Physics 505 | Electricity and Magnetism | Fall 2003

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Problem Set 6
Maximal score: 25 Points

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1. Jackson, Problem 4.1 **6 Points**

a): The charge distribution is

$$\rho(\mathbf{x}) = \frac{q}{r^2}\delta(r-a)\delta(\cos\theta)\left[\delta(\phi)+\delta(\phi+\frac{\pi}{2})-\delta(\phi-\pi)-\delta(\phi+\frac{3\pi}{2})\right]$$

and thus

$$q_{lm} = \int \rho(\mathbf{x}) r^l Y^*_{lm}(\theta,\phi) d^3x = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, q\, a^l\, P_l^m(0)\left\{1+\exp(-\mathrm{i}m\frac{\pi}{2})-\exp(-\mathrm{i}m\pi)-\exp(-\mathrm{i}m\frac{3\pi}{2})\right\} \quad .$$

Since $P_l^m(0) \neq 0$ only for $l+m$ even and the term in curly brackets $= (1-(-1)^m)(1+\exp(-\mathrm{i}m\frac{\pi}{2})) \neq 0$ only form m odd, the moments are $\neq 0$ only for both l and m odd, in which case

$$q_{lm} = 2\left\{1+\mathrm{i}(-1)^{k+1}\right\}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, q\, a^l\, P_l^m(0) \quad .$$

The first non-zero moments are

$$\begin{aligned} q_{1,\pm1} &= \mp(1\mp \mathrm{i})\sqrt{\frac{3}{2\,\pi}}qa \\ q_{3,\pm1} &= \pm(1\mp \mathrm{i})\sqrt{\frac{21}{16\,\pi}}qa^3 \\ q_{3,\pm3} &= \mp(1\pm \mathrm{i})\sqrt{\frac{35}{16\,\pi}}qa^3 \end{aligned}$$

b): The charge density is

$$\rho(\mathbf{x}) = \frac{q}{2\pi r^2}\left[\delta(r-a)\delta(1-\cos\theta)+\delta(r-a)\delta(1+\cos\theta)-\delta(r)\right]$$

and yields

$$q_{lm} = q\sqrt{\frac{2l+1}{4\pi}}\left\{a^l(1+(-1)^l)-2\delta_{l,0}\right\}\delta_{m,0} = \begin{cases} q\sqrt{\frac{2l+1}{\pi}}a^l & , \quad m=0 \text{ and } l \text{ even but } \neq 0 \\ 0 & , \quad \text{otherwise} \end{cases}$$

The lowest non-zero moments are

$$
\begin{aligned}
q_{2,0} &= \sqrt{\frac{5}{\pi}} q a^2 ; \quad q_{2,m\neq 0} = 0 \\
q_{4,0} &= \sqrt{\frac{9}{\pi}} q a^4 ; \quad q_{4,m\neq 0} = 0
\end{aligned}
$$

c) and d): The potential is $\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \sum_{l,m} \frac{4\pi}{2l+1} q_{l,m} \frac{1}{r^{l+1}} Y_{lm}(\theta, \phi)$. For the above case b) this is, when keeping only the lowest non-vanishing term,

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\epsilon_0} \frac{a^2}{r^3} (3\cos^2\theta - 1) + ...$$

In the xy-plane, this is

$$\Phi(r, \theta = \frac{\pi}{2}) = -\frac{q}{4\pi\epsilon_0} \frac{a^2}{r^3} + ...$$

while the exact potential is

$$\Phi_{\text{exact}}(r, \theta = \frac{\pi}{2}) = -\frac{q}{4\pi\epsilon_0} \left\{ \frac{2}{r} - \frac{2}{\sqrt{a^2 + r^2}} \right\} \approx -\frac{q}{4\pi\epsilon_0} \left\{ \frac{a^2}{r^3} - \frac{3a^4}{4r^5} + ... \right\} \quad ,$$

in agreement with the multipole expansion.

