Physics 506 Winter 2004

G. Raithel

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Disclaimer: The purpose of these notes is to provide you with a general list of topics that were covered in class. The notes are not a substitute for reading the textbook, nor is it guaranteed that they are complete. If you find typos, please report them to me.

1 1/6/2004

The microscopic and macroscopic Maxwell equations have been reviewed. From the microscopic equations and under the assumptions of harmonic time dependence of the fields, well-defined $\epsilon(\omega)$ and $\mu(\omega)$, and source-free conditions, one obtains a homogeneous Helmholtz equation for the fields,

$$\left(\nabla^2 + \epsilon\mu\omega^2\right)\begin{pmatrix}\mathbf{E}\\ \mathbf{B}\end{pmatrix} = 0 \tag{1}$$

Under the absence of boundary conditions, the equation can be solved,yielding, in cartesian coordinates, plane-wave solutions (see Chapter 7 of Jackson). Some basic properties have been reviewed. In particular, the dispersion relation of plane waves is

$$k = \frac{\omega}{c} = \sqrt{\epsilon\mu}\omega = n\omega \tag{2}$$

with refractive index n. There is no cutoff frequency, *i.e.* under absence of polarization damping plane waves with real k exist down to arbitrarily low frequency.

We consider a waveguide geometry invariant under translation in z-direction. The waveguide walls form a set of at least one closed surface S in the xy-plane. Using the ansatz

$$\mathbf{E}(x,y,z,t) = \mathbf{E}(x,y)\exp(\mathrm{i}kz - \mathrm{i}\omega t)$$

- same for $\mathbf{B}$ -, with $k \in \mathbb{C}$. Writing $\nabla := \nabla_t + \frac{\partial}{\partial z}$ we find after insertion into Eq. 1

$$\left(\nabla_t^2 + \epsilon\mu\omega^2 - k^2\right)\begin{pmatrix}\mathbf{E}\\ \mathbf{B}\end{pmatrix} = 0 \tag{3}$$

with boundary conditions on S tbd. Note that this equation is for the fields that depend only on x and y. Also, for the different solutions we will find dispersion relations $k(\omega)$ that are generally different from the free-space one (Eq. 2).

We decompose $\mathbf{E}(x,y)$ into transverse and longitudinal parts, $\mathbf{E}(x,y) = \hat{\mathbf{z}}E_z(x,y)+\mathbf{E}_t(x,y)$. For harmonic fields in a linear medium it follows then from the homogeneous Maxwell's equations

$$\begin{aligned}\mathbf{E}_t(x,y) &= \frac{\mathrm{i}}{\epsilon\mu\omega^2 - k^2}\left[k\nabla_t E_z(x,y) - \omega\hat{\mathbf{z}} \times \nabla_t B_z(x,y)\right] \\ \mathbf{B}_t(x,y) &= \frac{\mathrm{i}}{\epsilon\mu\omega^2 - k^2}\left[k\nabla_t B_z(x,y) + \omega\epsilon\mu\hat{\mathbf{z}} \times \nabla_t E_z(x,y)\right] \end{aligned} \qquad (4)$$

where k is positive or negative, dependent of the direction of propagation. Thus, the transverse fields follow from the longitudinal ones unless $\epsilon\mu\omega^2 - k^2 = 0$.

In the case $\epsilon\mu\omega^2 - k^2 = 0$ it is $E_z = B_z = 0$, and a special treatment is necessary to find the transverse fields. The electric field of these so-called **TEM-modes** follows from a 2D potential satisfying the 2D Laplace equation,

$$\nabla_t^2\Phi(x,y) = 0$$

with boundary conditions $\Phi(x,y)|_{S_i} = V_i = const.$ on the involved waveguide surfaces S_i. From the solution for $\Phi(x,y)$ one obtains $\mathbf{E}_t = -\nabla_t\Phi(x,y)$ and $\mathbf{B}_t(x,y) = \pm\sqrt{\epsilon\mu}\hat{\mathbf{z}} \times \mathbf{E}_t$. Thus, TEM-modes are largely found by solving equations analogous to those of 2D electrostatic problems. The dispersion relation of TEM modes is identical to that of plane waves ($k = \sqrt{\epsilon\mu}\omega$; see Eq. 2).

