Instructor: Jianming Qian
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Physics 506: Solutions to Assignment #1

Problem 8.2

(a) In a cylindrical coordinate system with the $z-$axis along the axes of the two circular cylinders, the TEM mode has fields that vary as $e^{i(kz-\omega t)}$, where $k^2 = \omega^2/v^2 = \mu\epsilon\omega^2$. Therefore, the magnetic field has the form

$$\vec{B} = B_\phi \hat{\phi} e^{i(kz-\omega t)}$$

where B_ϕ is determined from Ampere's law:

$$\oint \vec{B} \cdot d\vec{\ell} = \mu I(z,t) = \mu I_0 e^{i(kz-\omega)t} \qquad \Rightarrow \qquad B_\phi = \frac{\mu I_0}{2\pi\rho}$$

Note that

$$H_0 = \frac{B_\phi|_{\rho=a}}{\mu} \qquad \Rightarrow \qquad I_0 = 2\pi a H_0$$

Therefore

$$\vec{B} = \mu H_0 \frac{a}{\rho} e^{i(kz-\omega t)}\hat{\phi} \qquad \text{and} \qquad \vec{H} = \frac{vecB}{\mu} = H_0 \frac{a}{\rho} e^{i(kz-\omega t)}\hat{\phi}$$

The electric field in between the two cylinders can be determined from the magnetic field through the Ampere-Maxwell's equation:

$$\nabla \times \vec{B} = \mu\epsilon \frac{\partial \vec{E}}{\partial t} \qquad \Rightarrow \qquad \vec{E} = -\frac{\vec{k} \times \vec{B}}{\mu\epsilon\omega} = \sqrt{\frac{\mu}{\epsilon}} H_0 \frac{a}{\rho} e^{i(kz-\omega t)}\hat{\rho}$$

Here $\vec{k} = k\hat{z} = \omega\sqrt{\mu\epsilon}\hat{z}$ is the wave vector. The average Poynting vector

$$\langle \vec{S} \rangle = \frac{1}{2}\vec{E} \times \vec{H}^* = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}} |H_0|^2 \frac{a^2}{\rho^2}\hat{z}$$

The average power flow along the line (neglecting the wires) is

$$P = \int \langle \vec{S} \rangle \cdot d\vec{a} = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}} |H_0|^2 a^2 \int_a^b \frac{2\pi\rho d\rho}{\rho^2} = \sqrt{\frac{\mu}{\epsilon}} (\pi a^2 |H_0|^2) \ln(\frac{b}{a})$$

(b) The average power loss per unit area on the cylinder surfaces is given by Eq. (8.15):

$$\frac{dP}{da} = -\frac{1}{2\sigma\delta}|\vec{K}_{\rm eff}|^2 = -\frac{1}{2\sigma\delta}|\vec{n} \times \vec{H}_{||}|^2 = -\frac{1}{2\sigma\delta}|H_\phi|^2 = -\frac{1}{2\sigma\delta}\frac{|B_\phi|^2}{\mu^2} = -\frac{1}{2\sigma\delta}|H_0|^2 \frac{a^2}{\rho^2}$$

The average power loss per unit length along the $z-$direction

$$\frac{dP}{dz} = \frac{dP}{da}|_{\rho=a}(2\pi a) + \frac{dP}{da}|_{\rho=b}(2\pi b) = -\frac{\pi a^2 |H_0|^2}{\sigma\delta}\left\{\frac{1}{a} + \frac{1}{b}\right\}$$

From (a), one has

$$\pi a^2 |H_0|^2 = \sqrt{\frac{\epsilon}{\mu}} \frac{P}{\ln(b/a)}$$

Plugging into dP/dz:

$$\frac{dP}{dz} = -\frac{1}{\sigma\delta}\sqrt{\frac{\epsilon}{\mu}} \frac{\frac{1}{a} + \frac{1}{b}}{\ln(\frac{b}{a})} P = -2\gamma P$$

where

$$\gamma = \frac{1}{2\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\frac{\frac{1}{a}+\frac{1}{b}}{\ln(\frac{b}{a})}$$

