Physics 505 | Electricity and Magnetism | Fall 2003

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Problem Set 8

Maximal score: 25 Points

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1. Jackson, Problem 5.10 **9 Points**

a): In cylindrical coordinates, the 3D current density of a loop current I with radius a in the plane $z=0$ centered at the origin is

$$\begin{aligned} \mathbf{j}(\rho', z', \phi') &= \hat{\phi}' J_\phi(\rho', z') \\ \text{with} \quad J_\phi(\rho', z') &= I\delta(r'-a)\delta(z') \quad . \end{aligned}$$

(This applies because $I = \int \mathbf{j}(\rho', z', \phi') \cdot d\mathbf{a}'$, the integral taken over a plane of constant ϕ'.)

Thus, using the expansion of $\frac{1}{|\mathbf{x}-\mathbf{x}'|}$ in Eq. 3.149 of Jackson, it is

$$\begin{aligned} \mathbf{A}(\rho, z, \phi=0) &= \frac{\mu_0}{4\pi} \int \frac{\hat{\phi}' J_\phi(\rho', z')}{|\mathbf{x}-\mathbf{x}'|} d^3x' \\ A_\phi(\rho, z) &= \frac{\mu_0}{4\pi} \int \frac{(\hat{\phi}\cdot\hat{\phi}') J_\phi(\rho', z')}{|\mathbf{x}-\mathbf{x}'|} d^3x' \\ A_\phi(\rho, z) &= \frac{\mu_0 I}{\pi^2} \int_{k=0}^{\infty} \int_{\rho', z', \phi'} \cos\phi' \delta(r'-a)\delta(z') \cos(k(z-z')) \\ &\quad \times \left[\frac{1}{2} I_0(k\rho_<) K_0(k\rho_>) + \sum_{m=1}^{\infty} \{\cos(m(\phi-\phi')) I_m(k\rho_<) K_m(k\rho_>)\} \right] \rho' d\rho' d\phi' dz' dk \quad \text{use } \phi = 0 \\ A_\phi(\rho, z) &= \frac{\mu_0 I a}{\pi} \int_{k=0}^{\infty} \cos(kz) I_1(k\rho_<) K_1(k\rho_>) dk \quad \text{q.e.d.} \end{aligned}$$

b): Using the expansion of $\frac{1}{|\mathbf{x}-\mathbf{x}'|}$ in Problem 3.16b of Jackson, it is

$$\begin{aligned} A_\phi(\rho, z) &= \frac{\mu_0 I}{4\pi} \int_{k=0}^{\infty} \sum_{m=-\infty}^{\infty} \int_{\rho', z', \phi'} \cos\phi' \delta(r'-a)\delta(z') \exp(-k(z_> - z_<)) \\ &\quad \times \exp(im(\phi-\phi')) J_m(k\rho) J_m(k\rho') \rho' d\rho' d\phi' dz' dk \quad \text{use } \phi = 0 \\ A_\phi(\rho, z) &= \frac{\mu_0 I a}{4} \int_{k=0}^{\infty} \exp(-k(z_> - z_<)) J_1(k\rho) J_1(ka) dk \times 2 \\ A_\phi(\rho, z) &= \frac{\mu_0 I a}{2} \int_{k=0}^{\infty} \exp(-k|z|) J_1(k\rho) J_1(ka) dk \quad \text{q.e.d.} \end{aligned}$$

Not required, but good exercise: The utilized expansion of $G_{\rm free}(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|}$ is obtained as follows. Using completeness relations for δ-functions, it is

$$\begin{aligned}
\Delta G &= -4\pi\delta(\mathbf{x}-\mathbf{x}') = -\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(z-z')\delta(\phi-\phi') \\
\Delta G &= -2\int_{k=0}^{\infty} dk \sum_{m=-\infty}^{\infty} kJ_m(k\rho')\exp(-im\phi')\delta(z-z')J_m(k\rho)\exp(im\phi) \qquad (1)
\end{aligned}$$