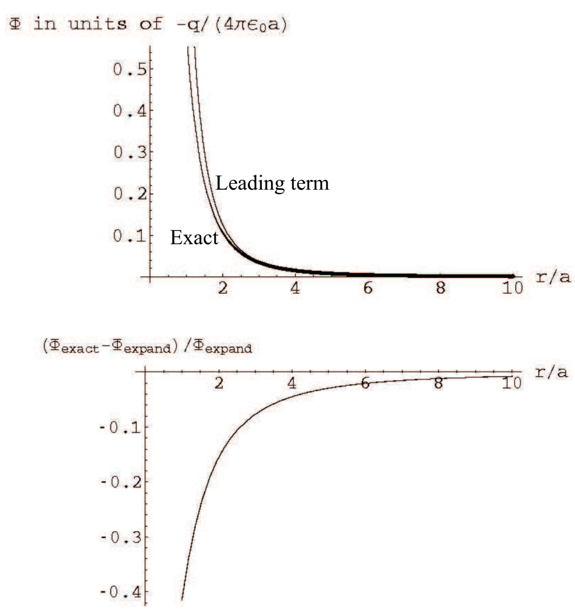

Figure 1: **Upper panel:** Exact potential and leading term vs. r/a. **Lower panel:** Relative deviation of exact potential from its asymptotic form vs. r/a.

2. Jackson, Problem 4.6 **6 Points**

a): The electric-quadrupole interaction is $W = -\frac{1}{6}\sum_{i,j=1}^{3} Q_{ij}\frac{\partial E_j}{\partial x_i}(0)$.

In a cylindrically symmetric electric field it is, under the here valid assumption that $\rho(0) = 0$,

$$\nabla \cdot \mathbf{E}(0) = 0 = \frac{\partial E_x}{\partial x}(0) + \frac{\partial E_y}{\partial y}(0) + \frac{\partial E_z}{\partial z}(0) = 2\frac{\partial E_x}{\partial x}(0) + \frac{\partial E_z}{\partial z}(0) \quad ,$$

and thus $\frac{\partial E_x}{\partial x}(0) = \frac{\partial E_y}{\partial y}(0) = -\frac{1}{2}\frac{\partial E_z}{\partial z}(0)$.

From the cylindrical symmetry of the nuclear charge distribution, explained in class and noted on p.151 of the textbook, it follows $Q_{ij} = 0$ unless $i = j$. Also, $Q_{11} = Q_{22}$. Further, since the trace $\mathrm{Tr}(Q) = Q_{11} + Q_{22} + Q_{33} = 0$, it is $Q_{11} = Q_{22} = -\frac{1}{2}Q_{33}$. The electric-quadrupole interaction thus is

$$W = -\frac{1}{6}\sum_{i,j=1}^{3} Q_{ij}\frac{\partial E_j}{\partial x_i}(0) = -\frac{1}{6}\left\{Q_{33}\frac{\partial E_z}{\partial z}(0) + 2\times(-\frac{1}{2}\frac{\partial E_z}{\partial z}(0))\times(\frac{-Q_{33}}{2})\right\} = -\frac{1}{4}\,eQ\,\frac{\partial E_z}{\partial z}(0)$$

where per definitonem $eQ = Q_{33}$.

b): The calculation yields $\frac{\partial E_z}{\partial z}(0) = -8.27\times 10^{20}\,\frac{\mathrm{V}}{\mathrm{m}^2} = -0.085\frac{e}{4\pi\epsilon_0 a_0^3}$. Note that $\frac{e}{4\pi\epsilon_0 a_0^2}$ is the atomic unit for electric field, and $\frac{e}{4\pi\epsilon_0 a_0^3}$ that of electric-field inhomogeneity.

c): The outer perimeter of the nucleus is defined by the equation $\frac{z^2}{a^2} + \frac{\rho^2}{b^2} = 1$. Thus, the volume of the nucleus is

$$V = 2\pi\int_{\rho=0}^{b}\left\{\int_{-a\sqrt{1-\frac{\rho^2}{b^2}}}^{a\sqrt{1-\frac{\rho^2}{b^2}}} dz\right\}\rho d\rho = 4\pi a\int_{\rho=0}^{b}\sqrt{1-\frac{\rho^2}{b^2}}\rho d\rho = \frac{4\pi}{3}ab^2 \quad .$$

