Prof. G. Raithel

Problem Set 5

Problem 1 **7 Points**

A cylindrical surface with radius a and infinite length in the z-direction carries a surface potential $V(z,\phi)$. Find a series solution for the potential anywhere **outside** the cylinder. Provide an integral solution for the expansion coefficients. There are no charges in the volume of interest.

The Laplace equation $\Delta\Phi(\rho,z,\phi)=0$ is to be solved for the boundary condition $V(z,\phi)$ on a cylinder with radius a and infinite length in both the $+z$- and $-z$-directions. We seek the solution in the exterior volume $\rho > a$. Using $\Delta = \frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+\frac{\partial^2}{\partial z^2}$ and $\Phi(\rho,z,\phi)=R(\rho)Q(\phi)Z(z)$, we find

$$\begin{aligned}
\frac{\partial^2}{\partial\rho^2}\Phi+\frac{1}{\rho}\frac{\partial}{\partial\rho}\Phi+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\Phi+\frac{\partial^2}{\partial z^2}\Phi &= 0\\
\frac{1}{R}R''+\frac{1}{\rho R}R'+\frac{1}{\rho^2}\frac{Q''}{Q}+\frac{Z''}{Z} &= 0 \quad ,
\end{aligned}$$

where in the second line we have divided by RQZ and used $'$ for derivatives. Since we are looking for solutions that are orthogonal on the cylinder mantle, the z-dependence is chosen to be of the form

$$Z''=-k^2 Z \quad \Rightarrow \quad Z(z)=\exp(\mathrm{i}kz) \quad \text{with k real}\,.$$

It follows

$$\frac{\rho^2}{R}R''+\frac{\rho}{R}R'+\frac{Q''}{Q}-k^2\rho^2=0 \quad .$$

Considering the fact that the potential must be a single-valued function of ϕ, and that there are no boundaries in the ϕ-coordinate, the solution for Q is

$$Q''=-m^2 Q \quad \Rightarrow \quad Q(\phi)=\exp(\mathrm{i}m\phi) \quad \text{with m integer}\,.$$

The resultant equation for R,

$$R''+\frac{1}{\rho}R'-\left(k^2+\frac{m^2}{\rho^2}\right)R=0$$

is solved by the modified Bessel functions $I_m(|k|\rho)$ and $K_m(|k|\rho)$. Since we are interested in the exterior volume, we only use the solution that is regular for $\rho \to \infty$; that solution is $K_m(|k|\rho)$ (see Eqs. 3.98-3.104 in the textbook). Note that $K_m(x) = K_{-m}(x)$.

To summarize, $\boxed{\Phi(\rho, z, \phi) = \sum_{m=-\infty}^{\infty} \int_{k=-\infty}^{\infty} A_m(k)\, K_m(|k|\rho)\, \exp(\mathrm{i}kz)\, \exp(\mathrm{i}m\phi)\, dk}$.

To find the coefficient functions $A_m(k)$, we consider this expression on the boundary,

$$V(z, \phi) = \sum_{m=-\infty}^{\infty} \int_{k=-\infty}^{\infty} A_m(k)\, K_m(|k|a)\, \exp(\mathrm{i}kz)\, \exp(\mathrm{i}m\phi)\, dk \quad ,$$

multiply with $\exp(-\mathrm{i}k'z)\, \exp(-\mathrm{i}m'\phi)$ and integrate over the cylinder mantle,

$$\int_{z=-\infty}^{\infty} \int_{\phi=0}^{2\pi} V(z, \phi)\, \exp(-\mathrm{i}k'z)\, \exp(-\mathrm{i}m'\phi)\, dz\, d\phi =$$
$$= \sum_{m=-\infty}^{\infty} \int_{k=-\infty}^{\infty} A_m(k)\, K_m(|k|a) \int_{z=-\infty}^{\infty} \int_{\phi=0}^{2\pi} \exp(\mathrm{i}kz)\, \exp(\mathrm{i}m\phi) \exp(-\mathrm{i}k'z)\, \exp(-\mathrm{i}m'\phi)\, dz\, d\phi\, dk \quad \Leftrightarrow$$

$$\int_{z=-\infty}^{\infty} \int_{\phi=0}^{2\pi} V(z, \phi)\, \exp(-\mathrm{i}k'z)\, \exp(-\mathrm{i}m'\phi)\, dz\, d\phi =$$
$$\sum_{m=-\infty}^{\infty} \int_{k=-\infty}^{\infty} A_m(k)\, K_m(|k|a)\, 4\pi^2\, \delta_{m,m'}\, \delta(k-k')\, dk \quad \Leftrightarrow$$

$$\int_{z=-\infty}^{\infty} \int_{\phi=0}^{2\pi} V(z, \phi)\, \exp(-\mathrm{i}k'z)\, \exp(-\mathrm{i}m'\phi)\, dz\, d\phi \quad = \quad A_{m'}(k')\, K_{m'}(|k'|a)\, 4\pi^2 \quad .$$

Thus, it is $\boxed{A_m(k) = \frac{1}{4\pi^2\, K_m(|k|a)} \int_{z=-\infty}^{\infty} \int_{\phi=0}^{2\pi} V(z, \phi)\, \exp(-\mathrm{i}kz)\, \exp(-\mathrm{i}m\phi)\, dz\, d\phi}$.

