Physics 505 | Electricity and Magnetism | Fall 2003

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Problem Set 9

Maximal score: 25 Points

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1. Jackson, Problem 5.22 **6 Points**

Hints given: Consider the result of the previous homework problem 5.19 and the discussion in Sec. 5.14 of Jackson to construct the magnetic field by an image method.

When calculating the force, note that Eq. 5.151 cannot be used, because the medium is nonlinear. Use a simpler equation for the energy of an object of fixed magnetization in an external B-field.

Magnetic field. Consider two identical rods with identical, constant magnetization $\mathbf{M} = \hat{\mathbf{z}} M_0$, length L and an arbitrary cross section in the xy-plane, one extending from $z = -L$ to $z = -0$, and the other from $z = +0$ to $z = L$. By symmetry, the magnetic-field lines of the arrangement intersect the plane $z = 0$ at a right angle. To see this, consider two identical magnetic dipoles, equivalent to volume elements of the magnetized rods located at (x', y', z') and $(x', y', -z')$. At any location $(x, y, z = 0)$, the z-components of the B-fields of these dipoles add up, while the x- and y-components cancel. Following the discussion in Sec. 5.14, the boundary condition for the B-field outside a highly permeable medium with surface $z = 0$ is that $B_z = 0$ at all locations $(x, y, z = 0)$. This is exactly what the described arrangement of two rods also provides. Thus, in the volume of interest $z > 0$ the B-fields of the following cases are indistinguishable:

1) $\mu = \infty$ in the volume $z < 0$, $\mu = \mu_0$ in the volume $z \geq 0$, and one magnetized rod extending from $z = 0$ to $z = L$.

2) $\mu = \mu_0$ everywhere and two magnetized rods, as described.

Force Method 1

Due to the absence of free currents, the $\mathbf{H}$-field of the image problem follows from $\mathbf{H} = -\nabla \Phi_M$. The magnetic potential Φ_M follows from magnetic surface charge densities $\sigma_M = M_0$ on the end faces at $z = L$ and $z = -0$, and $\sigma_M = -M_0$ on the end faces at $z = -L$ and $z = +0$. For long, thin rods, one can neglect the effect of the magnetic charges at $\pm L$, leaving only the charges at $z = \pm 0$. The magnetic potential is then analogous to the electric potential inside a plate capacitor, with σ_E/ϵ_0 replaced by σ_M. Similarly, the H-field is analogous to the E-field. Assuming an infinitesimally small gap $0 < z < z_0$ between the image rod and the real rod, it is

$$\mathbf{H}(x, y, z) = \begin{cases} \hat{\mathbf{z}} M_0 & , \quad 0 < z < z_0 \quad \text{and} \quad (x, y) \in A \\ 0 & , \quad \text{otherwise} \end{cases} .$$

There, A stands for the surface area of the end faces in the xy-plane. Since $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$,

$$\mathbf{B}(x,y,z) = \begin{cases} \hat{\mathbf{z}}\mu_0 M_0 & , \quad \forall\, z \quad \text{and} \quad (x,y) \in A \\ 0 & , \quad \text{otherwise} \end{cases} \qquad .$$

Since an increase of z_0 by dz_0 implies an increase of the magnetic-field volume by the gap volume Adz_0, the magnetostatic energy $W(z_0)$ satisfies

$$dW(z_0) = dz_0 A \frac{1}{2B^2\mu_0} = dz_0 A \frac{\mu_0 M_0^2}{2} \quad .$$

Note that the position of the image rod **remains fixed** and does not vary as a function of z_0. Then,

$$\mathbf{F} = -\hat{z}\frac{dW(z_0)}{dz_0} = -\hat{z}A\frac{\mu_0 M_0^2}{2} \quad \text{, q.e.d.}$$

The result is also valid for the real problem, because the real rod cannot distinguish between being attracted by another real rod or by its image rod; note the similarity of this argument with image problems in electrostatics. The significance of the $-$sign in the result is that the force pulls the rod towards the permeable medium (as expected).