The charge density ρ_C inside the nucleus is constant and given by $\rho_C = Z/V$, where Z is the order number (here $Z = 63$). Further,

$$Q = 2\pi\rho_C\int(2z^2-\rho^2)\rho d\rho dz \quad ,$$

where the integral goes over the volume of the nucleus. With

$$\begin{aligned}
2\pi\int(2z^2-\rho^2)\rho d\rho dz &= 2\pi\int_{\rho=0}^{b}\left\{\int_{-a\sqrt{1-\frac{\rho^2}{b^2}}}^{a\sqrt{1-\frac{\rho^2}{b^2}}}(2z^2-\rho^2)dz\right\}\rho d\rho \\
&= 4\pi\int_{\rho=0}^{b}\left\{\frac{2}{3}a^3\rho\sqrt{1-\frac{\rho^2}{b^2}}^{3} - a\rho^3\sqrt{1-\frac{\rho^2}{b^2}}\right\}d\rho \\
&= \frac{8}{15}\pi ab^2(a^2-b^2)
\end{aligned}$$

it is $Q = \frac{2Z}{5}(a^2-b^2) = \frac{4Z}{5}(a-b)R$, where here $R = (a+b)/2 = 7\times 10^{-15}$m. Thus,

$$\frac{a-b}{R} = \frac{5}{4ZR^2}Q = 0.101 \quad .$$

3. Jackson, Problem 4.8 **7 Points**

Let ϕ denote the angle with respect to the external field. Then, because of symmetry the solution will only contain terms $\propto \rho^{\pm n}\cos(n\phi)$. After elimination of diverging terms other than that producing the external field E_0, the potential is of the following form:

Outer region:

$$\Phi_1 = -\rho E_0 \cos\phi + \sum_{n=1}^{\infty} d_n \rho^{-n} \cos(n\phi)$$

Middle region:

$$\Phi_2 = \sum_{n=1}^{\infty} b_n \rho^{n} \cos(n\phi) + \sum_{n=1}^{\infty} c_n \rho^{-n} \cos(n\phi)$$

Inner region:

$$\Phi_3 = \sum_{n=1}^{\infty} a_n \rho^{n} \cos(n\phi)$$

Boundary condition on outer interface for D-field:

$$\begin{aligned} & \epsilon \left.\frac{\partial \Phi_2}{\partial \rho}\right|_b &=& \ \epsilon_0 \left.\frac{\partial \Phi_1}{\partial \rho}\right|_b \\ \forall n: \quad & \epsilon n b_n b^{n-1} - \epsilon n c_n b^{-n-1} &=& \ -\epsilon_0 E_0 \delta_{n,1} - \epsilon_0 n d_n b^{-n-1} \end{aligned}$$

Boundary condition on outer interface for E-field:

$$\begin{aligned} & \left.\frac{\partial \Phi_2}{\rho \partial \phi}\right|_b &=& \ \left.\frac{\partial \Phi_1}{\rho \partial \phi}\right|_b \\ \forall n: \quad & -n b_n b^{n-1} - n c_n b^{-n-1} &=& \ E_0 \delta_{n,1} - n d_n b^{-n-1} \end{aligned}$$

Boundary condition on inner interface for D-field:

$$\begin{aligned} & \epsilon_0 \left.\frac{\partial \Phi_3}{\partial \rho}\right|_a &=& \ \epsilon \left.\frac{\partial \Phi_2}{\partial \rho}\right|_a \\ \forall n: \quad & \epsilon_0 n a_n a^{n-1} &=& \ \epsilon n b_n a^{n-1} - \epsilon n c_n a^{-n-1} \end{aligned}$$