Integrating the above equation:

$$P(z) = P_0 e^{-2\gamma z}$$

(c) The characteristic impedance Z_0 is the ratio between the voltage and the current.

$$V = \int_a^b \vec{E}\cdot d\vec{\ell} = \sqrt{\frac{\mu}{\epsilon}}H_0 a\int_a^b \frac{d\rho}{\rho}e^{i(kz-\omega t)} = \sqrt{\frac{\mu}{\epsilon}}aH_0\ln(\frac{b}{a})e^{i(kz-\omega t)}$$

The current is given in (a) to be $I = 2\pi a H_0 e^{i(kz-\omega t)}$. The impedance is therefore

$$Z_0 = \frac{V}{I} = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\ln(\frac{b}{a})$$

(d) Series resistance per unit length can be calculated from the average power loss per unit length:

$$-\frac{1}{2}|I|^2 R = \frac{dP}{dz} \quad\Rightarrow\quad R = -\frac{2}{|I|^2}\frac{dP}{dz} = \frac{1}{2\pi\sigma\delta}\left\{\frac{1}{a}+\frac{1}{b}\right\}$$

The inductance per unit length can be calculated from the energy per unit length in the magnetic field:

$$\frac{1}{4}L|I|^2 = \int\frac{1}{4}\vec{B}\cdot\vec{H}^* da = \frac{1}{4}\left\{\int_0^a + \int_a^b + \int_b^\infty\right\}\vec{B}\cdot\vec{H}^*(2\pi\rho)d\rho$$

Note that inside the conductors,

$$\vec{H}(\xi,t) = \vec{H}_{||}e^{-(1-i)\xi/\delta}e^{-i\omega t} = H_\phi e^{-(1-i)\xi/\delta}e^{-i\omega t}\hat{\phi}$$

where ξ is the distance into the conductor and $\vec{H}_{||}$ is the tangential component of the field at the surface. Assuming $\delta \ll a$,

$$\int_0^a \vec{B}\cdot\vec{H}^* da = 2\pi\mu_c\int_0^a \mu_c|H_\phi|^2_{\rho=a}e^{-2\xi/\delta}\rho d\rho = 2\pi|H_0|^2\int_0^a e^{-2(a-\rho)/\delta}\rho d\rho \approx \pi\mu_c a\delta|H_0|^2$$

$$\int_a^b \vec{B}\cdot\vec{H}^* da = 2\pi\mu\int_a^b |H_\phi|^2\rho d\rho = 2\pi\mu a^2|H_0|^2\ln(\frac{b}{a})$$

$$\int_b^\infty \vec{B}\cdot\vec{H}^* da = 2\pi\int_b^\infty \mu_c|H_\phi|^2_{\rho=b}e^{-2\xi/\delta}\rho d\rho \approx \pi\mu_c\delta\frac{a^2}{b}|H_0|^2$$

The inductance per unit length

$$L = \frac{1}{|I|^2}\left\{\int_0^a + \int_a^b + \int_b^\infty\right\}\vec{B}\cdot\vec{H}^* da = \frac{1}{4\pi^2 a^2|H_0|^2}\left\{\int_0^a + \int_a^b + \int_b^\infty\right\}\vec{B}\cdot\vec{H}^* da = \frac{\mu}{2\pi}\ln(\frac{b}{a}) + \frac{\mu_c\delta}{4\pi}\left\{\frac{1}{a}+\frac{1}{b}\right\}$$

Problem 8.5

(a) Since the guide is a single conductor, there can be no TEM modes. To determine TM and TE modes, we choose a rectangular coordinate system with its origin at the middle of the side $\sqrt{2}a$ such that the three sides are described by $x = a/2$, $y = -a/2$, and $y = x$. For TM modes, the $\psi = 0$ on the surfaces. The boundary conditions at $x = a/2$ and $y = -a/2$ can be met by choosing $\psi_{mn}(x,y)$ to have the form:

$$\psi_{mn}(x,y) \sim \sin(\frac{m\pi(x-a/2)}{a})\sin(\frac{n\pi(y+a/2)}{a}) \sim \sin(\frac{m\pi(x+a/2)}{a})\sin(\frac{n\pi(y+a/2)}{a})$$

