Physics 505 Electricity and Magnetism Fall 2003
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Problem Set 7
Maximal score: 25 Points

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1. Jackson, Problem 5.1 **6 Points**

Consider the i-th cartesian component of the B-Field,

$$
\begin{aligned}
\frac{4\pi}{\mu_0 I}\mathbf{B}(\mathbf{x})\cdot\hat{\mathbf{x}}_i &= \oint_{\partial S}\hat{\mathbf{x}}_i\cdot\left[d\mathbf{l}'\times\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right] \\
&= \oint_{\partial S}\hat{\mathbf{x}}_i\cdot\left[d\mathbf{l}'\times\nabla_{x'}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right] \\
&= \oint_{\partial S}d\mathbf{l}'\cdot\left[\nabla_{x'}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\times\hat{\mathbf{x}}_i\right] \quad | \quad \text{Stokes' law} \\
&= \int_S d\mathbf{a}'\cdot\left\{\nabla_{x'}\times\left[\nabla_{x'}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\times\hat{\mathbf{x}}_i\right]\right\} \\
&= \int_S d\mathbf{a}'\cdot\left\{\left[\nabla_{x'}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right](\nabla_{x'}\cdot\hat{\mathbf{x}}_i)-\hat{\mathbf{x}}_i\left[\nabla^2_{x'}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right]\right. \\
&\quad \left.+(\hat{\mathbf{x}}_i\cdot\nabla_{x'})\left[\nabla_{x'}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right]-(\left[\nabla_{x'}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right]\cdot\nabla_{x'})\hat{\mathbf{x}}_i\right\} \quad \text{use} \quad \mathbf{x}\neq\mathbf{x}' \quad \text{always} \\
&= \int_S d\mathbf{a}'\cdot\left\{0-0+(\hat{\mathbf{x}}_i\cdot\nabla_{x'})\left[\nabla_{x'}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right]-0\right\} \\
&= \int_S d\mathbf{a}'\cdot\left\{0-0+(\hat{\mathbf{x}}_i\cdot\nabla_{x'})\left[\nabla_{x'}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right]-0\right\} \\
&= \int_S d\mathbf{a}'\cdot\left\{\frac{\partial}{\partial x'_i}\left[\nabla_{x'}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right]\right\} = -\int_S d\mathbf{a}'\cdot\left\{\frac{\partial}{\partial x_i}\left[\nabla_{x'}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right]\right\} \\
&= -\frac{\partial}{\partial x_i}\int_S d\mathbf{a}'\cdot\left\{\left[\nabla_{x'}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right]\right\} \quad | \quad \text{see Eqn. after 1.25 on page 33 of textbook} \\
&= \frac{\partial}{\partial x_i}\int_S d\Omega' = \frac{\partial}{\partial x_i}\Omega(\mathbf{x})
\end{aligned}
$$

Thus, $B_i = \frac{\mu_0 I}{4\pi}\frac{\partial}{\partial x_i}\Omega(\mathbf{x})$, and

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi}\nabla_x\Omega(\mathbf{x}) \quad \text{q.e.d.}$$

2. Jackson, Problem 5.3 **6 Points**

Consider a loop current with radius R around the z-axis. The loop is centered at location $\hat{\mathbf{z}}z'$. Then, the magnetic field at an observation point $\hat{\mathbf{z}}z$ is

$$\mathbf{B}(0,0,z) = \frac{\mu_0 I}{4\pi} \oint d\mathrm{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}$$

Insert $\mathbf{x}' = \begin{pmatrix} R\cos\phi' \\ R\sin\phi' \\ z' \end{pmatrix}$ and $d\mathbf{l}' = \begin{pmatrix} -R\sin\phi' \\ R\cos\phi' \\ 0 \end{pmatrix} d\phi'$ to find

$$\mathbf{B}(0,0,z) = \frac{\mu_0 I}{4\pi} \frac{1}{\sqrt{R^2+z'^2}^3} \int_0^{2\pi} \begin{pmatrix} -Rz'\cos\phi' \\ Rz'\sin\phi' \\ R^2 \end{pmatrix} d\phi' = \hat{\mathbf{z}} \frac{\mu_0 I}{2} \frac{R^2}{\sqrt{R^2+z'^2}^3}$$