Boundary condition on inner interface for E-field:

$$\begin{aligned} & \left.\frac{\partial \Phi_3}{\rho \partial \phi}\right|_a &=& \ \left.\frac{\partial \Phi_2}{\rho \partial \phi}\right|_a \\ \forall n: \quad & -n a_n a^{n-1} &=& \ -n b_n a^{n-1} - n c_n a^{-n-1} \end{aligned}$$

Note that the boundary conditions for the E-field are equivalent with setting the potentials on the interfaces equal. The system to be solved therefore is

$$\begin{pmatrix} a^{2n} & -a^{2n} & -1 & 0 \\ a^{2n} & -\epsilon_r a^{2n} & \epsilon_r & 0 \\ 0 & \epsilon_r b^{2n} & -\epsilon_r & 1 \\ 0 & b^{2n} & 1 & -1 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -E_0 b^2\, \delta_{n,1} \\ -E_0 b^2\, \delta_{n,1} \end{pmatrix} \qquad \forall n \tag{1}$$

where $\epsilon_r = \epsilon/\epsilon_0$. Since the determinant $D = a^{2n}(b^{2n}(\epsilon_r+1)^2 - a^{2n}(\epsilon_r-1)^2)$ is generally $\neq 0$, all a_n, b_n, c_n, d_n are zero unless $n = 1$. For $n = 1$, with Kramer's rule, Mathematica or equivalent one finds:

$$\begin{aligned} a_1 &= E_0 \frac{4b^2\epsilon_r}{a^2(\epsilon_r-1)^2 - b^2(\epsilon_r+1)^2} \\ b_1 &= E_0 \frac{2b^2(\epsilon_r+1)}{a^2(\epsilon_r-1)^2 - b^2(\epsilon_r+1)^2} \\ c_1 &= E_0 \frac{2a^2b^2(\epsilon_r-1)}{a^2(\epsilon_r-1)^2 - b^2(\epsilon_r+1)^2} \\ d_1 &= E_0 \frac{b^2(a^2-b^2)(\epsilon_r^2-1)}{a^2(\epsilon_r-1)^2 - b^2(\epsilon_r+1)^2} \end{aligned} \tag{2}$$

b:) In the inner region, there is a homogeneous electric field of a size less than the outer field E_0. In the intermediate region, the field is inhomogeneous and weakest (see figure).