Also, expanding the Green's function it is

$$\begin{aligned}
\Delta G &= \left[\frac{1}{\rho}\partial_\rho\rho\partial_\rho + \frac{1}{\rho^2}\partial_\phi^2 + \partial_z^2\right]\int_{k=0}^{\infty} dk \sum_{m=-\infty}^{\infty} A_{k,m}(z|\rho', z', \phi')J_m(k\rho)\exp(im\phi) \\
&= \int_{k=0}^{\infty} dk \sum_{m=-\infty}^{\infty} \left\{\left[\frac{d^2}{dz^2} - \frac{m^2}{\rho^2}\right]A_{k,m}(z|\rho', z', \phi')\right\}J_m(k\rho)\exp(im\phi) \\
&\quad + \int_{k=0}^{\infty} dk \sum_{m=-\infty}^{\infty} \left\{\frac{1}{\rho}\partial_\rho\rho\partial_\rho J_m(k\rho)\right\}A_{k,m}(z|\rho', z', \phi')\exp(im\phi) \quad \text{by Bessel equation} \\
&= \int_{k=0}^{\infty} dk \sum_{m=-\infty}^{\infty} \left\{\left[\frac{d^2}{dz^2} - k^2\right]A_{k,m}(z|\rho', z', \phi')\right\}J_m(k\rho)\exp(im\phi) \qquad (2)
\end{aligned}$$

Equating the terms in Eqs. 1 and 2 in front of the orthogonal functions $J_m(k\rho)\exp(im\phi)$, and defining

$$g_k(z, z') = \frac{A_{k,m}(z|\rho', z', \phi')}{-2kJ_m(k\rho')\exp(-im\phi')}$$

the equation for the reduced Green's function $g_k(z, z')$ is

$$\left[\frac{d^2}{dz^2} - k^2\right]g_k(z, z') = \delta(z - z')$$

To avoid divergence, the solution must be of the form

$$g_k(z, z') = C\exp(kz_<)\exp(-kz_>) \quad .$$

Inserting into that result into the differential equation for the reduced Green's function and integrating over an infinitesimal region that includes the δ-inhomogeneity, it is found

$$C = -\frac{1}{2k} \quad .$$

Inserting the results in reverse order, it is

$$\begin{aligned}
g_k(z,z') &= -\frac{1}{2k}\exp(-k(z_> - z_<)) \\
A_{k,m}(z|\rho',z',\phi') &= J_m(k\rho')\exp(-\mathrm{i}m\phi')\exp(-k(z_> - z_<)) \\
G_{\text{free}}(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} &= \int_{k=0}^{\infty} dk \sum_{m=-\infty}^{\infty} J_m(k\rho')J_m(k\rho)\exp(-k(z_> - z_<))\exp(\mathrm{i}m(\phi-\phi')) \quad \text{q.e.d.}
\end{aligned}$$

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c): case a):

$$\begin{aligned}
\mathbf{B}(\rho,z) &= \nabla\times\mathbf{A} = \text{(here)}\,\nabla\times(\hat{\phi}A_\phi(\rho,z)) \\
\mathbf{B}(\rho,z) &= \hat{\rho}\left[-\partial_z A_\phi(\rho,z)\right] + \hat{\mathbf{z}}\left[\frac{1}{\rho}\partial_\rho \rho A_\phi(\rho,z)\right] \\
\mathbf{B}(\rho,z) &= \frac{\mu_0 I a}{\pi}\left[\hat{\rho}\int_{k=0}^{\infty} k\sin(kz)I_1(k\rho_<)K_1(k\rho_>)dk + \hat{\mathbf{z}}\int_{k=0}^{\infty}\cos(kz)\left\{\frac{1}{\rho}\partial_\rho\rho I_1(k\rho_<)K_1(k\rho_>)\right\}dk\right]
\end{aligned}$$

On the z-axis, it is $I_1(k\rho_<)K_1(k\rho_>) = I_1(0)K_1(k\rho) = 0$ and thus $B_\rho = 0$.