Now, consider a solenoid the axis of which coincides with the z-axis. The solenoid has N windings per length, current I, and end points z_1 and z_2. Then, the amount of current flowing across a length dz' is $dI = INdz'$, and

$$\begin{aligned}
\mathbf{B}(0,0,z) &= \int \hat{\mathbf{z}} \frac{\mu_0 dI}{2} \frac{R^2}{\sqrt{R^2+z'^2}^3} \\
&= \hat{\mathbf{z}} \frac{\mu_0 R^2 IN}{2} \int_{z_1}^{z^2} \frac{dz'}{\sqrt{R^2+z'^2}^3} \\
&= \hat{\mathbf{z}} \frac{\mu_0 R^2 IN}{2} \left[\frac{z'}{R^2\sqrt{R^2+z'^2}^3} \right]_{z_1}^{z^2} \\
&= \hat{\mathbf{z}} \frac{\mu_0 IN}{2} \left[\cos\theta_1 + \cos\theta_2\right] \quad ,
\end{aligned}$$

with angles θ_1 and θ_2 as shown in the problem statement.

3. Jackson, Problem 5.8 **7 Points**

a): In the Coulomb gauge, $\nabla^2 \mathbf{A} = -\mu_0 J_\phi(r,\theta)\hat{\phi}$. Using a variable separation method, we construct a solution of the form $\mathbf{A} = A_\phi(r,\theta)\hat{\phi}$. (By finding the solution of that form, it is shown that it exists.) The following derivatives of spherical unit vectors are useful:

$$\begin{aligned}
\partial_r \hat{\phi} &= 0 \\
\partial_\theta \hat{\phi} &= 0 \\
\partial_\phi \hat{\mathbf{r}} &= \hat{\phi}\sin\theta \\
\partial_\phi \hat{\theta} &= \hat{\phi}\cos\theta \\
\partial_\phi \hat{\phi} &= -\hat{\mathbf{r}}\sin\theta - \hat{\theta}\cos\theta \\
\partial_\phi^2 \hat{\phi} &= -\hat{\phi} \quad .
\end{aligned}$$

Thus, writing out $\nabla^2\left(A_\phi(r,\theta)\hat{\phi}\right) = -\mu_0 J_\phi(r,\theta)\hat{\phi}$ in spherical coordinates yields:

$$\begin{aligned}
\left(\frac{1}{r^2}\partial_r r^2 \partial_r + \frac{1}{r^2\sin\theta}\partial_\theta \sin\theta\partial_\theta + \frac{1}{r^2\sin^2\theta}\partial_\phi^2\right)\left(A_\phi(r,\theta)\hat{\phi}\right) &= -\mu_0 J_\phi(r,\theta)\hat{\phi} \\
\left(\left[\frac{1}{r^2}\partial_r r^2 \partial_r + \frac{1}{r^2\sin\theta}\partial_\theta \sin\theta\partial_\theta\right] A_\phi(r,\theta)\right)\hat{\phi} - \left(\frac{1}{r^2\sin^2\theta}A_\phi(r,\theta)\right)\hat{\phi} &= -\mu_0 J_\phi(r,\theta)\hat{\phi} \\
\left[\frac{1}{r^2}\partial_r r^2 \partial_r + \frac{1}{r^2\sin\theta}\partial_\theta \sin\theta\partial_\theta - \frac{1}{r^2\sin^2\theta}\right] A_\phi(r,\theta) &= -\mu_0 J_\phi(r,\theta)
\end{aligned}$$

This is a 2-nd order, linear, inhomogeneous PDE for $A_\phi(r,\theta)$, similar to the Poisson equation, which is solvable. To identify the behavior inside and outside the current distribution, we solve the homogeneous equation by separation of variables. Writing $A_\phi(r,\theta) = \frac{U(r)}{r}\Theta(\theta)$, it is

$$\underbrace{\frac{r^2\frac{d^2}{r^2}U(r)}{U(r)}}_{=l(l+1)} + \underbrace{\frac{\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}\Theta(\theta) - \frac{\Theta(\theta)}{\sin^2\theta}}{\Theta(\theta)}}_{=-l(l+1)} = 0$$