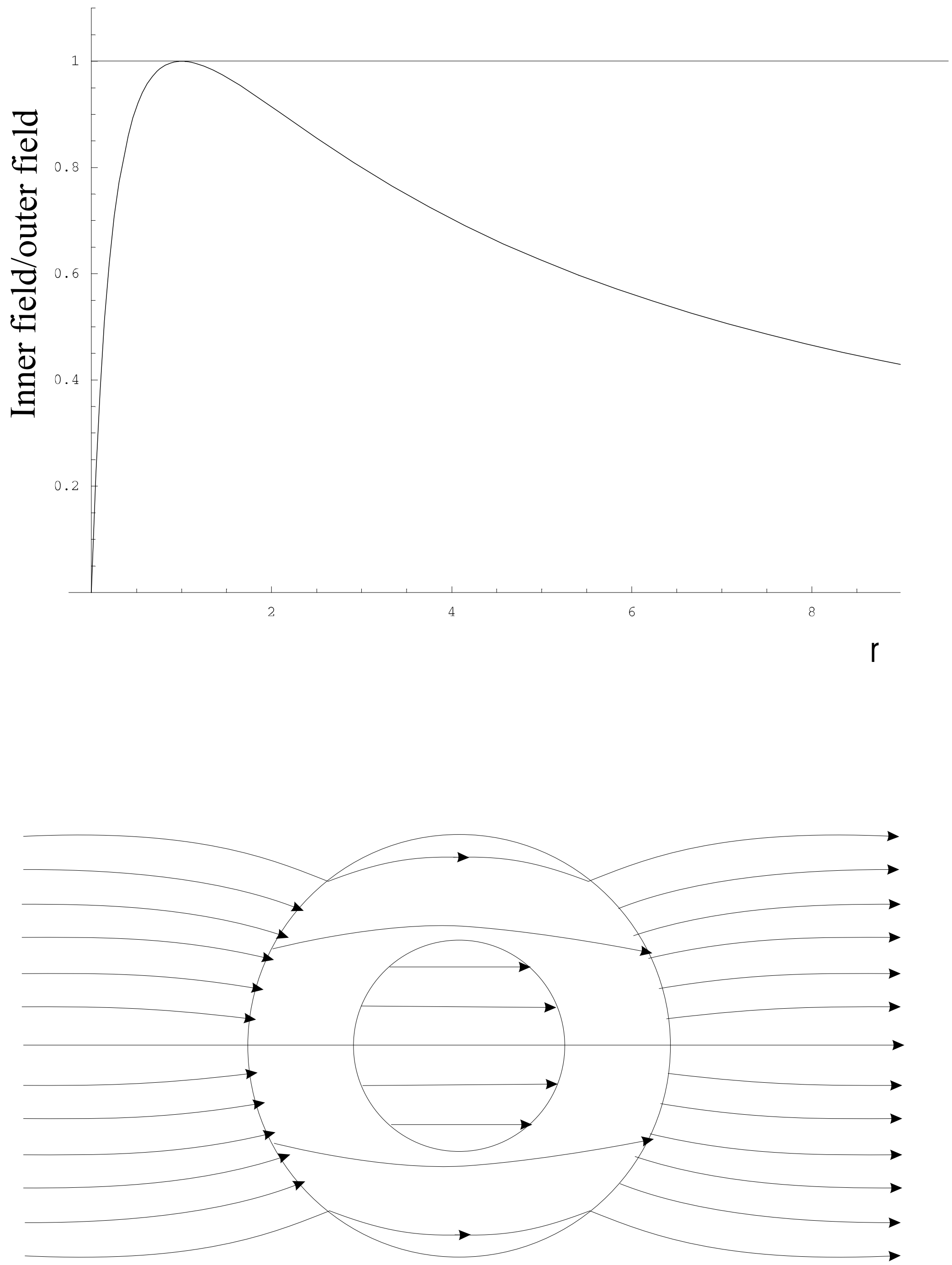

Figure 2: **Upper panel:** Strength of inner field relative to outer field for $b = 2a$ vs. dielectric constant of the shell. The presence of the dielectric shell attenuates the field. **Lower panel:** Qualitative drawing of electric field lines.

c): Solid cylinder (case $a = 0$). We find

$$\begin{aligned} a_1 &= \text{(irrelevant)} \\ b_1 &= -E_0 \frac{2}{\epsilon_r + 1} \\ c_1 &= 0 \\ d_1 &= E_0 \frac{b^2(\epsilon_r - 1)}{\epsilon_r + 1} \end{aligned} \tag{3}$$

The inside field, given by b_1 and c_1, is homogeneous and attenuated by a factor $\frac{2}{\epsilon_r+1}$ relative to the outside field. The d_1-term reflects a "2D dipole moment" proportional to area, external field, and contrast $\frac{(\epsilon_r-1)}{(\epsilon_r+1)}$ in the dielectric constant.

Cylindrical cavity in bulk dielectric (case $b \to \infty$). We find

$$\begin{aligned} a_1 &= -E_0 \frac{4\epsilon_r}{(\epsilon_r + 1)^2} \\ b_1 &= -E_0 \frac{2}{\epsilon_r + 1} \\ c_1 &= -E_0 \frac{2a^2(\epsilon_r - 1)}{(\epsilon_r + 1)^2} \\ d_1 &= \text{(irrelevant)} \end{aligned} \tag{4}$$

Here, the outside field is given by b_1 and c_1. At large distances, the outside field is homogeneous and has a magnitude given by b_1. The cavity field, given by a_1, is homogeneous and amplified by a factor $\frac{2\epsilon_r}{\epsilon_r+1}$ relative to the asymptotic outside field. The c_1-term reflects a "2D dipole moment" of the cavity proportional to area, the asymptotic outside field b_1, and the contrast $\frac{(\epsilon_r-1)}{(\epsilon_r+1)}$ in the dielectric constant.