The boundary condition at $\psi(x,y)|_{y=x} = 0$ can be met by requiring $\psi(x,y)$ be antisymmetric under the exchange of $x \leftrightarrow y$, i.e.,

$$\psi(x,y) \sim \sin(\frac{m\pi(x+a/2)}{a})\sin(\frac{n\pi(y+a/2)}{a}) - \sin(\frac{n\pi(x+a/2)}{a})\sin(\frac{m\pi(y+a/2)}{a})$$

Thus, the TM waves have the general form

$$\psi(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\left\{\sin(\frac{m\pi(x+a/2)}{a})\sin(\frac{n\pi(y+a/2)}{a}) - \sin(\frac{n\pi(x+a/2)}{a})\sin(\frac{m\pi(y+a/2)}{a})\right\}$$

Here $A_{mn} = 0$ for $m = n$ (ψ vanishes if $m = n$). The corresponding cutoff frequencies are given by Eq. (8.44):

$$\omega_{mn} = \frac{\pi}{\sqrt{\mu\epsilon}}\sqrt{\frac{m^2}{a^2} + \frac{n^2}{a^2}} = \frac{\pi c}{a}\sqrt{m^2+n^2} \quad \text{with} \quad m \neq n \text{ and } m,n > 0$$

The dominant mode is $m = 1, n = 2$ or $m = 2, n = 1$:

$$\omega_{1,2} = \omega_{2,1} = \frac{\pi c}{a}\sqrt{5}$$

For TE waves, $\partial\psi/\partial n = 0$ on the surfaces. The conditions at $x = a/2$ and $y = -a/2$ are met by choosing $\psi_{mn}(x,y)$ to have the form:

$$\psi_{mn}(x,y) \sim \cos(\frac{m\pi(x+a/2)}{a})\cos(\frac{n\pi(y+a/2)}{a})$$

The boundary condition at $y = x$ is met if $\partial\psi/\partial n$

$$\frac{\partial\psi}{\partial n} = \vec{n}\cdot\nabla_t\psi = -\frac{1}{\sqrt{2}}(\frac{\partial\psi}{\partial x} - \frac{\partial\psi}{\partial y})$$

is antisymmetric under exchange $x \leftrightarrow y$, *i.e.*;

$$-\frac{1}{2}\left\{\frac{\partial\psi(x,y)}{\partial x} - \frac{\partial\psi(x,y)}{\partial y}\right\} = \frac{1}{2}\left\{\frac{\partial\psi(y,x)}{\partial y} - \frac{\partial\psi(y,x)}{\partial x}\right\}$$

Thus $\psi(x,y)$ must be symmetric under $x \leftrightarrow y$. Therefore, the TE waves have the general form

$$\psi(x,y) = \sum_{m,n=0}^{\infty} A_{mn}\left\{\cos(\frac{m\pi(x+a/2)}{a})\cos(\frac{n\pi(y+a/2)}{a}) + \cos(\frac{n\pi(x+a/2)}{a})\cos(\frac{m\pi(y+a/2)}{a})\right\}$$

The corresponding cutoff frequencies are

$$\omega_{mn} = \frac{\pi}{\sqrt{\mu\epsilon}}\sqrt{\frac{m^2}{a^2} + \frac{n^2}{a^2}} = \frac{\pi c}{a}\sqrt{m^2+n^2} \quad \text{with and } m,n \geq 0$$