The angular equation is the generalized Legendre differential equation,

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta} - \frac{1}{\sin^2\theta} + l(l+1)\right]\Theta(\theta) = 0 \quad ,$$

which has the regular solution $P_l^1(\cos\theta)$; note that $P_l^{-1}(\cos\theta)$ is linearly dependent. This finding justifies a posteriori that $l(l+1)$ with $l = 1, 2...$ is a good choice for the separation variable. The radial equation,

$$\frac{d^2}{r^2}U(r) - \frac{l(l+1)}{r^2}U(r) = 0$$

has the solution

$$U(r) = A_l r^{l+1} + B_l r^{-l} \quad .$$

Summarizing, the interior and exterior solutions are found to be

$$\begin{aligned} A_{\phi,\text{interior}} &= \sum_{l=1}^{\infty} A_l r^l P_l^1(\cos\theta) = -\frac{\mu_0}{4\pi}\sum_{l=1}^{\infty} m_l r^l P_l^1(\cos\theta) \\ A_{\phi,\text{exterior}} &= \sum_{l=1}^{\infty} B_l r^{-l-1} P_l^1(\cos\theta) = -\frac{\mu_0}{4\pi}\sum_{l=1}^{\infty} \mu_l r^{-l-1} P_l^1(\cos\theta) \quad \text{q.e.d.} \end{aligned} \tag{1}$$

There, we also define the multipole moments m_l and μ_l. Note that the $\left\{P_l^1(x), l = 1, 2, 3..\right\}$ form a complete orthogonal set on the interval $[-1, 1]$.

b): In analogy with electrostatics, spherical multipole moments are obtained by expanding $\frac{1}{|\mathbf{x}-\mathbf{x}'|}$ in spherical harmonics. For azimuthal current distributions it is, for an observation point with $\phi = 0$,

$$\begin{aligned} A_\phi(r,\theta) &= \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(r',\theta',\phi')\cdot\hat{\phi}}{|\mathbf{x}-\mathbf{x}'|}d^3x' = \frac{\mu_0}{4\pi}\int \frac{J_\phi(r',\theta')\left[\hat{\phi}'\cdot\hat{\phi}\right]}{|\mathbf{x}-\mathbf{x}'|}d^3x' = \frac{\mu_0}{4\pi}\int \frac{J_\phi(r',\theta')\cos\phi'}{|\mathbf{x}-\mathbf{x}'|}d^3x' \\ &= \frac{\mu_0}{4\pi}\sum_{l,m}\frac{4\pi}{2l+1}Y_{lm}(\theta,\phi=0)\int \frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')\frac{1}{2}\left[\exp(\mathrm{i}\phi') + \exp(-\mathrm{i}\phi')\right]J_\phi(r',\theta')d^3x' \end{aligned}$$

Upon integration over ϕ', only $m = \pm 1$ give non-zero contributions. For each l, the $m = 1$ and $m = -1$ terms are equal; to show this, use the fact that $\phi = 0$, and Eqs. 3.51 and 3.53 of the textbook. Thus,

$$\begin{aligned} A_\phi(r,\theta) &= \frac{\mu_0}{4\pi}\sum_{l,m=1}\frac{4\pi}{2l+1}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-1)!}{(l+1)!}}P_l^1(\cos\theta)\ \times \\ &\quad \int \frac{r_<^l}{r_>^{l+1}}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-1)!}{(l+1)!}}P_l^1(\cos\theta')\frac{1}{2}J_\phi(r',\theta')d^3x' \times 2 \\ &= \frac{\mu_0}{4\pi}\sum_{l,m=1}\frac{1}{l(l+1)}P_l^1(\cos\theta)\int \frac{r_<^l}{r_>^{l+1}}P_l^1(\cos\theta')J_\phi(r',\theta')d^3x' \end{aligned} \tag{2}$$

For the interior region, $r_< = r$ and $r_> = r'$, and

$$A_{\phi,\text{interior}}(r,\theta) = -\frac{\mu_0}{4\pi}\sum_l\left\{-\frac{1}{l(l+1)}\int \frac{1}{r'^{l+1}}P_l^1(\cos\theta')J_\phi(r',\theta')d^3x'\right\}r^l P_l^1(\cos\theta)$$

Comparison with Eq. 1 shows that

$$m_l = -\frac{1}{l(l+1)}\int \frac{1}{r'^{l+1}}P_l^1(\cos\theta')J_\phi(r',\theta')d^3x' \quad \text{q.e.d.}$$