Notes. Various examples of waveguide geometries supporting TEM modes have been discussed. One requires at least two non-connected surfaces for TEM-modes to exist.

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2 1/8/2004

We consider the case of infinite conductivity of the walls, $\sigma \to \infty$. The skin depth then is $\delta = \sqrt{\frac{2}{\sigma\mu_c\omega}} \to 0$, and the boundary conditions are simple.

The remaining solutions fall into two types. For **TE-modes**, it is $E_z = 0$, $B_z \neq 0$ and $\frac{\partial B_z}{\partial n}|_S = 0$. For **TM-modes**, it is $B_z = 0$, $E_z \neq 0$ and $\frac{\partial E_z}{\partial n}|_S = 0$. In both cases, the equation to be solved is

$$(\nabla_t^2 + \mu\epsilon\omega^2 - k^2)\psi(x,y) = 0$$

where $\psi = B_z$ or $\psi = E_z$, respectively. Note the different respective boundary conditions.

Generic solution method. For both types of modes, the problem is an eigenvalue problem. Defining $\gamma^2 = \mu\epsilon\omega^2 - k^2$, the equation

$$(\nabla_t^2 + \gamma^2)\psi(x, y) = 0$$

with boundary conditions has a countable number of solutions (spectrum) γ_i^2 with eigenfunctions ψ_i (mode index i). Note that all $\gamma_i^2 > 0$. For given ω, the dispersion relations have the universal form

$$k(\omega) = \sqrt{\epsilon\mu}\sqrt{\omega^2 - \frac{\gamma_i^2}{\epsilon\mu}} =: \sqrt{\epsilon\mu}\sqrt{\omega^2 - \omega_i^2}$$

with cutoff frequencies ω_i (which depend on the details of the problem). For $\omega > \omega_i$, the respective mode propagates because it has real k, while for $\omega < \omega_i$ k is imaginary, and the mode is exponentially damped (hence the name cutoff frequency). The phase velocity

$$v_P = \frac{\omega}{k_i} > c$$

and the group velocity

$$v_G = \frac{d\omega}{dk_i} = \frac{c^2}{v_P} < c \quad .$$

Following Eq. 4, the transverse components of the fields with non-vanishing z-components are

$$\mathbf{E}_t = \pm\frac{\mathrm{i}k_i}{\gamma_i^2}\nabla_t E_{z,i}$$

for TM-waves, and

$$\mathbf{H}_t = \pm\frac{\mathrm{i}k_i}{\gamma_i^2}\nabla_t H_{z,i}$$

for TE-waves. There, the $\pm$-signs correspond to z-dependences $\exp(\pm\mathrm{i}kz)$.

The transverse components of the fields with vanishing z-components are then

$$\mathbf{H}_t = \frac{\pm 1}{Z}\hat{\mathbf{z}} \times \mathbf{E}_t \quad \text{with} \quad Z = \frac{k_i}{\epsilon\omega}$$

for TM-modes, and

$$\mathbf{E}_t = \mp Z\hat{\mathbf{z}} \times \mathbf{H}_t \quad \text{with} \quad Z = \frac{\mu\omega}{k_i}$$

for TE-modes. Note the different values of the wave impedance Z. The upper signs correspond to z-dependences $\exp(\mathrm{i}kz)$, and the lower to $\exp(-\mathrm{i}kz)$.

The example of a waveguide with rectangular cross section has been discussed (read in textbook).

The above equations present a recipe for the dispersion relations and fields of all modes - TEM, TE, TM - in guides with linear filling and infinite wall conductivity.