Though m and n can be equal in this case, however they cannot be both zero. Otherwise, $H_z = \psi$ is a constant, which leads to vanishing $\vec{H}_t$ and $\vec{E}_t$. Consequently there is no wave. The dominant mode is therefore $m = 1, n = 0$ or $m = 0, n = 1$:

$$\omega_{1,0} = \omega_{0,1} = \frac{\pi c}{a}$$

Problem 8.8

(a) Assume $h \ll a$, $\delta_i, \delta_e \ll a$, the electric and magnetic fields are approximately

$$E_r = -\frac{ic^2}{\omega_\ell r}\ell(\ell+1)\frac{u_\ell(r)}{r}P_\ell(\cos\theta) \approx -\frac{ic^2}{\omega_\ell}\ell(\ell+1)\frac{u_\ell(a)}{a^2}P_\ell(\cos\theta) = -\frac{ic}{a}\sqrt{\ell(\ell+1)}u_\ell(a)P_\ell(\cos\theta)$$

$$E_\theta = -\frac{ic^2}{\omega r}\frac{du_\ell(r)}{dr}P_\ell^1(\cos\theta) \approx -\frac{ic^2}{\omega a}\frac{du_\ell(r)}{dr}|_{r=a}P_\ell^1(\cos\theta) = 0$$

$$B_\phi = \frac{u_\ell(r)}{r}P_\ell^1(\cos\theta) \approx \frac{u_\ell(a)}{a}P_\ell^1(\cos\theta)$$

where $\omega_\ell = \sqrt{\ell(\ell+1)}c/a$ is the resonance frequency and $u_\ell(r)$ is the solution of the radial equation (8.103). The average energy stored in the fields

$$U = \frac{1}{4}\int(\vec{D}\cdot\vec{E}^* + \vec{B}\cdot\vec{H}^*)d\tau = \frac{1}{4}\int(\epsilon|\vec{E}|^2 + \frac{|\vec{B}|^2}{\mu})d\tau$$

$$= \frac{1}{4}\int_a^{a+h} r^2 dr \int d\Omega \left\{\epsilon\frac{c^2}{a^2}\ell(\ell+1)u_\ell^2(a)[P_\ell(\cos\theta)]^2 + \frac{1}{\mu}\frac{u_\ell^2(a)}{a^2}[P_\ell^1(\cos\theta)]^2\right\}$$

$$= \frac{1}{4}\frac{u_\ell^2(a)}{a^2}\cdot\frac{1}{3}\{(a+h)^3 - a^3\}\int d\Omega\left\{\epsilon c^2\ell(\ell+1)[P_\ell(\cos\theta)]^2 + \frac{1}{\mu}[P_\ell^1(\cos\theta)]^2\right\}$$

$$\approx \frac{1}{4}\frac{u_\ell^2(a)}{a^2}\frac{1}{3}a^3\left\{1 + 3\frac{h}{a} - 1\right\}(2\pi)\int_0^\pi \sin\theta d\theta\left\{\frac{1}{\mu}\ell(\ell+1)[P_\ell(\cos\theta)]^2 + \frac{1}{\mu}[P_\ell^1(\cos\theta)]^2\right\}$$

$$= \frac{\pi h u_\ell^2(a)}{2\mu}\int_{-1}^{+1} dx\left\{\ell(\ell+1)[P_\ell(x)]^2 + [P_\ell^1(x)]^2\right\}$$

Note that

$$\int_{-1}^{+1}[P_\ell(x)]^2 dx = \frac{2}{2\ell+1}, \qquad \int_{-1}^{+1}[P_\ell^m(x)]^2 dx = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}$$

Plugging into U:

$$U = \frac{2\pi h u_\ell^2(a)}{\mu}\frac{\ell(\ell+1)}{2\ell+1}$$

(Note that the average energies in electric and magnetic fields are equal). The average power loss is given by Eq. (8.15):

$$\frac{dP}{da} = \frac{1}{2\sigma\delta}|\vec{K}_{\text{eff.}}|^2 = \frac{1}{2\sigma\delta}|\vec{n}\times\vec{H}_{||}|^2 = \frac{1}{2\sigma\delta}|\vec{H}_{||}|^2$$