Also, using L'Hopital's rule, it is for $\rho \to 0$

$$\begin{aligned}
\frac{1}{\rho}\partial_\rho \rho I_1(k\rho)K_1(ka) &= K_1(ka)\left[\frac{1}{\rho}I_1(k\rho) + kI_1'(k\rho)\right] \\
&= 2kK_1(ka)I_1'(k\rho) = 2kK_1(ka)\frac{1}{2}\left[I_0(k\rho)+I_2(k\rho)\right] \\
&= kK_1(ka)I_0(k\rho) \\
&= kK_1(ka) \quad \text{for } \rho \to 0
\end{aligned}$$

and thus, using an integral table or Mathematica or equivalent, it is verified that

$$B_z = \frac{\mu_0 I a}{\pi}\int_{k=0}^{\infty} k\cos(kz)K_1(ka)dk = \frac{\mu_0 I a^2}{2\sqrt{z^2+a^2}^3}$$

$$\mathbf{B}(\rho=0,z) = \hat{\mathbf{z}}\frac{\mu_0 I a^2}{2\sqrt{z^2+a^2}^3}$$

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case b):

$$\mathbf{B}(\rho,z) = \frac{\mu_0 I a}{2}\left[\hat{\rho}\,\text{Sign}(z)\int_{k=0}^{\infty} k\exp(-k|z|)J_1(k\rho)J_1(ka)dk + \hat{\mathbf{z}}\int_{k=0}^{\infty}\exp(-k|z|)J_1(ka)\left\{\frac{1}{\rho}\partial_\rho\rho J_1(k\rho)\right\}dk\right]$$

On the z-axis, it is $B_\rho = 0$ and, taking the limit $\rho \to 0$,

$$\begin{aligned}
\frac{1}{\rho}\partial_\rho \rho J_1(k\rho) &= \left[\frac{1}{\rho}J_1(k\rho) + kJ_1'(k\rho)\right] \\
&= 2kJ_1'(k\rho) = 2k\frac{1}{2}\left[J_0(k\rho) + J_2(k\rho)\right] \\
&= k \quad \text{for}\, \rho \to 0
\end{aligned}$$

and thus, using an integral table or Mathematica or equivalent, it is verified that

$$B_z = \frac{\mu_0 I a}{2}\int_{k=0}^{\infty} \exp(-k\,|z|)J_1(ka)kdk = \frac{\mu_0 I a}{2}\frac{a}{\sqrt{z^2+a^2}^3}$$

$$\mathbf{B}(\rho = 0, z) = \hat{\mathbf{z}}\frac{\mu_0 I a^2}{2\sqrt{z^2+a^2}^3}$$

2. Jackson, Problem 5.15 **8 Points**

Consider first a single wire with current $\mathbf{I} = I\hat{\mathbf{z}}$ along the z-axis. Through variable separation of $\Delta\Phi_M = 0$ in cylindrical coordinates and subsequent consideration of the $\nu = 0$ terms it is seen that the magnetic potential is $\Phi_M = -\frac{I\phi}{2\pi}$ (see Section 2.11 of Jackson). The validity of this result is verified by noting that the correct **H**-field follows:

$$\mathbf{H} = -\nabla\Phi_M = \hat{\phi}\frac{I}{2\pi\rho}$$

Note that the x-axis or another plane of constant ϕ needs to be "cut out" of the volume of interest.