<u>Force Method 2:</u> We consider a rod with a circular cross section with radius a; everything else as before. Explicit expressions for the magnetic field in the volume of interest can then be imported from Problem 5.19. To obtain the force, consider the potential energy $W(z_0)$ of the real rod in an external field presented by the image rod.

$$W(z_0) = -\int \mathbf{M}(\mathbf{x}) \cdot \mathbf{B}_\mathrm{i}(\mathbf{x}) d^3x = -M_0 \int_{V_\mathrm{r}} B_{\mathrm{z,i}}(\mathbf{x}) d^3x \quad = -M_0 \int_{z_0}^{z_0+L} \left\{ \int_{xy} B_{\mathrm{z,i}}(\mathbf{x}) dxdy \right\} dz,$$

where $\mathbf{B}_\mathrm{i}(\mathbf{x})$ is the field of the image rod at locations $\mathbf{x}$ in the real rod, and V_r is the volume of the real rod. The position of the image rod is held fixed. For a long, thin rod, $B_{\mathrm{z,i}}(\mathbf{x}) \approx B_{\mathrm{z,i}}(0,0,z)$, and

$$W(z_0) \approx -M_0 \int_{z_0}^{z_0+L} B_{\mathrm{z,i}}(0,0,z) \left\{ \int_{xy} dxdy \right\} dz = -M_0 A \int_{z_0}^{z_0+L} B_{\mathrm{z,i}}(0,0,z) dz$$

where A is the rod cross section. Thus, the adhesion force is

$$F_z = -\left.\frac{dW(z_0)}{dz_0}\right|_{z_0=0} = M_0 A \left[B_{\mathrm{z,i}}(0,0,L) - B_{\mathrm{z,i}}(0,0,0) \right]$$

By Problem 5.19, and by shifting the origin to the $z = 0$-end of the image rod, it is

$$B_{\mathrm{z,i}}(0,0,z) \quad = \quad \frac{\mu_0 M_0}{2} \left(\frac{z+L}{\sqrt{a^2 + (z+L)^2}} - \frac{z}{\sqrt{a^2+z^2}} \right)$$

and

$$F_z = \frac{\mu_0 M_0^2 A}{2} \left[\frac{2L}{\sqrt{a^2 + 4L^2}} - \frac{L}{\sqrt{a^2 + L^2}} - \frac{L}{\sqrt{a^2 + L^2}} + 0 \right] \quad .$$

For $a \ll L$

$$F_z = -\frac{\mu_0 M_0^2 A}{2} \quad \text{q.e.d.}$$

The result also holds for long, thin shapes with arbitrary cross sections, because one can consider them as superpositions of sub-portions that have cylindrical cross sections.

2. **7 Points**

An infinitely long cylindrical region with radius a and constant permeability $\mu \gg \mu_0$ carries a volume current density

$$\mathbf{j}_z(\rho,\phi) = \hat{\mathbf{z}}\, j_0\, \cos\phi J_1(\frac{x_{12}}{a}\rho) \quad .$$

(J_1 is a Bessel function and x_{12} one of its roots; notation as usual).

a) Based on the discussion in Sec. 5.14 of the textbook, specify a Poisson-like equation with Dirichlet boundary conditions suitable to find the vector potential in the region $\rho < a$.

b) Find an eigenfunction expansion for the corresponding two-dimensional Green's function $G(\rho,\rho',\phi,\phi')$.

c) Using the previous result, calculate the vector potential.

d) Find the magnetic field in the region $\rho < a$.

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a): The vector potential can be chosen such that it only has a z-component $A_z(\rho,\phi)$. The equation to solve and the appropriate boundary condition then are:

$$\Delta A_z(\rho,\phi) = -\mu j_z(\rho,\phi) \qquad \text{Boundary condition}: \qquad A_z(a,\phi) = 0$$

b): Eigenfunctions and eigenvalues. We separate $\psi(\rho,\phi) = R(\rho)\Phi(\phi)$ to obtain Bessel's differential equation for $R(\rho)$:

$$\begin{aligned}
(\Delta + \lambda)\psi(\rho,\phi) &= 0 \\
\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \lambda\right) R(\rho)\Phi(\phi) &= 0 \\
\left(\frac{\rho}{R(\rho)}\frac{d}{d\rho}\rho\frac{d}{d\rho}R(\rho) + \lambda\rho^2\right) + \left(\frac{1}{\Phi(\phi)}\frac{d^2}{d\phi^2}\Phi(\phi)\right) &= 0 \qquad (1)
\end{aligned}$$

A single-valued vector potential requires

$$\frac{1}{\Phi(\phi)}\frac{d^2}{d\phi^2}\Phi(\phi) = -m^2$$

with integer m; the solution is $\Phi(\phi) = \exp(im\phi)$.