Now note that

$$|\vec{H}_{||}| = \frac{B_\phi}{\mu} = \frac{u_\ell(a)}{\mu a}P_\ell^1(\cos\theta)$$

The average total power loss

$$P_{\text{loss}} = \int_{\text{interior}}\frac{dP}{da}da + \int_{\text{exterior}}\frac{dP}{da}da = \frac{\mu_\ell^2(a)}{2\mu^2}\left\{\frac{1}{\sigma_i\delta_i} + \frac{1}{\sigma_e\delta_e}\right\}\int[P_\ell^1(\cos\theta)]^2 d\Omega$$

$$= \frac{2\pi u_\ell^2(a)}{\mu^2}\left\{\frac{1}{\sigma_i\delta_i} + \frac{1}{\sigma_e\delta_e}\right\}\frac{\ell(\ell+1)}{2\ell+1}$$

Note that

$$\delta^2 = \frac{2}{\mu\omega_\ell\sigma} \qquad \Rightarrow \qquad \frac{1}{\sigma\delta} = \frac{\mu\omega_\ell\delta}{2}$$

Therefore,

$$P_{\rm loss} = \frac{2\pi u_\ell^2(a)}{\mu^2}\left\{\frac{\mu\omega\delta_i}{2} + \frac{\mu\omega\delta_e}{2}\right\} = \frac{\pi u_\ell^2(a)\omega_\ell}{\mu}\frac{\ell(\ell+1)}{2\ell+1}(\delta_i+\delta_e)$$

The Schumann resonance Q value:

$$Q = \omega_\ell \frac{U}{P_{\rm loss}} = \omega_\ell \frac{2\pi h u_\ell^2(a)}{\mu}\frac{\ell(\ell+1)}{2\ell+1}\frac{\mu}{\pi u_\ell^2(a)\omega_\ell}\frac{2\ell+1}{\ell(\ell+1)}\frac{1}{\delta_i+\delta_e} = \frac{2h}{\delta_i+\delta_e}$$

independent of ℓ and $N=2$.
(b) For the lowest Schumann resonance,

$$\ell = 1 \;\Rightarrow\; \omega_1 = \sqrt{2}\frac{c}{a} = \sqrt{2}\frac{3\cdot 10^8}{6.4\cdot 10^6} = 66.3 \text{ Hz}$$

$$\delta_e = \sqrt{\frac{2}{\mu\omega_1\sigma_e}} = \sqrt{\frac{2}{4\pi\cdot 10^{-7}\times 66.3\times 0.1}} = 4.9\cdot 10^2 \text{ m}$$

$$\delta_i = \sqrt{\frac{2}{\mu\omega_1\sigma_i}} = \sqrt{\frac{2}{4\pi\cdot 10^{-7}\times 66.3\times 10^{-5}}} = 4.9\cdot 10^4 \text{ m}$$

$$Q = \frac{2h}{\delta_e+\delta_i} = \frac{2\times 10^5}{4.9\cdot 10^2(1+100)} = 4.0$$

(c) With $\sigma_i \approx 10^{-5}$ $(\Omega\text{m})^{-1}$, $\delta_i \approx 49$ km is not small compared with $h \approx 100$ km. However, the fields vary over distances of order a, at least for $\ell = 1$. Thus, the approximation of Section 8.1 are valid, at least for small ℓ values. When a/ℓ becomes of order of δ_i, these approximations won't be adequate. In this case, it occurs at $\ell \sim 100$.