Problem 3.17 **9 Points**

To demonstrate the general method, this problem is worked out in a verbose manner.

Preparation: We use the completeness (=closure) relations

$$\begin{aligned}
\delta(\phi-\phi') &= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} \exp(-\mathrm{i}m\phi')\,\exp(\mathrm{i}m\phi) \\
\delta(z-z') &= \frac{2}{L}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) \\
\frac{\delta(\rho-\rho')}{\rho'} &= \int_{k=0}^{\infty} kJ_m(k\rho')J_m(k\rho)dk \quad \forall\, m \quad ,
\end{aligned} \tag{1}$$

where the last one is obtained by treating Eq. 3.108 in the textbook with the replacements explained after Eq. 3.112. In doubt, for any complete set of functions with given orthogonality relation the appropriate closure relation can be obtained using the procedure outlined in class. The procedure is demonstrated again in the following:

Completeness of the set $\{J_m(k\rho), k\in[0,\infty[\,, m \text{ fixed}\}\Rightarrow$ any $f(\rho)$ can be expanded as

$$f(\rho)=\int_{k=0}^{\infty} A(k)J_m(k\rho)dk \tag{2}$$

Obtain expansion coefficient function $A(k)$ using the orthogonality relation Eq. 3.108 of the textbook:

$$\int_{\rho=0}^{\infty} f(\rho)J_m(k'\rho)\rho d\rho = \int_{k=0}^{\infty} A(k)\int_{\rho=0}^{\infty} J_m(k\rho)J_m(k'\rho)\rho d\rho dk$$

$$\int_{\rho=0}^{\infty} f(\rho)J_m(k'\rho)\rho d\rho = \int_{k=0}^{\infty} A(k)\frac{1}{k}\delta(k-k')dk$$

$$A(k') = k'\int_{\rho=0}^{\infty} f(\rho)J_m(k\rho)\rho d\rho$$

To simplify writing in what follows, swap primes from k to ρ:

$$A(k) = k\int_{\rho=0}^{\infty} f(\rho')J_m(k\rho')\rho' d\rho'$$

Insert into Eq. 2:

$$f(\rho)=\int_{k=0}^{\infty} k\int_{\rho=0}^{\infty} f(\rho')J_m(k\rho')\rho' d\rho' J_m(k\rho)dk$$

$$f(\rho) = \int_{\rho=0}^{\infty} \left\{ \int_{k=0}^{\infty} kJ_m(k\rho')J_m(k\rho)dk \right\} \rho' f(\rho')d\rho'$$

From the last line, it is obvious that

$$\frac{\delta(\rho-\rho')}{\rho'} = \int_{k=0}^{\infty} kJ_m(k\rho')J_m(k\rho)dk \quad \forall\, m$$

---

a): Obtain given expansion of Green's function.

Step 1: Write down Equation for Green's function with δ-function in cylindrical coordinates,

$$\Delta G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}') = -4\pi\frac{\delta(\rho-\rho')}{\rho'}\delta(\phi-\phi')\delta(z-z')$$

Step 2: On right side, use completeness relations for two out of the three δ-functions. From the given result, we suspect that we need to use the completeness relations for $\delta(z-z')$ and $\delta(\phi-\phi')$:

$$\Delta G(\mathbf{x},\mathbf{x}') = -\frac{4}{L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} \exp(-im\phi')\,\exp(im\phi)\,\sin\left(\frac{n\pi z'}{L}\right)\,\sin\left(\frac{n\pi z}{L}\right)\,\frac{\delta(\rho-\rho')}{\rho'}$$

$$\Delta G(\mathbf{x},\mathbf{x}') = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\left\{-\frac{4}{L}\,\exp(-im\phi')\,\sin\left(\frac{n\pi z'}{L}\right)\,\frac{\delta(\rho-\rho')}{\rho'}\right\}\exp(im\phi)\,\sin\left(\frac{n\pi z}{L}\right) \tag{3}$$

Step 3: On left side, expand the Green's function using the orthogonal function sets that have also been used in Step 2. Note that $\mathbf{x}'$ only enters as a parameter of the calculation; Δ acts on $\mathbf{x}$.

$$\Delta G(\mathbf{x},\mathbf{x}') = \Delta\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\exp(im\phi)\,\sin\left(\frac{n\pi z}{L}\right)\,A_{mn}(\rho|\rho',z',\phi')$$

Step 4: Write Laplacian in the proper coordinates and take derivatives of the orthogonal functions:

$$\Delta G(\mathbf{x},\mathbf{x}') = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\left\{\left[\frac{1}{\rho}\frac{d}{d\rho}\,\rho\,\frac{d}{d\rho} - \frac{m^2}{\rho^2} - \frac{n^2\pi^2}{L^2}\right]A_{mn}(\rho|\rho',z',\phi')\right\}\exp(im\phi)\,\sin\left(\frac{n\pi z}{L}\right) \tag{4}$$