Energy flow. Inserting the fields in terms of $\psi = B_z$ or $\psi = E_z$ for TE- and TM-modes, respectively, one can determine the complex Poynting vector $\mathbf{S} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^*$, and integrate its z-component to obtain the transmitted power (for real ϵ, μ),

$$P = \int_A \hat{\mathbf{z}} \cdot S da = \frac{1}{2\sqrt{\epsilon\mu}} \left(\frac{\omega}{\omega_i}\right)^2 \sqrt{1 - \frac{\omega_i^2}{\omega^2}} \left\{ \begin{array}{c} \epsilon \\ \mu \end{array} \right\} \int_A \psi_i^* \psi_i da \tag{5}$$

The upper line is for TM , the lower for TE-modes. With regard to units, note the physical difference of the ψ in the two cases.

Similarly, the linear energy density $U = \int u da$ with $u = \frac{1}{4}(\epsilon \mathbf{E} \cdot \mathbf{E}^* + \mu \mathbf{H} \cdot \mathbf{H}^*$ is found to be

$$U = \frac{1}{2} \left(\frac{\omega}{\omega_i}\right)^2 \left\{ \begin{array}{c} \epsilon \\ \mu \end{array} \right\} \int_A \psi_i^* \psi_i da \tag{6}$$

The upper line is for TM , the lower for TE-modes. The group velocity equals $v_G = \frac{P}{U}$, which can be confirmed to be identical with $\frac{d\omega}{dk_i}$ (as required).

3 1/13/2004

The effect of waveguide losses due to Ohm-type resistance is that in all fields

$$k_i \rightarrow k_i + \mathrm{i}\beta_i + \alpha_i$$

with real α and β. We first calculate the damping constant β, and then the change in wavenumber, α.

It has been sketched how to derive an expression for the power loss per unit area,

$$\frac{dP}{da} = \frac{1}{2\sigma\delta} \left|\mathbf{H}_{||}\right|^2$$

with wall conductivity σ, skin depth δ and surface H-field $\mathbf{H}_{||}$, which - according to the boundary conditions for H - for reasonably well conducting walls is parallel to the surface. It follows that

$$\left|\frac{dP}{dz}\right| = \frac{1}{2\sigma\delta} \oint_C \left|\mathbf{H}_{||}\right|^2 dl$$

where the line integral goes over the waveguide surface in the xy-plane. This can be worked out in terms of the mode function of the longitudinal field, ψ. The result,

$$\left|\frac{dP}{dz}\right| = \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_i}\right)^2 \oint_C \left\{ \begin{array}{c} \frac{1}{\mu^2\omega_i^2}\left|\frac{\partial\psi}{\partial n}\right|^2 \\ \frac{1}{\epsilon\mu\omega_i^2}\left(1-\frac{\omega_i^2}{\omega^2}\right)\left|\hat{\mathbf{n}}\times\nabla_t\psi\right|^2 + \frac{\omega_i^2}{\omega^2}\left|\psi\right|^2 \end{array} \right\} dl$$

(upper line for TM, lower for TE) and Eq. 5 can be used to calculate β_i,

$$\beta_i = \left|\frac{dP}{dz}\right| \frac{1}{2P} \quad .$$

It is noted that generally losses are large close to cutoff frequencies. This fact has an intuitive explanation, which was discussed.

To obtain the loss-induced change α in the (real) wavenumber, one follows a procedure known as perturbation of boundary conditions. The method was explained in some detail for a non-degenerate TM mode. In that case, express the magnetic field on the guide surface, $\mathbf{H}_{||}$, in terms of $\psi = E_z$. In the case of $\sigma < \infty$ the field $\mathbf{H}_{||}$ is accompanied by an electric field

$$\mathbf{E}_{||} = \hat{\mathbf{z}}E_{z,\mathrm{wall}} = \sqrt{\frac{\mu_c\omega}{2\sigma}}(1-\mathrm{i})(\hat{\mathbf{n}}\times\mathbf{H}_{||})$$

($\hat{\mathbf{n}}$ is inward and μ_c is the permeability of the wall), which for TM-modes is in the $\hat{\mathbf{z}}$-direction and thus represents a perturbation of the boundary condition for ψ (which for $\sigma = \infty$ reads $\psi = 0$ on S). Explicitly, the perturbed eigenvalue problem for eigenvalue γ^2 and perturbed function ψ is

$$(\nabla_t^2 + \gamma^2)\psi = 0 \quad \text{with} \quad \psi = E_{z,\mathrm{wall}} = \quad (1+\mathrm{i})\frac{\mu_c\delta}{2\mu}\left(\frac{\omega}{\omega_i}\right)^2 \left|\frac{\partial\psi_0}{\partial n}\right|_S$$

with unperturbed cutoff frequency ω_i and unperturbed modefunction ψ_0. It has been explained why, with the use of Green's II theorem and assuming $k \gg \alpha$, it follows $\alpha = \beta$.

