frac{1}{2}\frac{\{(4\Delta k)(k^2-k_z^2)-(2\Delta k^2)(2k-2k_z(dk_z/dk))\}/(k^2-k_z^2)^2}{\sqrt{2\Delta k^2/(k^2-k_z^2)-1}}$$

After some algebra, the above equation can be written as

$$\frac{\sqrt{2\Delta k^2/(k^2-k_z^2)-1}}{\cos^2\left\{a\sqrt{k^2-k_z^2}-p\pi/2\right\}}\frac{a}{2}(k-k_z\frac{dk_z}{dk}) = \frac{\Delta k}{\sqrt{k^2-k_z^2}} - \frac{\Delta k^2}{(k^2-k_z^2)^{3/2}}(k-k_z\frac{dk_z}{dk})$$

We now note

$$k^2-k_z^2 = k_x^2 = k^2\sin^2\theta_p \quad \text{and} \quad \cos^2\left\{a\sqrt{k^2-k_z^2}-\frac{p\pi}{2}\right\} = \frac{1}{1+\tan^2\left\{a\sqrt{k^2-k_z^2}-p\pi/2\right\}} = \frac{\sin^2\theta_p}{2\Delta}$$

Plugging into the above equation:

$$\sqrt{\frac{2\Delta}{\sin^2\theta_p}-1}\left\{\frac{2\Delta}{\sin^2\theta_p}\right\}\frac{ka}{2}(1-\cos\theta_p\frac{dk_z}{dk}) = \frac{\Delta}{\sin\theta_p} - \frac{\Delta}{\sin^3\theta_p}(1-\cos\theta_p\frac{dk_z}{dk})$$

Solving for dk_z/dk:

$$\frac{dk_z}{dk} = \frac{1}{\cos\theta_p}\frac{\cos^2\theta_p + ka\sqrt{2\Delta-\sin^2\theta_p}}{1+ka\sqrt{2\Delta-\sin^2\theta_p}}$$

Note that $k = n_1\omega/c$, therefore the axial group velocity

$$v_g = \frac{d\omega}{dk_z} = \frac{d\omega}{dk}\frac{dk}{dk_z} = \frac{c\cos\theta_p}{n_1}\frac{1+ka\sqrt{2\Delta-\sin^2\theta_p}}{\cos^2\theta_p+ka\sqrt{2\Delta-\sin^2\theta_p}} = \frac{c\cos\theta_p}{n_1}\frac{1+\beta_p a}{\cos^2\theta_p+\beta_p a}$$

where $\beta_p = k\sqrt{2\Delta-\sin^2\theta_p}$. The group velocity is greater than the expected $c\cos\theta_p/n_1$. This is consistent with the Goos-Hanchen effect that the right ray is shifted forward after total internal reflection, resulting a greater group velocity.

Problem 8.20

(a) The field in the waveguide can be written as

$$\vec{E}^{(\pm)} = \sum_\lambda A_\lambda^{(\pm)}\vec{E}_\lambda^{(\pm)}$$

where the coefficients $A_\lambda^{(\pm)}$ are given by Eq. (8.146):

$$A_\lambda^{(\pm)} = -\frac{Z_\lambda}{2}\int_V \vec{J}\cdot\vec{E}_\lambda^{(\mp)}d\tau = -\frac{Z_\lambda}{2}\int \vec{I}\cdot\vec{E}_\lambda^{(\mp)}d\ell$$

Choose the bottom-left corner of the guide as the coordinate origin with the $x-$axis along the edge a and the $y-$axis along the edge b.

$$\vec{I} = I_0(-\sin\phi\hat{x} + \cos\phi\hat{y})$$

Here ϕ is the polar angle with respect to the center-of-the loop. Thus

$$A_\lambda^{(\pm)} = -\frac{Z_\lambda}{2}\int_{-\pi/2}^{\pi/2} I_0(-\sin\phi\hat{x} + \cos\phi\hat{y}) \cdot \vec{E}_\lambda^{(\mp)}(Rd\phi) = -\frac{1}{2}RI_0Z_\lambda\int_{-\pi/2}^{\pi/2}\left\{-\sin\phi\{E_\lambda^{(\mp)}\}_x + \cos\phi\{E_\lambda^{(\mp)}\}_y\right\}d\phi$$

where $\{E_\lambda^{(\mp)}\}_x$ and $\{E_\lambda^{(\mp)}\}_y$ are $x-$ and $y-$ components of the eigen-field along the loop.
For TM waves, the electric field components are given by Eq. (8.135):

$$\{E_{mn}^{(\mp)}\}_x = \frac{2\pi m}{\gamma_{mn}a\sqrt{ab}}\cos\{\frac{m\pi(R\cos\phi)}{a}\}\sin\{\frac{n\pi(h+R\sin\phi)}{b}\}$$

$$\{E_{mn}^{(\mp)}\}_y = \frac{2\pi n}{\gamma_{mn}b\sqrt{ab}}\sin\{\frac{m\pi(R\cos\phi)}{a}\}\cos\{\frac{n\pi(h+R\sin\phi)}{b}\}$$