The radial equation then is

$$\left[\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}R(\rho) + \left(\lambda - \frac{m^2}{\rho^2}\right)\right] R(\rho) = 0$$

The solutions that are regular at $\rho = 0$ are $J_m(\sqrt{\lambda}\rho)$. To match the boundary condition $R(a) = 0$, we set

$$\sqrt{\lambda}a = x_{mn}$$

From Eq. 3.95 it follows that the normalized eigenfunctions are

$$\psi_{mn}(\rho,\phi) = \frac{1}{a\sqrt{\pi}} J_{m+1}(x_{mn})\, J_m\left(\frac{x_{mn}}{a}\rho\right)\, \exp(\mathrm{i}m\phi) \quad ; \tag{2}$$

the eigenvalues are $\lambda_{mn} = \left(\frac{x_{mn}}{a}\right)^2$.

Using Eq. 3.160, the eigenfunction expansion of the Green's function is

$$G(\mathbf{x},\mathbf{x}') = 4\pi \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{1}{\pi\, x_{mn}^2\, J_{m+1}^2(x_{mn})}\ \exp(\mathrm{i}m(\phi-\phi'))\, J_m\left(\frac{x_{mn}}{a}\rho\right) J_m\left(\frac{x_{mn}}{a}\rho'\right)$$

c): In analogy with the solution of the Poisson equation with Dirichlet boundary conditions, it is

$$A_z(\rho,\phi) = \frac{\mu}{4\pi}\int_V G(\rho,\phi,\rho',\phi') j_z(\rho,\phi)\rho' d\rho' d\phi' - \frac{1}{4\pi}\int_{\partial V} A_z(a,\phi')\frac{\partial}{\partial\rho'}\, G(\rho,\phi,\rho',\phi')\big|_{\rho'=a}\, a d\phi'$$

Since the potential is zero on the boundary, the surface term vanishes, and

$$A_z(\rho,\phi) = \mu\, j_0 \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{1}{\pi\, x_{mn}^2\, J_{m+1}^2(x_{mn})}\ \exp(\mathrm{i}m\phi)\, J_m\left(\frac{x_{mn}}{a}\rho\right) \int_V \exp(-\mathrm{i}m\phi')\, J_m\left(\frac{x_{mn}}{a}\rho'\right)\, \cos\phi'\, J_1(\frac{x_{12}}{a}\rho')\rho' d\rho' d\phi'$$

The integral equals

$$\begin{aligned}
& \frac{1}{2}\left(\int_0^{2\pi}(\exp(\mathrm{i}\phi')+\exp(-\mathrm{i}\phi'))\exp(-\mathrm{i}m\phi')d\phi'\right)\left(\int_0^a J_m\left(\frac{x_{mn}}{a}\rho'\right)\, J_1(\frac{x_{12}}{a}\rho')\rho' d\rho'\right)\\
= \quad & \pi(\delta_{m,1}+\delta_{m,-1})\left((-1)^{(|m|-m)/2}\int_0^a J_{|m|}\left(\frac{x_{mn}}{a}\rho'\right)\, J_1(\frac{x_{12}}{a}\rho')\rho' d\rho'\right)\\
= \quad & \pi(\delta_{m,1}-\delta_{m,-1})\left(\int_0^a J_1\left(\frac{x_{1n}}{a}\rho'\right)\, J_1(\frac{x_{12}}{a}\rho')\rho' d\rho'\right)\\
= \quad & \frac{a^2\pi}{2}(\delta_{m,1}-\delta_{m,-1})\delta_{n,2}J_2^2(x_{12})
\end{aligned}$$

which, when inserted into the previous equation, yields

$$\begin{aligned}
A_z(\rho,\phi) &= \frac{a^2\pi\,\mu\, j_0}{2}\frac{1}{\pi\, x_{mn}^2\, J_2^2(x_{12})}\, J_2^2(x_{12})\left[\exp(\mathrm{i}\phi)\, J_1\left(\frac{x_{12}}{a}\rho\right) - \exp(-\mathrm{i}\phi)\, J_{-1}\left(\frac{x_{12}}{a}\rho\right)\right]\\
A_z(\rho,\phi) &= \frac{a^2\,\mu\, j_0}{2\, x_{12}^2}\, J_1\left(\frac{x_{12}}{a}\rho\right)[\exp(\mathrm{i}\phi)\, + \exp(-\mathrm{i}\phi)]\\
A_z(\rho,\phi) &= \frac{a^2\,\mu\, j_0}{x_{12}^2}\, J_1\left(\frac{x_{12}}{a}\rho\right)\cos\phi
\end{aligned} \tag{3}$$

As a test, one may insert the solution into the equation $\Delta A_z(\rho,\phi) = -\mu j_z(\rho,\phi)$ and verify its validity. It is also, as required, $A_z(a,\phi) = 0$.