Similarly, for the exterior region, $r_> = r$ and $r_< = r'$, and

$$A_{\phi,\text{interior}}(r,\theta) = -\frac{\mu_0}{4\pi}\sum_l \left\{-\frac{1}{l(l+1)}\int r'^l P_l^1(\cos\theta')J_\phi(r',\theta')d^3x'\right\}\frac{1}{r^{l+1}}P_l^1(\cos\theta)$$

Comparison with Eq. 1 shows that

$$\mu_l = -\frac{1}{l(l+1)}\int r'^l P_l^1(\cos\theta')J_\phi(r',\theta')d^3x' \quad \text{q.e.d.}$$

4. Jackson, Problem 5.13 **7 Points**

There is an azimuthal surface current $\mathbf{K}(\theta') = \hat{\phi}'\sigma \sin\theta' a\omega$. The corresponding three-dimensional current density is

$$\mathbf{j}(r', \theta') = \hat{\phi}'\mathbf{K}(\theta')\delta(r' - a) = \hat{\phi}'\sigma \sin\theta' a\omega\delta(r' - a) = \hat{\phi}' J_\phi(r', \theta') \quad .$$

Using Eq. 2 of the previous problem and $\int_{-1}^{-1} P_l^m(x)P_{l'}^m(x)dx = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{l,l'}$ and $P_l^1 = -\sin\theta$, it is

$$\begin{aligned}
A_\phi(r, \theta) &= \frac{\mu_0}{4\pi}\sum_{l,m=1}\frac{1}{l(l+1)}P_l^1(\cos\theta)\int\frac{r_<^l}{r_>^{l+1}}P_l^1(\cos\theta')J_\phi(r', \theta')d^3x' \\
&= \frac{\mu_0}{4\pi}\sum_{l,m=1}\frac{1}{l(l+1)}P_l^1(\cos\theta)\int\frac{r_<^l}{r_>^{l+1}}P_l^1(\cos\theta')\sigma\sin\theta'\delta(r' - a)\, a\,\omega\, r'^2 d\cos\theta' d\phi' \\
&= -\frac{\mu_0}{4\pi}\sum_{l,m=1}\frac{1}{l(l+1)}P_l^1(\cos\theta)\,\frac{2\pi\sigma a^3\omega r_<^l}{r_>^{l+1}}\int P_l^1(x)P_1^1(x)dx \\
&= -\frac{\mu_0\sigma a^3\omega}{4}P_l^1(\cos\theta)\frac{r_<}{r_>^2}\frac{4}{3} \\
&= \frac{\mu_0\sigma a^3\omega}{3}\sin\theta\frac{r_<}{r_>^2}
\end{aligned}$$

Thus, it is outside the sphere

$$\mathbf{A}_{\text{exterior}}(r, \theta) = \hat{\phi}\frac{\mu_0\sigma a^4\omega}{3}\,\frac{1}{r^2}\,\sin\theta$$

and inside

$$\mathbf{A}_{\text{interior}}(r, \theta) = \hat{\phi}\frac{\mu_0\sigma a\omega}{3}\,r\sin\theta$$

Using that for azimuthal $\mathbf{A}$ it is $\mathbf{B} = \nabla\times\mathbf{A} = \hat{\mathbf{r}}\frac{1}{r\sin\theta}\partial_\theta\left[\sin\theta A_\phi\right] - \hat{\theta}\frac{1}{r}\partial_r\left[rA_\phi\right]$ it is found:

$$\mathbf{B}_{\text{exterior}}(r, \theta) = \frac{\mu_0\sigma a^4\omega}{3}\left[\hat{\mathbf{r}}\frac{2\cos\theta}{r^3} + \hat{\theta}\frac{\sin\theta}{r^3}\right] \quad ,$$

which is the field of a magnetic dipole $\mathbf{m} = \hat{\mathbf{z}}\,\frac{4\pi\sigma a^4\omega}{3}$, and

$$\mathbf{B}_{\text{interior}}(r, \theta) = \frac{2\mu_0\sigma a\omega}{3}\left[\hat{\mathbf{r}}\cos\theta - \hat{\theta}\sin\theta\right] \quad ,$$

which is a homogeneous magnetic field in z-direction.