4. Jackson, Problem 4.10 — 6 Points

a): We first identify the solutions for E and D. Since there cannot be any potential differences on the conductor surfaces, the electric fields in the regions with and without dielectric must essentially be the same. The only question is whether there are any non-trivial structures in the field near the interface between the dielectric and the free space between the two shells. We first claim that the E-field is very simple, namely that it is identical with the field of a free point charge located at the origin. The claim is proved later by showing that the corresponding solution satisfies all boundary conditions.

We assume that the E-field has the form $\mathbf{E}(r) \propto \frac{\mathbf{r}}{r^3}$ both in the regions with and without dielectric. Using Gauss's law on a sphere with radius $a < r < b$, it is then the case that

$$\begin{aligned} \oint \mathbf{D} \cdot \mathbf{da} &= Q = 2\pi r^2(\epsilon_0 E(r) + \epsilon E(r)) \\ \mathbf{E}(r) &= \frac{Q}{2\pi(\epsilon_0 + \epsilon)} \frac{\mathbf{r}}{r^3} \end{aligned} \tag{5}$$

The free charge on the outer shell is then $-Q$, as can be seen by considering a Gaussian surface inside the outer conductor.

The specified field is the only solution, because it satisfies the equations $\nabla \cdot \mathbf{D} = \rho$ and $\nabla \times \mathbf{E} = 0$ in the volume of interest, it satisfies the general boundary conditions for E- and D-fields at the interface between the dielectric and the free space, and it produces the correct charges on the inner and outer shells.

b): We use the usual boundary condition $\hat{\mathbf{n}} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma_{\text{free}}$, where $\hat{\mathbf{n}}$ is a unit vector pointing from region 1 to region 2. Since $D = 0$ inside conductors, on the part of the inner surface facing the dielectric-free region the condition reads

$$\sigma_{\text{free}} = \hat{\mathbf{r}} \cdot \mathbf{D}(a) = \frac{Q\epsilon_0}{2\pi(\epsilon_0 + \epsilon)} \hat{\mathbf{r}} \cdot \frac{\mathbf{r}}{a^2} = \frac{Q\epsilon_0}{2\pi a^2(\epsilon_0 + \epsilon)}$$

and on the part facing the dielectric it is

$$\sigma_{\text{free}} = \frac{Q\epsilon}{2\pi a^2(\epsilon_0 + \epsilon)}$$

c): Here, $\mathbf{P}(r) = (\epsilon - \epsilon_0)\mathbf{E}(r)$. The volume polarization charge density $\rho_{\text{pol}} = -\nabla \cdot \mathbf{P} = 0$ everywhere. The surface polarization charge density generally is given by $\hat{\mathbf{n}} \cdot (\mathbf{P}_1 - \mathbf{P}_2) = \sigma_{\text{pol}}$, where $\hat{\mathbf{n}}$ is a unit vector pointing from region 1 to region 2. On the part of the inner surface facing the dielectric the condition reads

$$\sigma_{\text{pol}} = -\hat{\mathbf{r}} \cdot \mathbf{P}(a) = -\frac{Q(\epsilon - \epsilon_0)}{2\pi a^2(\epsilon_0 + \epsilon)} \quad ;$$

on the part facing the dielectric-free space it is $\sigma_{\text{pol}} = 0$.

Not required: On the interface between the dielectric and the dielectric-free volume it is $\mathbf{P} \perp \hat{\mathbf{n}}$ and thus $\sigma_{\text{pol}} = 0$. On the part of the outer surface facing the dielectric it is $\sigma_{\text{pol}} = +\hat{\mathbf{r}} \cdot \mathbf{P}(b) = \frac{Q(\epsilon-\epsilon_0)}{2\pi b^2(\epsilon_0+\epsilon)}$.