Note: (Unnormalized) eigenfunctions equivalent to Eq. 2 are

$$\psi_{mn1}(\rho,\phi) = J_m\left(\frac{x_{mn}}{a}\rho\right)\sin(m\phi) \quad \text{and} \quad \psi_{mn2}(\rho,\phi) = J_m\left(\frac{x_{mn}}{a}\rho\right)\cos(m\phi) \quad ,$$

where $m = 0, 1, 2, ..$ and $n = 1, 2, ...$ It may thus be noted that the specified current distribution is an eigenfunction of Δ. Insertion of $A_z(\rho,\phi) = \alpha\psi_{122}(\rho,\phi)$ then quickly leads to the result:

$$\begin{aligned}
\Delta A_z(\rho,\phi) &= -\mu j_z(\rho,\phi) \\
\alpha\Delta\psi_{122}(\rho,\phi) &= -\mu\, j_0 \,\cos\phi J_1(\frac{x_{12}}{a}) = -\mu\, j_0\psi_{122} \\
\alpha\left(\frac{x_{12}}{a}\right)^2\psi_{122}(\rho,\phi) &= \mu\, j_0\Delta\psi_{122}(\rho,\phi) \\
\alpha &= \mu\, j_0\left(\frac{a}{x_{12}}\right)^2 \\
A_z(\rho,\phi) &= \left(\frac{a^2\,\mu\, j_0}{x_{12}^2}\right) J_1(\frac{x_{12}}{a})\,\cos\phi
\end{aligned} \tag{4}$$

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d): Since **A** only has a z-component, it is

$$\begin{aligned}
\mathbf{B}(\rho,\phi) &= \nabla\times\mathbf{A} \\
&= \hat{\rho}\frac{1}{\rho}\frac{\partial A_z(\rho,\phi)}{\partial\phi} - \hat{\phi}\frac{\partial A_z(\rho,\phi)}{\partial\rho} \\
&= -\frac{a^2\,\mu\, j_0}{x_{12}^2}\left\{\hat{\rho}\left[\frac{1}{\rho}J_1\left(\frac{x_{12}}{a}\rho\right)\sin\phi\right] + \hat{\phi}\left[\frac{x_{12}}{a}J_1'\left(\frac{x_{12}}{a}\rho\right)\cos\phi\right]\right\}
\end{aligned} \tag{5}$$

For $\rho = a$, this field only has a ϕ-component and therefore is parallel to the surface, as required by the boundary condition inside highly permeable media.

Note: In the geometry considered, the B-lines are parallel to contour lines of A_z. Also, the density of contour lines of A_z is proportional to $|\mathbf{B}(\rho,\phi)|$. Thus, field line plots are most easily obtained by plotting contour lines of A_z:

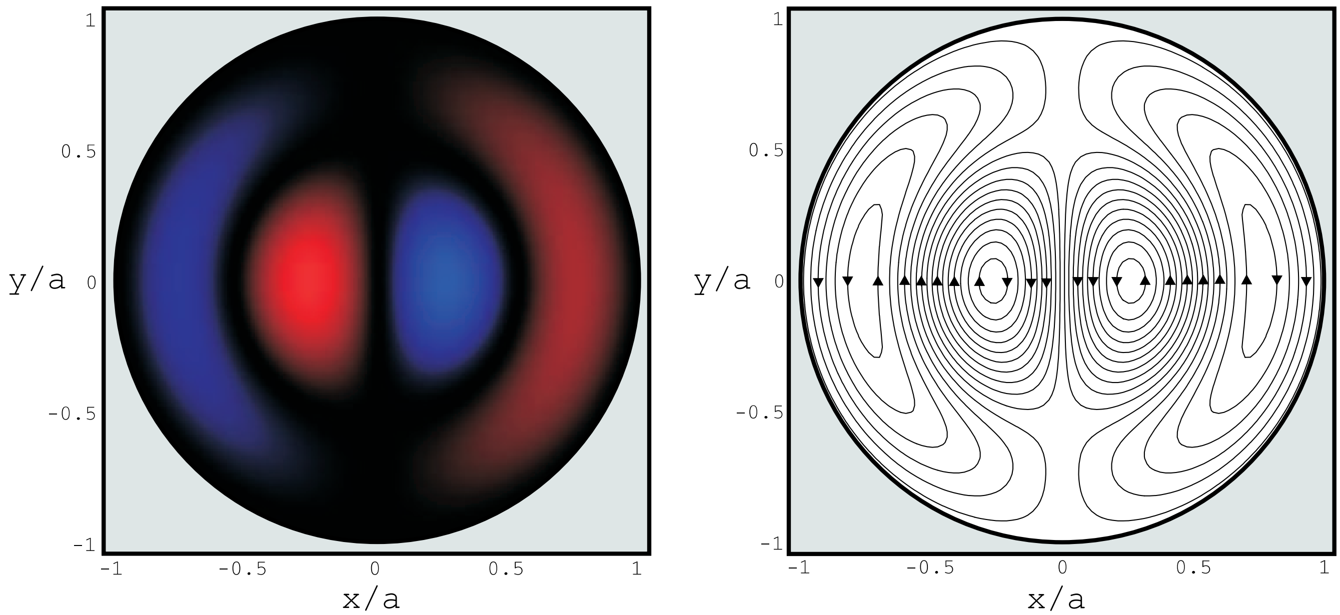

Figure 1: **Left:** Current density (black=0, blue=out of page, red = into page). **Right:** Magnetic-field lines.

3. Jackson, Problem 6.1 **6 Points**

a): Under absence of sources other than the specified "flash" on the z-axis at time $t' = 0$, the retarded solution is

$$
\begin{aligned}
\psi(\mathbf{x},t) &= \psi^{+}(\mathbf{x},t) = \int_V \int_{t'=-\infty}^{\infty} f(\mathbf{x}',t') \frac{\delta(t' - \left[t - \frac{R}{c}\right])}{R} d^3x' dt' \\
\psi(\mathbf{x},t) &= \int_V \int_{t'=-\infty}^{\infty} \delta(x')\delta(y')\delta(t') \frac{\delta(t' - \left[t - \frac{R}{c}\right])}{R} d^3x' dt' \\
&= \int \frac{\delta\left(t - \sqrt{x^2+y^2+(z-z')^2}/c\right)}{\sqrt{x^2+y^2+(z-z')^2}} dz'
\end{aligned}
$$

Case $\sqrt{x^2+y^2} > ct$: The argument of the δ-function is always zero, and

$$
\psi(\mathbf{x}, t < \frac{\sqrt{x^2+y^2}}{c}) = 0 \quad .
$$

This corresponds to the case that no part of signal flash has arrived yet.

Case $\sqrt{x^2+y^2} < ct$:

$$
\psi(\mathbf{x}, t > \frac{\sqrt{x^2+y^2}}{c}) = \sum_{z_0} \int \frac{\delta(z'-z_0)}{\sqrt{x^2+y^2+(z-z')^2} \left| \frac{d}{dz'} \left(t - \frac{\sqrt{x^2+y^2+(z-z')^2}}{c} \right) \right|_{z'=z_0}} dz'
$$

where the sum goes over the solutions of $t - \frac{\sqrt{x^2+y^2+(z-z_0)^2}}{c} = 0$, *i.e.* $z_0 = z \pm \sqrt{c^2t^2 - x^2 - y^2}$. Thus,

$$
\begin{aligned}
\psi(\mathbf{x},t) &= \sum_{z_0} \frac{1}{\sqrt{x^2+y^2+(z-z_0)^2} \left| \frac{z-z'}{\sqrt{x^2+y^2+(z-z')^2}} \right|_{z'=z_0}} \\
\psi(\mathbf{x},t) &= \sum_{z_0} \frac{1}{ct \frac{\sqrt{c^2t^2-x^2-y^2}}{c^2 t}}
\end{aligned}
$$

Since the sum has two identical terms, it is

$$
\psi(\mathbf{x}, t > \frac{\sqrt{x^2+y^2}}{c}) = \frac{2c}{\sqrt{c^2t^2 - x^2 - y^2}} \quad .
$$

Summary of cases: Using $\rho = \sqrt{x^2+y^2}$ and $\Theta(x) := \begin{cases} 0 & , \quad x < 0 \\ 1 & , \quad x \geq 0 \end{cases}$, it is

$$
\psi(\mathbf{x},t) = \frac{2c}{\sqrt{c^2t^2 - \rho^2}} \Theta(ct - \rho) \quad \text{, q.e.d.}
$$

b):