Step 5: Expansions in orthogonal sets of functions are unique. Thus, we can separately equate the coefficients of the $\exp(im\phi)\,\sin\left(\frac{n\pi z}{L}\right)$ in Eq. 3 and Eq. 4. Dividing by $\exp(-im\phi')\,\sin\left(\frac{n\pi z'}{L}\right)$ and using $D_\rho = \left[\frac{1}{\rho}\frac{d}{d\rho}\,\rho\,\frac{d}{d\rho} - \frac{m^2}{\rho^2} - \frac{n^2\pi^2}{L^2}\right]$ as abbreviation for the involved linear differential operator in ρ, we find

$$D_\rho\left\{\frac{A_{mn}(\rho|\rho',z',\phi')}{\exp(-im\phi')\,\sin\left(\frac{n\pi z'}{L}\right)}\right\} = -\frac{4}{L}\,\frac{\delta(\rho-\rho')}{\rho'}$$

Step 6: Noticing that the expression in the curly brackets of the last equation can only depend on the parameters of D_ρ (which are m and n), on ρ and on ρ', we define the reduced Green's function

$$g_{mn}(\rho,\rho') = \frac{A_{mn}(\rho|\rho',z',\phi')}{\exp(-\mathrm{i}m\phi')\,\sin\left(\frac{n\pi z'}{L}\right)}$$

and proceed to solve

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} - \frac{m^2}{\rho^2} - \frac{n^2\pi^2}{L^2}\right] g_{mn}(\rho,\rho') = -\frac{4}{L}\frac{\delta(\rho-\rho')}{\rho'} \tag{5}$$

Inspection of Eq. 3.98ff of the textbook shows that the differential equation is the modified Bessel differential equation (except the δ-function inhomogeneity), and that the linearly independent solutions of the homogeneous equation are $I_m(k\rho)$ and $K_m(k\rho)$, with $k = \frac{n\pi}{L}$. In the domain $\rho < \rho'$ only $I_m(k\rho)$ is regular, while in the domain $\rho > \rho'$ only $K_m(k\rho)$ is regular. Thus, for the reduced Green's function to be regular, symmetric in ρ and ρ', and continuous at $\rho = \rho'$, it must be of the form

$$g_{mn}(\rho,\rho') = C\, I_m(k\rho_<)\, K_m(k\rho_>) \quad ,$$

with a constant C to be determined to match the inhomogeneity. Also, $\rho_< = \min(\rho,\rho')$ and $\rho_> = \max(\rho,\rho')$.

Step 7: Find C. Integrating Eq. 5 from $\rho' - \epsilon$ to $\rho' + \epsilon$ with $\epsilon \to 0$ and dropping vanishing terms we find

$$\frac{d}{d\rho} g_{mn}(\rho,\rho')|_{\rho=\rho'+\epsilon} - \frac{d}{d\rho} g_{mn}(\rho,\rho')|_{\rho=\rho'-\epsilon} = -\frac{4}{\rho' L}$$

The left side equals $C\,k\,W\,[I_m(x), K_m(x)]$ with $x = k\rho'$ and the Wronski determinant $W = I_m(x)\left(\frac{d}{dx}K_m(x)\right) - \left(\frac{d}{dx}I_m(x)\right)K_m(x)$. The Wronsky determinant can be evaluated in the asymptotic region, where both $I_m(x)$ and $K_m(x)$ have simple forms given in Eqs. 3.102ff of the textbook, leading to $W\,[I_m(x), K_m(x)] = -\frac{1}{x}$ (see Eq. 3.147 in textbook). We thus find

$$-Ck\frac{1}{x} = -Ck\frac{1}{k\rho'} = -\frac{4}{\rho' L}$$

and therfore $C = \frac{4}{L}$.

Step 8:

Going backward through all steps, the Green's function expansion is assembled into the final result:

$$g_{mn}(\rho,\rho') = \frac{4}{L}\, I_m(k\rho_<)\, K_m(k\rho_>) \quad ,$$

$$A_{mn}(\rho|\rho',z',\phi') = \frac{4}{L}\, I_m(k\rho_<)\, K_m(k\rho_>)\, \exp(-\mathrm{i}m\phi')\, \sin\left(\frac{n\pi z'}{L}\right)$$

and, using $k = \frac{n\pi}{L}$,

$$G(\mathbf{x}, \mathbf{x}') = \frac{4}{L} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \exp(-\mathrm{i}m\phi') \exp(\mathrm{i}m\phi) \sin\left(\frac{n\pi z'}{L}\right) \sin\left(\frac{n\pi z}{L}\right) I_m(\frac{n\pi}{L}\rho_<) K_m(\frac{n\pi}{L}\rho_>) \quad \text{, q.e.d.}$$

---

b:

Step 1: Write down Equation for Green's function with δ-function in cylindrical coordinates,

$$\Delta G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}') = -4\pi \frac{\delta(\rho - \rho')}{\rho'} \delta(\phi - \phi')\delta(z - z')$$