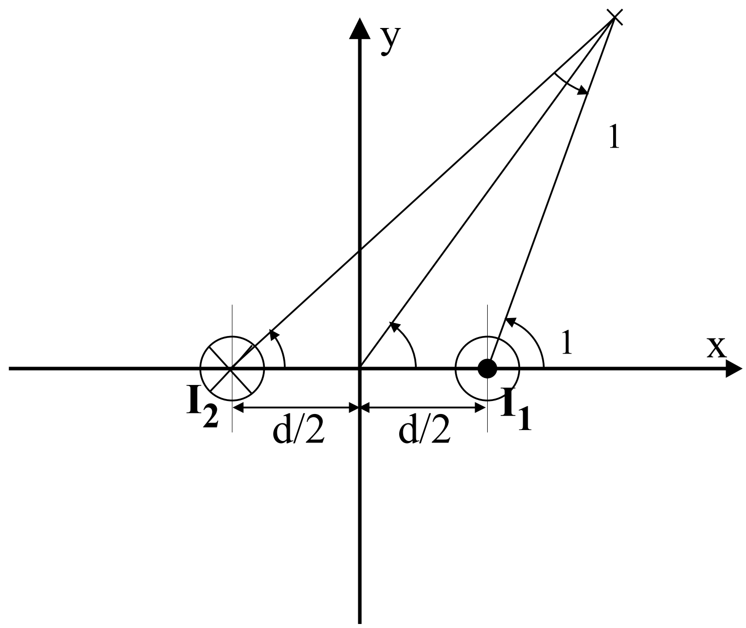


Figure 1: Geometry of the problem.

Now, consider two wires parallel to the z-axis, one with current $\mathbf{I}_1 = I\hat{\mathbf{z}}$ at $(d/2, 0)$ and one with current $\mathbf{I}_2 = -I\hat{\mathbf{z}}$ at $(-d/2, 0)$. Then, by superposition it is found that

$$\Phi_M = \frac{I}{2\pi}(\phi_2 - \phi_1)$$

where the ϕ_i describe the azimuthal angles of the observation point with respect to the respective currents $\mathbf{I}_i$. Simple geometry shows that for $\rho \gg d$ it is $\phi_2 - \phi_1 = -\frac{d\sin\phi}{\rho} + \mathcal{O}(\frac{d^2}{\rho^2})$, where ρ and ϕ are the coordinates of the observation point.

Thus

$$\Phi_M = -\frac{Id\sin\phi}{2\pi\rho} + \mathcal{O}(\frac{d^2}{\rho^2}) \quad \text{, q.e.d.}$$

Note that in the limit $\rho \gg d$ the magnetostatic potential is valid without restriction, because the currents through the volume of interest are limited to a small region in the center and add up to zero.

b): Through variable separation of the Laplace equation in cylindrical coordinates (2D) it is seen that the potentials in the regions 1, 2 and 3 can be expanded as follows:

$$\begin{aligned}
\Phi_1 &= -\frac{Id}{2\pi}\rho^{-1}\sin\phi + \sum_{n=1}^{\infty} A_n\rho^n \sin(n\phi) \\
\Phi_2 &= \sum_{n=1}^{\infty}\left[B_n\rho^n + C_n\rho^{-n}\right]\sin(n\phi) \\
\Phi_3 &= \sum_{n=1}^{\infty}\left[D_n\rho^{-n}\right]\sin(n\phi)
\end{aligned}$$

The boundary conditions on the interfaces are, due to the absence of free currents,

$$\begin{array}{cc}
1-2 & 2-3 \\
\hat{\mathbf{n}}\cdot\mathbf{B}_1|_{\rho=a} = \hat{\mathbf{n}}\cdot\mathbf{B}_2|_{\rho=a} & \hat{\mathbf{n}}\cdot\mathbf{B}_2|_{\rho=b} = \hat{\mathbf{n}}\cdot\mathbf{B}_3|_{\rho=b} \\
\hat{\mathbf{n}}\times\mathbf{H}_1|_{\rho=a} = \hat{\mathbf{n}}\times\mathbf{H}_2|_{\rho=a} & \hat{\mathbf{n}}\times\mathbf{H}_2|_{\rho=b} = \hat{\mathbf{n}}\times\mathbf{H}_3|_{\rho=b}
\end{array}$$