$$\begin{aligned}
\psi(\mathbf{x},t) &= \int_V \int_{t'=-\infty}^{\infty} f(\mathbf{x}',t') \frac{\delta(t' - \left[t - \frac{R}{c}\right])}{R} d^3x'dt' \\
\psi(\mathbf{x},t) &= \int_V \int_{t'=-\infty}^{\infty} \delta(x')\delta(t') \frac{\delta(t' - \left[t - \frac{R}{c}\right])}{R} d^3x'dt' \\
&= \int_{y'} \left\{ \int_{z'} \frac{\delta\left(t - \sqrt{x^2 + (y-y')^2 + (z-z')^2}/c\right)}{\sqrt{x^2 + (y-y')^2 + (z-z')^2}} dz' \right\} dy' \\
&= \int_{y'} \frac{2c}{\sqrt{c^2t^2 - x^2 - (y-y')^2}} \, \Theta\left(ct - \sqrt{x^2 + (y-y')^2}\right) dy'
\end{aligned} \tag{6}$$

where the last line follows from part a) by substituting y with $y - y'$.

Case $ct < |x|$:

$$\begin{aligned}
&\forall y' &:& \quad ct - \sqrt{x^2 + (y-y')^2} < 0 \\
\Rightarrow &\forall y' &:& \quad \Theta\left(ct - \sqrt{x^2 + (y-y')^2}\right) = 0 \\
\psi(\mathbf{x}, t < |x|/c) &= 0
\end{aligned}$$

Case $ct > |x|$:

$$\begin{aligned}
\psi(\mathbf{x}, t > |x|/c) &= \int_{y'} \frac{2c}{\sqrt{c^2t^2 - x^2 - (y-y')^2}} \, \Theta\left(ct - \sqrt{x^2 + (y-y')^2}\right) dy' \\
&= \int_{y'=y-\sqrt{c^2t^2-x^2}}^{y+\sqrt{c^2t^2-x^2}} \frac{2c}{\sqrt{c^2t^2 - x^2 - (y-y')^2}} dy' \\
&= \int_{\tilde{y}=\sqrt{c^2t^2-x^2}}^{-\sqrt{c^2t^2-x^2}} \frac{-2c}{\sqrt{c^2t^2 - x^2 - \tilde{y}^2}} d\tilde{y} \\
&= \left[-2c \, \sin^{-1}\left(\frac{\tilde{y}}{\sqrt{c^2t^2 - x^2}} \right) \right]_{\sqrt{c^2t^2-x^2}}^{-\sqrt{c^2t^2-x^2}} \\
&= 2\pi c
\end{aligned}$$

Summary of cases:

$$\psi(\mathbf{x},t) = 2\pi c \, \Theta(ct - |x|) \quad , \text{q.e.d.}$$

4. Jackson, Problem 6.2a+b **6 Points**

a): We consider

$$\int_V \delta^3\left(\mathbf{x}' - \mathbf{r}(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c})\right) d^3x'$$

for a particle with trajectory $\mathbf{r}(t)$. We define

$$\mathbf{y}'(\mathbf{x}') = \mathbf{x}' - \mathbf{r}(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c})$$

with zeros $\mathbf{x}'_0$ defined by $\mathbf{y}'(\mathbf{x}'_0) = 0$. The observation coordinates $\mathbf{x}$ and t are fixed parameters of the calculation. It is

$$\mathbf{x}'_0 = \mathbf{r}(t - \frac{|\mathbf{x}-\mathbf{x}'_0|}{c}) = [\mathbf{r}(t)]_{\rm ret} \quad .$$

Then,

$$\int_V \delta^3\left(\mathbf{x}' - \mathbf{r}(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c})\right) d^3x' = \int_V \delta^3(\mathbf{x}' - \mathbf{x}'_0) \left|\frac{\partial \mathbf{y}'}{\partial \mathbf{x}'}\right|^{-1}_{\mathbf{x}'_0} d^3x'$$

The elements of the Jacobi matrix are

$$\begin{aligned}
\left.\frac{\partial y'_j}{\partial x'_i}\right|_{\mathbf{x}'_0} &= \frac{\partial}{\partial x'_i}\left(x'_j - r_j(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c})\right)\Bigg|_{\mathbf{x}'_0} \\
&= \delta_{ij} - \left[\frac{dr_j}{dt}\right]_{\rm ret} \frac{\partial}{\partial x'_i}\left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\Bigg|_{\mathbf{x}'_0} \\
&= \delta_{ij} - \left[\frac{dr_j}{dt}\right]_{\rm ret} \left(\frac{1}{c}\frac{(x_i - x'_i)}{|\mathbf{x}-\mathbf{x}'|}\right)\Bigg|_{\mathbf{x}'_0} \\
&= \delta_{ij} - [v_j(t)]_{\rm ret} \left[\left(\frac{1}{c}\frac{(x_i - r_i(t))}{|\mathbf{x}-\mathbf{r}(t)|}\right)\right]_{\rm ret} \\
&= \delta_{ij} - \frac{1}{c}\left[v_j(t)\,\hat{R}_i(\mathbf{x},t)\right]_{\rm ret} \\
&=: \delta_{ij} - v_j\,k_i \quad ,
\end{aligned}$$