Step 2: Use completeness relations for $\frac{\delta(\rho-\rho')}{\rho'}$ and $\delta(\phi - \phi')$:

$$\Delta G(\mathbf{x}, \mathbf{x}') = -2 \sum_{m=-\infty}^{\infty} \int_{k=0}^{\infty} \exp(-\mathrm{i}m\phi') \exp(\mathrm{i}m\phi) J_m(k\rho) J_m(k\rho')\delta(z - z') \, k \, dk$$

$$\Delta G(\mathbf{x}, \mathbf{x}') = \sum_{m=-\infty}^{\infty} \int_{k=0}^{\infty} \{-2k \exp(-\mathrm{i}m\phi') J_m(k\rho')\delta(z - z')\} \exp(\mathrm{i}m\phi) J_m(k\rho) \, dk \tag{6}$$

Step 3: Expand the Green's function using the orthogonal function sets that have also been used in Step 2.

$$\Delta G(\mathbf{x}, \mathbf{x}') = \Delta \sum_{m=-\infty}^{\infty} \int_{k=0}^{\infty} \exp(\mathrm{i}m\phi) J_m(k\rho) A_{mk}(z|\rho', z', \phi') dk$$

Step 4: Write Laplacian in the proper coordinates and take the obvious derivatives of the orthogonal functions:

$$\Delta G(\mathbf{x}, \mathbf{x}') = \sum_{m=-\infty}^{\infty} \int_{k=0}^{\infty} \left[\frac{\partial^2}{\partial z^2} - \frac{m^2}{\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho}\right] \exp(\mathrm{i}m\phi) J_m(k\rho) A_{mk}(z|\rho', z', \phi') dk$$

The ordinary Bessel diff. equation reads $\left[-\frac{m^2}{\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho}\right] J_m(k\rho) = -k^2 J_m(k\rho)$, and thus:

$$\Delta G(\mathbf{x}, \mathbf{x}') = \sum_{m=-\infty}^{\infty} \int_{k=0}^{\infty} \left\{\left[\frac{d^2}{dz^2} - k^2\right] A_{mk}(z|\rho', z', \phi')\right\} \exp(\mathrm{i}m\phi) J_m(k\rho) dk \tag{7}$$

Step 5: We equate coefficients in Eqs. 6 and Eq. 7. Dividing by $\exp(-\mathrm{i}m\phi') J_m(k\rho')$ we find

$$\left[\frac{d^2}{dz^2} - k^2\right] \left\{\frac{A_{mk}(z|\rho', z', \phi')}{\exp(-\mathrm{i}m\phi') J_m(k\rho')}\right\} = -2k \, \delta(z - z')$$

Step 6: Noting that the expression in the curly brackets of the last equation can only depend on k, z and z', we define the reduced Green's function

$$g_k(z,z') = \frac{A_{mk}(z|\rho',z',\phi')}{\exp(-\mathrm{i}m\phi')\,J_m(k\rho')}$$

and proceed to solve

$$\left[\frac{d^2}{dz^2} - k^2\right] g_k(z,z') = -2k\,\delta(z-z')$$

A reduced Green's function that matches the boundary conditions at $z = 0$ and $z = L$, that is symmetric in z and z' and that is continuous at $z = z'$ must be of the form

$$g_k(z,z') = C\,\sinh(k\,z_<)\sinh(k\,(L - z_>)) \quad ,$$

where the constant C needs to be determined to match the inhomogeneity at $z = z'$. Also, $z_< = \min(z,z')$ and $z_> = \max(z,z')$.

Step 7: Find C. Integrating the differential equation for $g_k(z,z')$ from $z'-\epsilon$ to $z'+\epsilon$ with $\epsilon \to 0$ and dropping vanishing terms we find

$$\frac{d}{dz} g_k(z,z')|_{z=z'+\epsilon} - \frac{d}{dz} g_k(z,z')|_{z=z'-\epsilon} = -2k$$

Direct calculation yields

$$\begin{aligned}\frac{d}{dz} g_k(z,z')|_{z=z'+\epsilon} - \frac{d}{dz} g_k(z,z')|_{z=z'-\epsilon} &= -kC\left[\sinh(kz')\cosh(k\,(L-z_>)) + \cosh(kz')\sinh(k\,(L-z_>))\right] \\ &= -kC\sinh(kL)\end{aligned}$$

and thus, with the equation before, $C = \frac{2}{\sinh(kL)}$.

Step 8:

Going backward through all steps, the Greens function expansion is assembled into the final result

$$G(\mathbf{x},\mathbf{x}') = 2\sum_{m=-\infty}^{\infty}\int_{k=0}^{\infty} \exp(-\mathrm{i}m\phi')\,\exp(\mathrm{i}m\phi)\,J_m(k\rho)\,J_m(k\rho')\,\frac{\sinh(k\,z_<)\sinh(k\,(L-z_>))}{\sinh(kL)}\,dk \quad \text{,q.e.d.}$$

Problem 3.23 **9 Points**

Hint: Use Green's function expansion techniques. For the second line, note Eq. 3.147.