Problem 8.18
(a) For TM modes, we have

$$(\nabla_t^2 + \gamma_\lambda^2)E_{z\lambda} = 0, \quad \text{and} \quad E_{z\lambda}|_C = 0$$

where the subscript C denotes boundary contour. Applying Green's theorem in two dimension:

$$\int_S (\phi\nabla_t^2\psi - \psi\nabla_t^2\phi)da = -\oint_C (\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n})d\ell$$

where the $-$ on the right side is due to the difference in the normal definition. With $\phi = E_{z\lambda}$ and $\psi = E_{z\mu}$, we gets

$$\int_S (E_{z\lambda}\nabla_t^2 E_{z\mu} - E_{z\mu}\nabla_t^2 E_{z\lambda})da = -\oint_C (E_{z\lambda}\frac{\partial E_{z\mu}}{\partial n} - E_{z\mu}\frac{\partial E_{z\lambda}}{\partial n})d\ell$$

The line integrals on the right-hand side vanish due to the boundary conditions. Therefore,

$$0 = \int_S (E_{z\lambda}\nabla_t^2 E_{z\mu} - E_{z\mu}\nabla_t^2 E_{z\lambda})da = \int_S \left\{E_{z\lambda}(-\gamma_\mu^2 E_{z\mu}) - E_{z\mu}(-\gamma_\lambda^2 E_{z\lambda})\right\} da = (\gamma_\lambda^2 - \gamma_\mu^2)\int_S E_{z\lambda}E_{z\mu}da$$

For the case $\gamma_\lambda \neq \gamma_\mu$, the integral must vanish:

$$\int_S E_{z\lambda}E_{z\mu}da = 0$$

Same argument applies to $H_{z\lambda}$ and $H_{z\mu}$, except in this case, the line integrals vanishes due to boundary conditions

$$\frac{\partial H_z}{\partial n}|_C = 0$$

(b) *Proof for TM modes only*
Applying Green's first identity

$$\int_S (\phi\nabla_t^2\psi + \nabla_t\phi\cdot\nabla_t\psi)da = -\oint_C \phi\frac{\partial\psi}{\partial n}d\ell$$

with $\phi = E_{z\lambda}$ and $\psi = E_{z\mu}$ for the TM modes, we get

$$\int_S (E_{z\lambda}\nabla_t^2 E_{z\mu} + \nabla_t E_{z\lambda}\cdot\nabla_t E_{z\mu})da = -\oint_C E_{z\lambda}\frac{\partial}{\partial n}E_{z\mu}d\ell$$

Again, the line integral on the right vanishes due to the boundary condition. $\nabla_t E_z$ and $\nabla_t^2 E_z$ are given by Eqs. (8.33, 8.34):

$$\nabla_t^2 E_{z\lambda} = -\gamma_\lambda^2 E_{z\lambda}, \qquad \nabla_t^2 E_{z\mu} = -\gamma_\mu^2 E_{z\mu}$$

$$\nabla_t E_{z\lambda} = -i\frac{\gamma_\lambda^2}{k_\lambda}\vec{E}_\lambda, \qquad \nabla_t E_{z\mu} = -i\frac{\gamma_\mu^2}{k_\mu}\vec{E}_\mu$$

where $\vec{E}_\lambda$ and $\vec{E}_\mu$ are transverse electric fields. The Green's first identify becomes

$$\gamma_\mu^2\int_S E_{z\lambda}E_{z\mu}da + \frac{\gamma_\lambda^2\gamma_\mu^2}{k_\lambda k_\mu}\int_S \vec{E}_\lambda\cdot\vec{E}_\mu da = 0 \quad \Rightarrow \quad \int_S E_{z\lambda}E_{z\mu}da = -\frac{\gamma_\mu^2}{k_\lambda k_\mu}\int_S \vec{E}_\lambda\cdot\vec{E}_\mu da$$

Assuming non-degeneracy and from (a), we obtain:

$$\text{For } \lambda\neq\mu; \quad \int_S \vec{E}_\lambda\cdot\vec{E}_\mu da = 0$$

By properly normalizing $\vec{E}_\lambda$, we have

$$\int_S \vec{E}_\lambda\cdot\vec{E}_\mu da = \delta_{\lambda\mu} \qquad \text{Eq.(8.131)}$$

$$\int_S E_{z\lambda}E_{z\mu}da = -\frac{\gamma_\mu^2}{k_\lambda k_\mu}\int_S \vec{E}_\lambda\cdot\vec{E}_\mu da = -\frac{\gamma_\lambda^2}{k_\lambda^2}\delta_{\lambda\mu} \qquad \text{Eq.(8.134) for TM waves}$$