Result, valid for non-degenerate TE and TM modes and $k \gg \alpha$: $\alpha = \beta$, *i.e.* to obtain the wavenumber change α it is sufficient to calculate β (which does not require the consideration of perturbed B/C).

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4 1/15/2004

Cavities. The only type of cavity that's of interest for this course is obtained by taking a waveguide of the geometry described so far (invariance under z-translation), and closing it off with conducting walls that are transverse to the z-axis and have a distance d. As a result of the additional boundary conditions on the ends, each waveguide mode i can exist in the cavity only at certain resonance frequencies ω_{ip}, where p is an integer counting index.

A straightforward consideration of the boundary conditions on the end faces leads to:

TM-modes (infinite conductivity):

Guide solutions are labeled as before. $\psi = E_z$. Solutions of the eigenvalue problem satisfy $(\nabla_t^2 + \gamma_i^2)\psi_i = 0$ with $\psi_i = 0$ on the surface S in the xy-plane. Then, in the corresponding cavity problem it is:

$$\begin{aligned}
E_z &= \psi_i(x,y)\cos\left(\frac{p\pi}{d}z\right) \\
\mathbf{E}_t &= -\frac{p\pi}{d\gamma_i^2}\left(\nabla_t\psi_i(x,y)\right)\sin\left(\frac{p\pi}{d}z\right) \\
\mathbf{H}_t &= \frac{\mathrm{i}\epsilon\omega_{ip}\pi}{\gamma_i^2}\left(\hat{\mathbf{z}}\times\nabla_t\psi_i(x,y)\right)\cos\left(\frac{p\pi}{d}z\right) \\
\omega_{ip} &= \frac{1}{\sqrt{\epsilon\mu}}\sqrt{\gamma_i^2 + \left(\frac{p\pi}{d}\right)^2}
\end{aligned} \tag{7}$$

There, $p = 0, 1, 2, ...$

TE-modes (infinite conductivity):

Guide solutions are labeled as before. $\psi = H_z$. Solutions of the eigenvalue problem satisfy $(\nabla_t^2 + \gamma_i^2)\psi_i = 0$ with $\frac{\partial\psi_i}{\partial n} = 0$ on the surface S in the xy-plane. Then, in the corresponding cavity problem it is:

$$\begin{aligned}
H_z &= \psi_i(x,y)\sin\left(\frac{p\pi}{d}z\right) \\
\mathbf{H}_t &= \frac{p\pi}{d\gamma_i^2}\left(\nabla_t\psi_i(x,y)\right)\cos\left(\frac{p\pi}{d}z\right) \\
\mathbf{E}_t &= -\frac{\mathrm{i}\mu\omega_{ip}\pi}{\gamma_i^2}\left(\hat{\mathbf{z}}\times\nabla_t\psi_i(x,y)\right)\cos\left(\frac{p\pi}{d}z\right) \\
\omega_{ip} &= \frac{1}{\sqrt{\epsilon\mu}}\sqrt{\gamma_i^2 + \left(\frac{p\pi}{d}\right)^2}
\end{aligned} \tag{8}$$

There, $p = 1, 2, ...$ Note that TE and TM-modes start counting with different values of p.

The spatial and temporal phase relations of the transverse and the longitudinal fields were explained.

As an example, the modes of a cylindrical resonator were discussed. The field patterns of the fundamental modes $TE_{mnp} = TE_{111}$ and $TM_{mnp} = TM_{010}$ were shown. Polarization degeneracy was discussed using the example of the TE_{111}-modes.

Reading: Chapter 8.7 of Jackson.