We use the completeness relations

$$\begin{aligned} \delta(\phi-\phi') &= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} \exp(-im\phi')\exp(im\phi) \\ \delta(z-z') &= \frac{2}{L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) \\ \frac{\delta(\rho-\rho')}{\rho'} &= \sum_{n=1}^{\infty}\frac{2}{a^2 J_{m+1}^2(x_{mn})}\,J_m(x_{mn}\frac{\rho}{a})J_m(x_{mn}\frac{\rho'}{a}) \quad \forall m \quad . \end{aligned} \tag{8}$$

a):

<u>Step 1:</u> Write down Equation for Green's function with δ-function in cylindrical coordinates,

$$\Delta G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}') = -4\pi\frac{\delta(\rho-\rho')}{\rho'}\delta(\phi-\phi')\delta(z-z')$$

<u>Step 2:</u> We use the completeness relations for $\delta(\phi-\phi')$ and $\frac{\delta(\rho-\rho')}{\rho'}$ to write:

$$\Delta G(\mathbf{x},\mathbf{x}') = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\left\{-\frac{4}{a^2 J_{m+1}^2(x_{mn})}\exp(-im\phi')\,J_m(x_{mn}\frac{\rho'}{a})\,\delta(z-z')\right\}\exp(im\phi)\,J_m(x_{mn}\frac{\rho}{a}) \tag{9}$$

<u>Step 3:</u> Expansion of the Green's function in $\exp(im\phi)\,J_m(x_{mn}\frac{\rho}{a})$:

$$\Delta G(\mathbf{x},\mathbf{x}') = \Delta\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\exp(im\phi)\,J_m(\frac{x_{mn}}{a}\rho)\,A_{mn}(z|\rho',z',\phi')$$

<u>Step 4:</u> Apply Laplacian:

$$\Delta G(\mathbf{x},\mathbf{x}') = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\left[\frac{\partial^2}{\partial z^2}-\frac{m^2}{\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\right]\exp(im\phi)\,J_m(\frac{x_{mn}}{a}\rho)\,A_{mn}(z|\rho',z',\phi')$$

The ordinary Bessel differential equation reads, in the present case, $\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}-\frac{m^2}{\rho^2}\right]J_m(\frac{x_{mn}}{a}\rho) = -\left(\frac{x_{mn}}{a}\right)^2 J_m(\frac{x_{mn}}{a}\rho)$, and thus:

$$\Delta G(\mathbf{x},\mathbf{x}') = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\left\{\left[\frac{d^2}{dz^2}-\left(\frac{x_{mn}}{a}\right)^2\right]A_{mn}(z|\rho',z',\phi')\right\}\exp(im\phi)\,J_m(\frac{x_{mn}}{a}\rho) \tag{10}$$

<u>Step 5:</u> We equate coefficients in Eqs. 9 and Eq. 10. Dividing by $-\frac{4}{a^2 J_{m+1}^2(x_{mn})}\exp(-im\phi')\,J_m(k\rho')$ yields

$$\left[\frac{d^2}{dz^2} - \left(\frac{x_{mn}}{a}\right)^2\right] g_{mn}(z,z') = \delta(z-z')$$

with the reduced Green's function

$$g_{mn}(z,z') = -\frac{a^2\, J_{m+1}^2(x_{mn})\, A_{mn}(z|\rho',z',\phi')}{4\, \exp(-im\phi')\, J_m(\frac{x_{mn}}{a}\rho')}$$

Step 6: A reduced Green's function that matches the boundary conditions at $z=0$ and $z=L$, that is symmetric in z and z' and that is continuous at $z=z'$ must be of the form

$$g_{mn}(z,z') = C\, \sinh(\frac{x_{mn}}{a}\, z_<)\, \sinh(\frac{x_{mn}}{a}\, (L-z_>)) \quad ,$$

where the constant C needs to be determined to match the inhomogeneity at $z=z'$. Also, $z_< = \min(z,z')$ and $z_> = \max(z,z')$.

Step 7: Find C. Integrating the differential equation for $g_{mn}(z,z')$ from $z'-\epsilon$ to $z'+\epsilon$ with $\epsilon \to 0$ and dropping vanishing terms we find

$$\frac{d}{dz} g_{mn}(z,z')|_{z=z'+\epsilon} - \frac{d}{dz} g_{mn}(z,z')|_{z=z'-\epsilon} = 1$$

Direct calculation similar to Problem 3.17 b) yields

$$C = -\frac{a}{x_{mn}\, \sinh(\frac{x_{mn}L}{a})} \quad .$$

Step 8:

Going backward through all steps, the Greens function expansion is assembled into

$$G(\mathbf{x},\mathbf{x}') = \frac{4}{a} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \exp(-im\phi')\, \exp(im\phi)\, J_m(\frac{x_{mn}}{a}\rho)\, J_m(\frac{x_{mn}}{a}\rho')\, \frac{\sinh(\frac{x_{mn}}{a}\, z_<)\, \sinh(\frac{x_{mn}}{a}\, (L-z_>))}{x_{mn}\, J_{m+1}^2(x_{mn})\, \sinh(\frac{x_{mn}L}{a})} \quad .$$