In the given geometry and expressed with the magnetostatic potential, they are

$$\begin{array}{cc}
1-2 & 2-3 \\
\mu_0\partial_\rho\Phi_1|_{\rho=a} = \mu\partial_\rho\Phi_2|_{\rho=a} & \mu\partial_\rho\Phi_2|_{\rho=b} = \mu_0\partial_\rho\Phi_3|_{\rho=b} \\
\mu_0\frac{1}{a}\partial_\phi\Phi_1|_{\rho=a} = \mu\frac{1}{b}\partial_\phi\Phi_2|_{\rho=a} & \mu\frac{1}{a}\partial_\phi\Phi_2|_{\rho=b} = \mu_0\frac{1}{b}\partial_\phi\Phi_3|_{\rho=b}
\end{array}$$

The resultant equations are

$$\begin{aligned}
\sum_{n=1}^{\infty}\left[\mu_0\frac{Id}{2\pi}a^{-2}\delta_{n,1} + \mu_0 A_n n a^{n-1}\right]\sin(n\phi) &= \sum_{n=1}^{\infty}\left[\mu B_n n a^{n-1} - \mu C_n n a^{-n-1}\right]\sin(n\phi) \\
\sum_{n=1}^{\infty}\left[-\frac{Id}{2\pi}a^{-2}\delta_{n,1} + A_n n a^{n-1}\right]\sin(n\phi) &= \sum_{n=1}^{\infty}\left[B_n n a^{n-1} + C_n n a^{-n-1}\right]\sin(n\phi) \\
\sum_{n=1}^{\infty}\left[\mu B_n n b^{n-1} - \mu C_n n b^{-n-1}\right]\sin(n\phi) &= \sum_{n=1}^{\infty}\left[-\mu_0 D_n n b^{-n-1}\right]\sin(n\phi) \\
\sum_{n=1}^{\infty}\left[B_n n b^{n-1} + C_n n b^{-n-1}\right]\sin(n\phi) &= \sum_{n=1}^{\infty}\left[D_n n b^{-n-1}\right]\sin(n\phi)
\end{aligned}$$

Using the orthogonality of the $\sin(n\phi)$ and $\mu_r = \frac{\mu}{\mu_0}$, the resultant set of equations for the coefficients of the Φ_i is

$$\begin{pmatrix}
a^{n-1} & -\mu_r a^{n-1} & \mu_r a^{-n-1} & 0 \\
a^{n-1} & -a^{n-1} & -a^{-n-1} & 0 \\
0 & \mu_r b^{n-1} & -\mu b^{-n-1} & b^{-n-1} \\
0 & b^{n-1} & b^{-n-1} & -b^{-n-1}
\end{pmatrix}
\begin{pmatrix} A_n \\ B_n \\ C_n \\ D_n \end{pmatrix}
=
\begin{pmatrix} -\frac{Id}{2\pi}a^{-2}\delta_{n,1} \\ \frac{Id}{2\pi}a^{-2}\delta_{n,1} \\ 0 \\ 0 \end{pmatrix}
\quad \forall \quad n = 1, 2, 3...$$

This system can be solved with Kramer's rule, Mathematica or similar. For $n \neq 1$ all coefficients are zero. For $n = 1$ one finds

$$D_1 = -\frac{Id}{2\pi} \frac{4\mu_r b^2}{b^2(1+\mu_r)^2 - a^2(1-\mu_r)^2}$$

and

$$\Phi_3 = -\frac{Id\sin\phi}{2\pi\rho} f \quad \text{with} \quad f = \frac{4\mu_r b^2}{b^2(1+\mu_r)^2 - a^2(1-\mu_r)^2}$$

Thus, the field is attenuated by the factor f, q.e.d. (No comparison with problem 5.14 required.)

c): The exact field reduction factor for $\mu_r = 200$, $b = 12.5\text{mm}$ and wall thickness $t = b - a = 3\text{mm}$ is $f = 4.56\%$. For $\mu_r \gg 1$ and $b \gg t$ it is

$$f \approx \frac{2b}{\mu_r t} \quad ,$$

which yields $f \approx 4.17\%$.