where we use the usual notation for retarded quantities and, in the last line, define some abbreviations for the following. The Jacobi determinant then is

$$\begin{aligned}
\left|\frac{\partial \mathbf{y}'}{\partial \mathbf{x}'}\right|_{\mathbf{x}'_0} &= \begin{vmatrix} 1 - v_1k_1 & -v_2k_1 & -v_3k_1 \\ -v_1k_2 & 1 - v_2k_2 & -v_3k_2 \\ -v_1k_3 & -v_2k_3 & 1 - v_3k_3 \end{vmatrix} = 1 - \mathbf{v}\cdot\mathbf{k} \\
&= \left[1 - \frac{\mathbf{v}(t)\cdot\hat{\mathbf{R}}(\mathbf{x},t)}{c}\right]_{\rm ret} \\
\text{where} \quad [\mathbf{v}(t)]_{\rm ret} &= \left[\frac{d}{dt}\mathbf{r}(t)\right]_{\rm ret} \quad \text{and} \quad \left[\hat{\mathbf{R}}(\mathbf{x},t)\right]_{\rm ret} = \left[\frac{\mathbf{x}-\mathbf{r}(t)}{|\mathbf{x}-\mathbf{r}(t)|}\right]_{\rm ret} \quad .
\end{aligned} \tag{7}$$

Thus,

$$\int_V \delta^3\left(\mathbf{x}' - \mathbf{r}(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c})\right) d^3x' = \left[\frac{1}{1-\frac{\mathbf{v}\cdot\hat{\mathbf{R}}}{c}}\right]_{\mathrm{ret}} =: \frac{1}{\kappa} \quad \text{, q.e.d.}$$

Note that the result is a function of the observation coordinates $\mathbf{x}$ and t and of the (known) particle trajectory.

b): Electric field:

We insert $\rho(\mathbf{x}', t') = q\delta^3(\mathbf{x}' - \mathbf{r}(t'))$ and $\mathbf{j}(\mathbf{x}', t') = q\mathbf{v}(t')\delta^3(\mathbf{x}' - \mathbf{r}(t'))$ into Eq. 6.55:

$$\begin{aligned}
\mathbf{E}(\mathbf{x}, t) &= \frac{1}{4\pi\epsilon_0}\int\left\{\frac{\hat{\mathbf{R}}}{R^2}q\left[\delta^3(\mathbf{x}'-\mathbf{r}(t'))\right]_{\mathrm{ret}} + \frac{\hat{\mathbf{R}}}{cR}\left[\frac{\partial}{\partial t'}q\delta^3(\mathbf{x}'-\mathbf{r}(t'))\right]_{\mathrm{ret}} - \frac{1}{c^2R}\left[\frac{\partial}{\partial t'}q\mathbf{v}(t')\delta^3(\mathbf{x}'-\mathbf{r}(t'))\right]_{\mathrm{ret}}\right\} d^3x' \\
&= \frac{1}{4\pi\epsilon_0}\int\left\{\frac{\hat{\mathbf{R}}}{R^2}q\left[\delta^3(\mathbf{x}'-\mathbf{r}(t'))\right]_{\mathrm{ret}} + \frac{\hat{\mathbf{R}}}{cR}\frac{\partial}{\partial t}\left[q\delta^3(\mathbf{x}'-\mathbf{r}(t'))\right]_{\mathrm{ret}} - \frac{1}{c^2R}\frac{\partial}{\partial t}\left[q\mathbf{v}(t')\delta^3(\mathbf{x}'-\mathbf{r}(t'))\right]_{\mathrm{ret}}\right\} d^3x'
\end{aligned}$$

where we have used Eq. 6.57 of the textbook. Since *before integration* $\mathbf{R} = \mathbf{R}(\mathbf{x}, \mathbf{x}')$, before integration it is $\frac{\partial}{\partial t}\mathbf{R} = 0$. Thus, functions of $\mathbf{R}$ can be dragged through $\frac{\partial}{\partial t}$-operators. Further, due to the time independence of $\mathbf{R}$, it is also allowed to take functions of $\mathbf{R}$ from outside $[*]_{\mathrm{ret}}$ to the inside. Example:

$$\frac{\hat{\mathbf{R}}}{R}\left[\frac{\partial}{\partial t'}q\delta^3(\mathbf{x}'-\mathbf{r}(t'))\right]_{\mathrm{ret}} = \frac{\partial}{\partial t}\left[\frac{\hat{\mathbf{R}}}{R}q\delta^3(\mathbf{x}'-\mathbf{r}(t'))\right]_{\mathrm{ret}}$$