The potential of a point charge q at $\mathbf{x}'$ follows from its charge density, $\rho(\mathbf{x}'') = q\delta(\mathbf{x}''-\mathbf{x}')$, and

$$\Phi(\mathbf{x},\mathbf{x}') = \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{x}'') G(\mathbf{x},\mathbf{x}'') d^3x'' = \frac{q}{4\pi\epsilon_0} \int_V \delta(\mathbf{x}''-\mathbf{x}') G(\mathbf{x},\mathbf{x}'') d^3x'' = \frac{q}{4\pi\epsilon_0} G(\mathbf{x},\mathbf{x}')$$

Thus, with the above obtained $G(\mathbf{x},\mathbf{x}')$ it is

$$\Phi(\mathbf{x},\mathbf{x}') = \frac{q}{a\pi\epsilon_0} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \exp(-im\phi')\, \exp(im\phi)\, J_m(\frac{x_{mn}}{a}\rho)\, J_m(\frac{x_{mn}}{a}\rho')\, \frac{\sinh(\frac{x_{mn}}{a}\, z_<)\, \sinh(\frac{x_{mn}}{a}\, (L-z_>))}{x_{mn}\, J_{m+1}^2(x_{mn})\, \sinh(\frac{x_{mn}L}{a})} \quad \text{, q.e.d.}$$

---

b):

Step 1: Write down Equation for Green's function with δ-function in cylindrical coordinates,

$$\Delta G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}') = -4\pi \frac{\delta(\rho - \rho')}{\rho'} \delta(\phi - \phi')\delta(z - z')$$

Step 2: We use the completeness relations for $\delta(\phi - \phi')$ and $\delta(z - z')$ to write:

$$\Delta G(\mathbf{x}, \mathbf{x}') = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \left\{ -\frac{4}{L} \exp(-im\phi') \sin\left(\frac{n\pi z'}{L}\right) \frac{\delta(\rho - \rho')}{\rho'} \right\} \exp(im\phi) \sin\left(\frac{n\pi z}{L}\right) \tag{11}$$

Step 3: Expansion of the Green's function in $\exp(im\phi) \sin\left(\frac{n\pi z}{L}\right)$:

$$\Delta G(\mathbf{x}, \mathbf{x}') = \Delta \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \exp(im\phi) \sin\left(\frac{n\pi z}{L}\right) A_{mn}(\rho|\rho', z', \phi')$$

Step 4: Apply Laplacian:

$$\Delta G(\mathbf{x}, \mathbf{x}') = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \left\{ \left[\frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d}{d\rho} - \left(\frac{n\pi}{L}\right)^2 - \frac{m^2}{\rho^2} \right] A_{mn}(\rho|\rho', z', \phi') \right\} \exp(im\phi) \sin\left(\frac{n\pi z}{L}\right) \tag{12}$$

Step 5: We equate coefficients in Eqs. 11 and Eq. 12, and divide by $-\frac{4}{L} \exp(-im\phi') \sin\left(\frac{n\pi z'}{L}\right)$. We find

$$\left[\frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d}{d\rho} - \left(\frac{n\pi}{L}\right)^2 - \frac{m^2}{\rho^2} \right] g_{mn}(\rho, \rho') = \frac{\delta(\rho - \rho')}{\rho'}$$

for the reduced Green's function

$$g_{mn}(\rho, \rho') = -\frac{L\, A_{mn}(\rho|\rho', z', \phi')}{4 \exp(-im\phi') \sin\left(\frac{n\pi z'}{L}\right)}$$

Step 6: The differential equation for g_{mn} is the modified Bessel equation (except the inhomogeneity). Thus, a reduced Green's function that solves the homogeneous equation, is symmetric in ρ and ρ', is continuous at $\rho = \rho'$, is regular at $\rho = 0$ and is vanishing at $\rho = a$ must be of the form

$$g_{mn}(\rho, \rho') = C\, I_m(\frac{n\pi}{L}\rho_<) \left(I_m(\frac{n\pi}{L}\rho_>) - \frac{I_m(\frac{n\pi a}{L})}{K_m(\frac{n\pi a}{L})} K_m(\frac{n\pi}{L}\rho_>) \right)$$

where the constant C needs to be determined to match the inhomogeneity at $\rho = \rho'$. Also, $\rho_< = \min(\rho, \rho')$ and $\rho_> = \max(\rho, \rho')$.

Step 7: Find C. Integrating the differential equation for $g_{mn}(\rho, \rho')$ from $\rho' - \epsilon$ to $\rho' + \epsilon$ with $\epsilon \to 0$ and dropping vanishing terms we find

$$\frac{d}{d\rho} g_{mn}(\rho, \rho')|_{\rho=\rho'+\epsilon} - \frac{d}{d\rho} g_{mn}(\rho, \rho')|_{\rho=\rho'-\epsilon} = \frac{1}{\rho'}$$

Straightforward calculation yields

$$\begin{aligned} 1 &= -C\rho' \frac{n\pi}{L} \frac{I_m(\frac{n\pi a}{L})}{K_m(\frac{n\pi a}{L})} \left[I_m(x) K'_m(x) - I'_m(x) K_m(x) \right|_{x=\frac{n\pi}{L}\rho'} \\ C &= \frac{K_m(\frac{n\pi a}{L})}{I_m(\frac{n\pi a}{L})} \end{aligned}$$

where in the second line Eq. 3.147 has been used.