Now turn into the relations of magnetic fields. Note that

$$\vec{H}_\lambda = \frac{\epsilon\omega}{k_\lambda}\hat{z}\times\vec{E}_\lambda = \frac{i\epsilon\omega}{\gamma_\lambda^2}\hat{z}\times\nabla_t E_{z\lambda}, \qquad \vec{H}_\mu = \frac{\epsilon\omega}{k_\mu}\hat{z}\times\vec{E}_\mu = \frac{i\epsilon\omega}{\gamma_\mu^2}\hat{z}\times\nabla_t E_{z\mu}$$

$$\vec{H}_\lambda\cdot\vec{H}_\mu = -\frac{(\epsilon\omega)^2}{\gamma_\lambda^2\gamma_\mu^2}(\hat{z}\times\nabla_t E_{z\lambda})\cdot(\hat{z}\times\nabla_t E_{z\mu}) = -\frac{(\epsilon\omega)^2}{\gamma_\lambda^2\gamma_\mu^2}\nabla_t E_{z\mu}\cdot\{(\hat{z}\times\nabla_t E_{z\lambda})\times\hat{z}\} = -\frac{(\epsilon\omega)^2}{\gamma_\lambda^2\gamma_\mu^2}\nabla_t E_{z\mu}\cdot\nabla_t E_{z\lambda}$$

Using Green's first identity with $\phi = E_{z\lambda}$ and $\psi = E_{z\mu}$, we have

$$\int \nabla_t E_{z\mu}\cdot\nabla_t E_{z\lambda}da = -\oint E_{z\mu}\frac{\partial E_{z\lambda}}{\partial n} - \int E_{z\mu}\nabla_t^2 E_{z\lambda}da = \gamma_\lambda^2\int E_{z\mu}E_{z\lambda}da = -\frac{\gamma_\lambda^4}{k_\lambda^2}\delta_{\lambda\mu}$$

Thus,

$$\int \vec{H}_\lambda\cdot\vec{H}_\mu da = -\frac{(\epsilon\omega)^2}{\gamma_\lambda^2\gamma_\mu^2}\int \nabla_t E_{z\mu}\cdot\nabla_t E_{z\lambda}da = -\frac{(\epsilon\omega)^2}{\gamma_\lambda^2\gamma_\mu^2}(-\frac{\gamma_\lambda^4}{k_\lambda^2}\delta_{\lambda\mu}) = \frac{(\epsilon\omega)^2}{k_\lambda^2}\delta_{\lambda\mu} = \frac{1}{Z_\lambda^2}\delta_{\lambda\mu} \qquad \text{Eq. (8.132)}$$

where $Z_\lambda = k_\lambda/(\epsilon\omega)$ is the wave impedance.

$$\frac{1}{2}\int(\vec{E}_\lambda\times\vec{H}_\mu)\cdot\hat{z}da = \frac{1}{2}\int\left\{\frac{ik_\lambda}{\gamma_\lambda^2}\nabla_t E_{z\lambda}\times\frac{i\epsilon\omega}{\gamma_\mu^2}(\hat{z}\times\nabla_t E_{z\mu})\right\}\cdot\hat{z}da$$

$$= -\frac{1}{2}\int\frac{k_\lambda\epsilon\omega}{\gamma_\lambda^2\gamma_\mu^2}(\nabla_t E_{z\lambda}\cdot\nabla_t E_{z\mu})da = -\frac{1}{2}\frac{k_\lambda\epsilon\omega}{\gamma_\lambda^2\gamma_\mu^2}(-\frac{\gamma_\lambda^4}{k_\lambda^2}\delta_{\lambda\mu}) = \frac{1}{2Z_\lambda}\delta_{\lambda\mu} \qquad \text{Eq. (8.133)}$$