3. Jackson, Problem 5.19 **8 Points**

a): Since there is no free currents, we use the magnetostatic potential. The potential of the described object is found from its volume magnetic charge density $\rho_M = -\nabla \cdot \mathbf{M} = 0$ and its surface magnetic charge density $\sigma_M = \hat{\mathbf{n}} \cdot \mathbf{M} = \pm M_0$ at $z = \pm L/2$, respectively:

$$\Phi_M = \frac{1}{4\pi} \int_{V \setminus \partial V} \frac{\rho_M(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' + \frac{1}{4\pi} \int_{\partial V} \frac{\sigma_M(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x'$$

In the given case, on the z-axis the potentials due to the top (T) and bottom (B) surfaces are

$$\begin{aligned} \Phi_{T/B} &= \pm \frac{M_0}{4\pi} \int_{\rho=0}^{a} 2\pi\rho d\rho \frac{1}{\sqrt{\rho^2 + \left(z \mp \frac{L}{2}\right)^2}} \\ &= \pm \frac{M_0}{2} \left[\sqrt{\rho^2 + \left(z \mp \frac{L}{2}\right)^2} \right]_0^a \\ &= \pm \frac{M_0}{2} \left(\sqrt{a^2 + \left(z \mp \frac{L}{2}\right)^2} - \left| z \mp \frac{L}{2} \right| \right) \end{aligned}$$

(upper signs for T, lower signs for B). The total potential $\Phi_M = \Phi_T + \Phi_B$, which is

$$\Phi_M(z) = \frac{M_0}{2} \left(\sqrt{a^2 + \left(z - \frac{L}{2}\right)^2} - \sqrt{a^2 + \left(z + \frac{L}{2}\right)^2} \right) + \frac{M_0}{2} \times \left\{ \begin{array}{ccc} L & , & z > L/2 \\ 2z & , & |z| \leq L/2 \\ -L & , & z < -L/2 \end{array} \right.$$

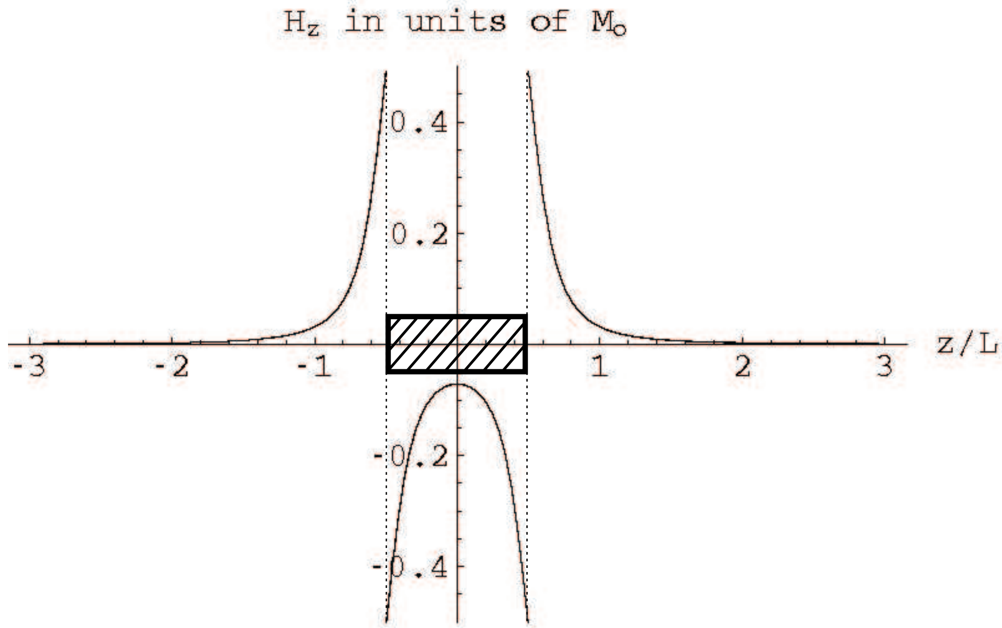
On the z-axis, the only non-zero component of $\mathbf{H}$ is