Step 8:

Going backward through all steps, the Greens function expansion is assembled into

$$G(\mathbf{x}, \mathbf{x}') = \frac{4}{L} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \left\{ \exp(im\phi) \exp(-im\phi') \sin\left(\frac{n\pi z}{L}\right) \sin\left(\frac{n\pi z'}{L}\right) \frac{I_m(\frac{n\pi \rho_<}{L})}{I_m(\frac{n\pi a}{L})} \left[I_m(\frac{n\pi a}{L}) K_m(\frac{n\pi \rho_>}{L}) - K_m(\frac{n\pi a}{L}) I_m(\frac{n\pi \rho_>}{L}) \right] \right\} \quad .$$

The potential of a point charge q at $\mathbf{x}'$ follows from $\Phi(\mathbf{x}, \mathbf{x}') = \frac{q}{4\pi\epsilon_0} G(\mathbf{x}, \mathbf{x}')$, leading to

$$\Phi(\mathbf{x}, \mathbf{x}') = \frac{q}{L\pi\epsilon_0} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \left\{ \exp(im\phi) \exp(-im\phi') \sin\left(\frac{n\pi z}{L}\right) \sin\left(\frac{n\pi z'}{L}\right) \frac{I_m(\frac{n\pi \rho_<}{L})}{I_m(\frac{n\pi a}{L})} \left[I_m(\frac{n\pi a}{L}) K_m(\frac{n\pi \rho_>}{L}) - K_m(\frac{n\pi a}{L}) I_m(\frac{n\pi \rho_>}{L}) \right] \right\} \quad \text{q.e.d.} \quad .$$

---

c):

The equation $(\Delta + \lambda)\Phi = 0$ for the given boundary conditions has the following complete set of **orthonormal** eigenfunctions (* see addendum below)

$$\Phi_{kmn} = \frac{1}{\sqrt{2\pi}} \exp(im\phi) \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi z}{L}\right) \frac{\sqrt{2}}{a J_{m+1}(x_{mn})} J_m\left(\frac{x_{mn}\rho}{a}\right) \quad .$$

Using the Bessel differential equation, it is seen that

$$\begin{aligned} \Delta\Phi_{kmn} &= \left(\frac{1}{\rho} \frac{\partial}{\partial\rho} \rho \frac{\partial}{\partial\rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} \right) \Phi_{kmn} \\ &= \left(\frac{1}{\rho} \frac{\partial}{\partial\rho} \rho \frac{\partial}{\partial\rho} - \left(\frac{k\pi}{L}\right)^2 - \frac{m^2}{\rho^2} \right) \Phi_{kmn} \end{aligned}$$

$$= \left(\frac{m^2}{\rho^2} - \frac{x_{mn}^2}{a^2} - \left(\frac{k\pi}{L}\right)^2 - \frac{m^2}{\rho^2} \right) \Phi_{kmn}$$
$$= \left(-\frac{x_{mn}^2}{a^2} - \left(\frac{k\pi}{L}\right)^2 \right) \Phi_{kmn} = -\lambda_{kmn}\Phi_{kmn} \quad , \quad .$$

The eigenvalue λ_{kmn} of the eigenfunction Φ_{kmn} is $\lambda_{kmn} = \frac{x_{mn}^2}{a^2} + \left(\frac{k\pi}{L}\right)^2$. Increasing the number of indices in Eq. 3.160 of the textbook and setting $\lambda = 0$, we find

$$G(\mathbf{x}, \mathbf{x}') = \frac{8}{La^2} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{\exp(im(\phi-\phi')) \sin\left(\frac{k\pi z}{L}\right) \sin\left(\frac{k\pi z'}{L}\right) J_m\left(\frac{x_{mn}\rho}{a}\right) J_m\left(\frac{x_{mn}\rho'}{a}\right)}{\left(\left(\frac{x_{mn}}{a}\right)^2 + \left(\frac{k\pi}{L}\right)^2\right) J_{m+1}^2(x_{mn})}$$

and for the potential $\Phi(\mathbf{x}, \mathbf{x}')$ of a point charge q at $\mathbf{x}'$ it follows from $\Phi(\mathbf{x}, \mathbf{x}') = \frac{q}{4\pi\epsilon_0} G(\mathbf{x}, \mathbf{x}')$ that

$$\Phi(\mathbf{x}, \mathbf{x}') = \frac{2q}{La^2\pi\epsilon_0} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{\exp(im(\phi-\phi')) \sin\left(\frac{k\pi z}{L}\right) \sin\left(\frac{k\pi z'}{L}\right) J_m\left(\frac{x_{mn}\rho}{a}\right) J_m\left(\frac{x_{mn}\rho'}{a}\right)}{\left(\left(\frac{x_{mn}}{a}\right)^2 + \left(\frac{k\pi}{L}\right)^2\right) J_{m+1}^2(x_{mn})} \quad \text{q.e.d.}$$