Here

$$\gamma_{mn}^2 = \pi^2\left\{\frac{m^2}{a^2} + \frac{n^2}{b^2}\right\}$$

Therefore,

$$A_{mn}^{(\pm)} = -\frac{\pi RI_0Z_{mn}}{\gamma_{mn}\sqrt{ab}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{-\frac{m}{a}\sin\phi\cos(\frac{m\pi R\cos\phi}{a})\sin(\frac{n\pi(h+R\sin\phi)}{b}) + \frac{n}{b}\cos\phi\sin(\frac{m\pi R\cos\phi}{a})\cos(\frac{n\pi(h+R\sin\phi)}{b})\right\}d\phi$$

$$= -\frac{\pi RI_0Z_{mn}}{\gamma_{mn}\sqrt{ab}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{\frac{1}{\pi R}\frac{d}{d\phi}\{\sin(\frac{m\pi R\cos\phi}{a})\}\sin(\frac{n\pi(h+R\sin\phi)}{b}) + \frac{1}{\pi R}\sin(\frac{m\pi R\cos\phi}{a})\frac{d}{d\phi}\{\sin(\frac{n\pi(h+R\sin\phi)}{b})\}\right\}d\phi$$

$$= -\frac{I_0Z_{mn}}{\gamma_{mn}\sqrt{ab}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d}{d\phi}\left\{\sin(\frac{m\pi R\cos\phi}{a})\sin(\frac{n\pi(h+R\sin\phi)}{b})\right\}d\phi$$

$$= -\frac{I_0Z_{mn}}{\gamma_{mn}\sqrt{ab}}\sin(\frac{m\pi R\cos\phi}{a})\sin(\frac{n\pi(h+R\sin\phi)}{b})|_{\phi=-\pi/2}^{\phi=\pi/2} = 0$$

Therefore, no TM modes are excited. This is because that a circular current in the transverse plane will always result in a non-vanishing longitudinal component of $\vec{H}$, *i.e.*, $H_z \neq 0$.
(b) For TE waves,

$$\{E_{mn}^{(\mp)}\}_x = -\frac{2\pi n}{\gamma_{mn}b\sqrt{ab}}\cos(\frac{m\pi R\cos\phi}{a})\sin(\frac{n\pi(h+R\sin\phi)}{b})$$

$$\{E_{mn}^{(\mp)}\}_y = \frac{2\pi m}{\gamma_{mn}a\sqrt{ab}}\sin(\frac{m\pi R\cos\phi}{a})\cos(\frac{n\pi(h+R\sin\phi)}{b})$$

with the normalization reduced by a factor of $\sqrt{2}$ if $m=0$ or $n=0$. Thus

$$A_{mn}^{(\pm)} = -\frac{\pi RI_0Z_{mn}}{\gamma_{mn}\sqrt{ab}}\int_{-\pi/2}^{\pi/2}\left\{\frac{n}{b}\sin\phi\cos(\frac{m\pi R\cos\phi}{a})\sin(\frac{n\pi(h+R\sin\phi)}{b}) + \frac{m}{a}\cos\phi\sin(\frac{m\pi R\cos\phi}{a})\cos(\frac{n\pi(h+R\sin\phi)}{b})\right\}d\phi$$

The lowest modes ($m=1, n=0$):

$$A_{1,0} = -\frac{\pi RI_0Z_{1,0}}{\gamma_{1,0}\sqrt{2a^3b}}\int_{-\pi/2}^{\pi/2}\left\{\cos\phi\sin(\frac{\pi R\cos\phi}{a})\right\}d\phi = -\frac{\pi RI_0Z_{1,0}}{\gamma_{1,0}\sqrt{2a^3b}}\left\{\pi J_1(\frac{\pi R}{a})\right\}$$

where $\gamma_{1,0} = \pi/a$. Here we have used the integral representation of Bessel functions:

$$\int_0^\pi \sin\theta \sin(x\sin\theta)d\theta = \int_{-\pi/2}^{\pi/2} \cos\phi \sin(x\cos\phi)d\phi = J_1(x)$$

The amplitude is independent of the height h. For $R \ll a$,

$$J_1(\frac{\pi R}{a}) \approx \frac{\pi R}{2a}, \qquad \Rightarrow \qquad A_{1,0} \approx -\frac{\pi^3 R^2 I_0 Z_{1,0}}{\gamma_{1,0}\sqrt{8a^5 b}}$$

(c) The average power radiated in either direction

$$P = \frac{1}{2}\int (\vec{E} \times \vec{H}^*) \cdot \hat{z}da = \frac{1}{2}\int \left\{ (\sum_\lambda A_\lambda \vec{E}_\lambda) \times (\sum_\mu A_\mu^* \vec{H}^*{}_\mu) \right\} \cdot \hat{z}da = \frac{1}{2}\sum_{\lambda\mu} A_\lambda A_\mu^* \int (\vec{E}_\lambda \times \vec{H}^*{}_\mu) \cdot \hat{z}da = \frac{1}{2}\sum_\lambda \frac{|A_\lambda|^2}{Z_\lambda}$$

In this case,

$$P = \frac{1}{2}\frac{|A_{1,0}|^2}{Z_{1,0}} = \frac{1}{2Z_{1,0}}|A_{1,0}|^2 \approx \frac{I_0^2}{16} Z_{1,0} \frac{a}{b} (\frac{\pi R}{a})^4$$