$$\begin{aligned} H_z &= -\partial_z \Phi_M(z) \\ &= -\frac{M_0}{2} \left(\frac{z - \frac{L}{2}}{\sqrt{a^2 + \left(z - \frac{L}{2}\right)^2}} - \frac{z + \frac{L}{2}}{\sqrt{a^2 + \left(z + \frac{L}{2}\right)^2}} \right) - M_0 \times \left\{ \begin{array}{ccc} 0 & , & |z| > L/2 \\ 1 & , & |z| \leq L/2 \end{array} \right. \end{aligned}$$

On the z-axis, the only non-zero component of $\mathbf{B}$ is

$$\begin{aligned} B_z &= \mu_0 \left(H_z + M_0 \times \left\{ \begin{array}{ccc} 0 & , & |z| > L/2 \\ 1 & , & |z| \leq L/2 \end{array} \right. \right) \\ &= -\frac{\mu_0 M_0}{2} \left(\frac{z - \frac{L}{2}}{\sqrt{a^2 + \left(z - \frac{L}{2}\right)^2}} - \frac{z + \frac{L}{2}}{\sqrt{a^2 + \left(z + \frac{L}{2}\right)^2}} \right) \end{aligned}$$

b):



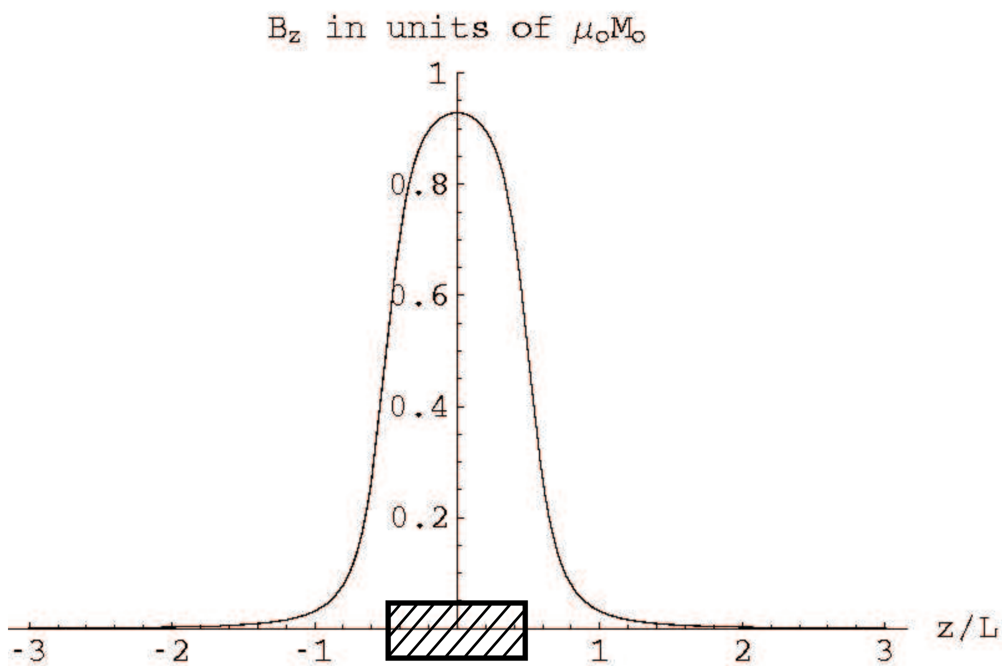


Figure 2: H_z and B_z vs. z/L for $L = 5a$.