Discussion. By the uniqueness theorem, all solutions must be identical. Therefore, some hard-to-come-by sum rules can be extracted. The comparison of a) with c) yields a discrete Fourier series expansion of the reduced Green's function in z:

$$g_{mn}(z, z') = -\frac{a \sinh(\frac{x_{mn}}{a} z_<) \sinh(\frac{x_{mn}}{a}(L - z_>))}{x_{mn} \sinh(\frac{x_{mn}L}{a})}$$
$$= -\frac{2}{L} \sum_{k=1}^{\infty} \frac{\sin\left(\frac{k\pi z}{L}\right) \sin\left(\frac{k\pi z'}{L}\right)}{\left(\frac{x_{mn}}{a}\right)^2 + \left(\frac{k\pi}{L}\right)^2}$$
$$\text{for} \quad 0 < z, z' < L \quad \text{and} \quad m = 0, \pm 1, \pm 2, ... \quad \text{and} \quad n = 1, 2, 3, ...$$

The comparison of b) with c) yields a discrete Bessel series expansion of the radial reduced Green's function:

$$g_{mn}(\rho, \rho') = -\frac{I_m(\frac{n\pi\rho_<}{L})}{I_m(\frac{n\pi a}{L})} \left[I_m(\frac{n\pi a}{L}) K_m(\frac{n\pi\rho_>}{L}) - K_m(\frac{n\pi a}{L}) I_m(\frac{n\pi\rho_>}{L}) \right]$$
$$= -\frac{2}{a^2} \sum_{k=1}^{\infty} \frac{J_m\left(\frac{x_{mk}\rho}{a}\right) J_m\left(\frac{x_{mk}\rho'}{a}\right)}{\left(\left(\frac{x_{mk}}{a}\right)^2 + \left(\frac{n\pi}{L}\right)^2\right) J_{m+1}^2(x_{mk})}$$
$$\text{for} \quad 0 < \rho, \rho' < a \quad \text{and} \quad m = 0, \pm 1, \pm 2, ... \quad \text{and} \quad n = 1, 2, 3, ...$$

Addendum. If not known or guessed from the statement of the problem, the eigenfunctions Φ (and corresponding eigenvalues λ) of $\Delta\Phi + \lambda = 0$ for the given boundary conditions are obtained as follows. Writing $\Phi = R(\rho)Q(\phi)Z(z)$ it is

$$
\begin{aligned}
\frac{\partial^2}{\partial\rho^2}\Phi + \frac{1}{\rho}\frac{\partial}{\partial\rho}\Phi + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\Phi + \frac{\partial^2}{\partial z^2}\Phi + \lambda\Phi &= 0 \\
\frac{1}{R}R'' + \frac{1}{\rho R}R' + \frac{1}{\rho^2}\frac{Q''}{Q} + \frac{Z''}{Z} + \lambda &= 0 \quad ,
\end{aligned}
$$

where in the second line we have divided by RQZ and used $'$ for derivatives. To match the boundary conditions at $z = 0$ and L, for the z-dependence we choose

$$
Z(z) = \sin\left(\frac{k\pi}{L}z\right) \quad \text{with } k = 1, 2, ... \quad .
$$

Then, $Z''/Z = -\left(\frac{k\pi}{L}\right)^2$, and it follows

$$
\frac{\rho^2}{R}R'' + \frac{\rho}{R}R' + \frac{Q''}{Q} + \left[\lambda - \left(\frac{k\pi}{L}\right)^2\right]\rho^2 = 0 \quad .
$$

Considering the fact that the eigenfunction must be single-valued functions of ϕ, the solution for Q is

$$
Q(\phi) = \exp(im\phi) \quad \text{with m integer} \quad .
$$

Then, $Q''/Q = -m^2$, and the remaining equation for R,

$$
R'' + \frac{1}{\rho}R' + \left[\lambda - \left(\frac{k\pi}{L}\right)^2 - \frac{m^2}{\rho^2}\right]R = 0
$$

is the standard Bessel equation (Eq. 3.75 in textbook) with k replaced by $\sqrt{\lambda - \left(\frac{k\pi}{L}\right)^2}$. Since the eigenfunctions must be regular at $\rho = 0$ and zero at $\rho = a$, the radial dependence is $J_m\left(\rho\sqrt{\lambda - \left(\frac{k\pi}{L}\right)^2}\right)$ with an eigenvalue λ such that

$$
a\sqrt{\lambda - \left(\frac{k\pi}{L}\right)^2} = x_{mn} \quad .
$$

Thus, $\lambda_{kmn} = \frac{x_{mn}^2}{a^2} + \left(\frac{k\pi}{L}\right)^2$, and the corresponding eigenfunction is

$$
\Phi_{kmn} = constant \times \exp(im\phi)\,\sin\left(\frac{k\pi z}{L}\right)\,J_m\left(\frac{x_{mn}\rho}{a}\right) \quad .
$$

The normalization constant follow from $\int \Phi^*_{kmn}(\rho,\phi,z)\Phi_{kmn}(\rho,\phi,z)\,\rho d\rho d\phi dz = 1$ and Eq. 3.95 of the textbook.