Thus, for the $\mathbf{E}$-field we can write

$$\begin{aligned}
\mathbf{E}(\mathbf{x}, t) \quad = \quad & \frac{q}{4\pi\epsilon_0}\left\{\int\left[\frac{\hat{\mathbf{R}}}{R^2}\delta^3(\mathbf{x}'-\mathbf{r}(t'))\right]_{\mathrm{ret}} d^3x'\right. \\
& + \frac{\partial}{\partial t}\int\left[\frac{\hat{\mathbf{R}}}{cR}\delta^3(\mathbf{x}'-\mathbf{r}(t'))\right]_{\mathrm{ret}} d^3x' \\
& \left. - \frac{\partial}{\partial t}\int\left[\frac{1}{c^2R}\mathbf{v}(t')\delta^3(\mathbf{x}'-\mathbf{r}(t'))\right]_{\mathrm{ret}} d^3x'\right\}
\end{aligned}$$

By the result of part a), the $\int * d^3x'$ can be executed by multiplying with $[1/\kappa]_{\mathrm{ret}}$ and replacing the $\mathbf{x}'$ in $\mathbf{R} = \mathbf{x} - \mathbf{x}'$ with the location $[\mathbf{r}(t')]_{\mathrm{ret}}$ of the particle at the retarded time. Also, $\mathbf{v}(t')$ is the particle velocity at the retarded time:

$$\mathbf{E}(\mathbf{x}, t) = \frac{q}{4\pi\epsilon_0}\left\{\left[\frac{\hat{\mathbf{R}}}{\kappa R^2}\right]_{\mathrm{ret}} + \frac{\partial}{c\partial t}\left[\frac{\hat{\mathbf{R}}}{\kappa R}\right]_{\mathrm{ret}} - \frac{\partial}{c^2\partial t}\left[\frac{\mathbf{v}}{\kappa R}\right]_{\mathrm{ret}}\right\} \quad \text{, q.e.d.}$$

Note that after the $\int * d^3x'$, the independent variables of $\hat{\mathbf{R}}$ have changed from $(\mathbf{x}, \mathbf{x}')$ to $(\mathbf{x}, t)$. In the final result, all quantities inside the $[*]_{\mathrm{ret}}$ can be written as functions of $(\mathbf{x}, t)$. Any explicit calculation will require knowledge of the particle trajectory $\mathbf{r}(t)$.

Magnetic field:

From Eqs. 6.56 and 6.57 in the textbook, the time-independence of $\mathbf{R}$ *before integration*, and by virtue of part a) it follows that

$$
\begin{aligned}
\mathbf{B}(\mathbf{x},t) &= \frac{q\,\mu_0}{4\pi}\int\left\{\left[\delta^3(\mathbf{x}'-\mathbf{r}(t'))\,\mathbf{v}(t')\times\frac{\hat{\mathbf{R}}}{R^2}\right]_{\mathrm{ret}}+\left[\frac{\partial}{\partial t'}\delta^3(\mathbf{x}'-\mathbf{r}(t'))\,\mathbf{v}(t')\times\frac{\hat{\mathbf{R}}}{cR}\right]_{\mathrm{ret}}\right\}d^3x' \\
&= \frac{q\,\mu_0}{4\pi}\left\{\left[\mathbf{v}\times\frac{\hat{\mathbf{R}}}{\kappa R^2}\right]_{\mathrm{ret}}+\frac{\partial}{\partial t}\int\left[\delta^3(\mathbf{x}'-\mathbf{r}(t'))\,\mathbf{v}(t')\times\frac{\hat{\mathbf{R}}}{cR}\right]_{\mathrm{ret}}\right\}d^3x' \\
&= \frac{q\,\mu_0}{4\pi}\left\{\left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa R^2}\right]_{\mathrm{ret}}+\frac{\partial}{c\partial t}\left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa R}\right]_{\mathrm{ret}}\right\} \quad \text{, q.e.d.}
\end{aligned}
$$

Comments on the electric-field result also apply to the magnetic field.