Q-values: If a well-defined cavity mode of frequency ω is "filled" with energy and subsequently left alone, the energy decays due to Ohm-type losses in the walls following a law

$$U(t) = U_0 \exp(-\omega t/Q)$$

This equation can be used as a definition of the cavity Q-factor. Any non-zero field component at any point in the cavity follows a law

$$E(t) = E_0 \exp(-\mathrm{i}(\omega + \Delta\omega)t) \exp(-\omega t/2Q) \quad ,$$

where $\Delta\omega$ accounts for a (negative) shift of the cavity-mode resonance frequency from its value ω that one would find for perfectly conducting walls. The power spectrum $I(\omega')$ of the decaying cavity field is proportional to the square of the magnitude of the Fourier transform of the field. It is a Lorentz curve with FWHM-value $\frac{\omega}{Q}$ centered at $\omega + \Delta\omega$,

$$I(\omega') \propto \frac{1}{(\omega' - \omega - \Delta\omega)^2 + \left(\frac{\omega}{2Q}\right)^2} \quad .$$

If one were to excite the cavity with a monochromatic drive of frequency ω' and fixed amplitude, the steady-state intracavity energy as a fucntion of ω' would follow that curve.

To **calculate** Q, use $Q = \omega \frac{U}{|\frac{dU}{dt}|}$. One finds by integration of the complex energy density $u = \frac{\epsilon}{4}\mathbf{E} \cdot \mathbf{E}^* + \frac{\mu}{4}\mathbf{H} \cdot \mathbf{H}^*$ over the cavity volume that

$$U = \frac{d}{4} \left\{ \begin{array}{c} \epsilon \\ \mu \end{array} \right\} \left[1 + \left(\frac{p\pi}{\gamma_i d} \right)^2 \right] \int_A |\psi|^2 da$$

For TM modes with p=0, the result must be multiplied with 2.

The loss power is obtained from a surface integral over the ideal (loss-free) magnetic field $\mathbf{H}$, which is parallel to the surface:

$$\left| \frac{dU}{dt} \right| = \frac{1}{2\sigma\delta} \left[\int_{\text{mantle+both ends}} |\mathbf{H}|^2 da \right]$$

The result for Q can be written in the form

$$Q = \frac{\mu}{\mu_c} \frac{V}{S_{tot}\delta} G_i$$

with a unit-less, mode-dependent G-factor, cavity volume V and total cavity surface S_{tot}. The result has been discussed.

One further finds from a calculation involving a variation of boundary conditions that in practical cases ($Q >> 1$) the frequency shift

$$\Delta\omega = -\frac{\omega}{2Q}$$

Thus, to find the frequency shift it suffices to calculate the Q-value from the idealized cavity field, and there is no variation of boundary conditions required.

Side discussion of TEM modes, the 2D Laplace equation, analytic functions, conformal mapping and other numerical methods (relaxation, finite-element method).

Note. In the equations involving Q and $\Delta\omega$, it is assumed that all damping and shifts originate in *Ohm*-heating. In particular, we neglect coupling losses and frequency shifts due to radiation leaking out trough cavity holes, which are of practical importance and often dominate cavity losses and frequency shifts.

A formalism can be developed that allows one to describe any harmonic waveguide field as a superposition of **normalized field modes** multiplied with amplitude coefficients. To find unique amplitude coefficients it is sufficient to specify the transverse fields $\mathbf{E}_t$ and $\mathbf{H}_t$ at an arbitrary location of z (longitudinal field components are not needed; they would actually over-specify the problem).

It is of great interest to determine the amplitude coefficients of the normalized field modes due to a localized harmonic current density $\mathbf{J}(\mathbf{x})\exp(-\mathrm{i}\omega t)$ in the guide. A simple expression that allows this calculation based on the $\mathbf{E}$-field of the normalized modes exists. Similarly, it is possible to calculate the amplitude coefficients of the normalized field modes due to localized apertures in the waveguide walls; there, to obtain a unique result it is sufficient to know the total tangential electric field in the apertures.

Reading. Chapter 8.12 of Jackson. This material is also covered by the last homework problem on Chapter 8.