Problem Set 4

Problem 2.13 **5 Points**

a): First, note that due to the the superposition principle the problem is equivalent to the sum of a constant potential $\frac{V_1+V_2}{2}$ and a problem of two half-cylinders on opposite potentials $\frac{V_1-V_2}{2}$ and $\frac{V_2-V_1}{2}$. Thus, we only need to consider the case of two half-cylinders on opposite potentials $V = \frac{V_1-V_2}{2}$ and $-V = \frac{V_2+V_1}{2}$. Assuming that $\phi = 0$ corresponds to the middle of the half-cylinder on potential V, dropping diverging terms, and considering the symmetry of the problem, for our case Eq. 2.71 reads

$$\Phi(\rho,\phi) = \sum_{n=1}^{\infty} a_n \rho^n \cos(n\phi) \tag{1}$$

To obtain the coefficients a_n, we write the potential on the surface $\rho = b$, multiply with $\cos(n'\phi)$ and integrate over ϕ:

$$\int_0^{2\pi} V(\phi)\cos(n'\phi)d\phi = \sum_{n=1}^{\infty} a_n b^n \int_0^{2\pi} \cos(n\phi)\cos(n'\phi)d\phi = \sum_{n=1}^{\infty} a_n b^n \pi \delta_{n,n'} = a_{n'} b^{n'} \pi \tag{2}$$

Thus,

$$\begin{aligned} a_n &= \frac{V}{b^n \pi n}\left(\int_{-\pi/2}^{\pi/2} \cos(n\phi)d\phi - \int_{\pi/2}^{(3\pi)/2} \cos(n\phi)d\phi\right) = \frac{4V}{b^n\pi n}\sin\left(\frac{n\pi}{2}\right) = \frac{4V}{b^n \pi n}\left\{ \begin{array}{ccc} 0 & , & \mathrm{n\,even} \\ (-1)^{(n+3)/2} & , & \mathrm{n\,odd} \end{array} \right. \\ a_n &= \left\{ \begin{array}{ccc} 0 & , & \mathrm{n\,even} \\ -\mathrm{i}\frac{4V}{b^n \pi n}\mathrm{i}^n & , & \mathrm{n\,odd} \end{array} \right. \end{aligned} \tag{3}$$

and:

$$\Phi(\rho,\phi) = -\mathrm{i}\frac{4V}{\pi}\sum_{n\,\mathrm{odd}}^{\infty}\frac{1}{n}\left(\frac{\rho}{a}\right)^n \mathrm{i}^n \cos(n\phi) = -\mathrm{i}\frac{4V}{\pi}\left(\mathrm{i}\,\mathrm{Im}\sum_{n\,\mathrm{odd}}^{\infty}\frac{1}{n}Z^n\right) \quad \text{with} \quad Z = \mathrm{i}r\exp(\mathrm{i}\phi) \quad \text{and} \quad r = \frac{\rho}{a} \tag{4}$$

Following the elaborations on p.74f of the textbook, it is $\sum_{n\,\mathrm{odd}}^{\infty}\frac{1}{n}Z^n = \frac{1}{2}\ln\frac{1+Z}{1-Z}$, and $\mathrm{Im}\left(\ln\frac{1+Z}{1-Z}\right)$ equals the phase of $\frac{1+Z}{1-Z}$. Since we find $\frac{1+Z}{1-Z} = \frac{1-r^2+\mathrm{i}2r\cos(\phi)}{1+r^2+2r\sin(\phi)}$ with $0 \le r \le 1$, the phase has a range $[-\pi/2, \pi/2]$ and is equal to $\tan^{-1}\left(\frac{2r\cos(\phi)}{1-r^2}\right) = \tan^{-1}\left(\frac{2\rho b\cos(\phi)}{b^2-\rho^2}\right)$. Thus,

$$\Phi(\rho,\phi) = -\mathrm{i}\frac{4V}{\pi}\left(\frac{\mathrm{i}}{2}\tan^{-1}\left(\frac{2\rho b\cos(\phi)}{b^2-\rho^2}\right)\right) = \frac{2V}{\pi}\tan^{-1}\left(\frac{2\rho b\cos(\phi)}{b^2-\rho^2}\right) \quad . \tag{5}$$

Using $V = \frac{V_1 - V_2}{2}$ and the superposition explained at the beginning, it is

$$\Phi(\rho,\phi) = \frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi}\tan^{-1}\left(\frac{2\rho b\cos(\phi)}{b^2-\rho^2}\right) \quad \text{, q.e.d.} \tag{6}$$

It is noted that the result has the correct limit for $\rho \to b$: Writing $\rho = b - \epsilon$ with $\epsilon > 0$, it is $\tan^{-1}\left(\frac{2\rho b\cos(\phi)}{b^2-\rho^2}\right) = \tan^{-1}\left(\frac{b\cos(\phi)}{\epsilon}\right) = \frac{\pi}{2} \times \mathrm{Sign}(\cos\phi)$, and

$$\Phi(\rho,\phi) = \left\{\begin{array}{lll} \frac{V_1+V_2}{2} + \frac{V_1-V_2}{2} = V_1 & \text{for} & -\frac{\pi}{2} < \phi < \frac{\pi}{2} \\ \frac{V_1+V_2}{2} - \frac{V_1-V_2}{2} = V_2 & \text{for} & \frac{\pi}{2} < \phi < \frac{3\pi}{2} \end{array}\right\} \text{and } \rho \to b \quad . \tag{7}$$

b): The charge density is $\sigma = \epsilon_0 E_\perp = -\epsilon_0 \frac{\partial\Phi}{\partial n} = +\epsilon_0 \frac{\partial\Phi}{\partial\rho}|_{\rho=b}$. Thus,

$$\begin{aligned} \sigma(\phi) &= \epsilon_0 \frac{V_1-V_2}{\pi} \frac{1}{1+\left(\frac{2\rho b\cos\phi}{b^2-\rho^2}\right)^2} \frac{(b^2-\rho^2)2b\cos\phi + 4\rho^2 b\cos\phi}{(b^2-\rho^2)^2}|_{\rho=b} \\ &= \epsilon_0 \frac{V_1-V_2}{\pi} \frac{2b\cos\phi(b^2+\rho^2)}{(b^2-\rho^2)^2 + 4\rho^2 b^2\cos^2\phi}|_{\rho=b} \\ &= \epsilon_0 \frac{V_1-V_2}{\pi} \frac{1}{b\cos\phi} \end{aligned} \tag{8}$$

Problem 2.23 **5 Points**

a): To simplify the calculation, we choose the origin such that the upper and lower walls are at $z = \pm a/2$, and the other four are at $x = 0$ or a and $y = 0$ or a. We transform into the frame of the problem statement after the calculation.

For the given boundary conditions, the potential is of the form

$$\Phi(x,y,z) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\frac{n\pi x}{a}) \sin(\frac{m\pi y}{a}) \cosh(\gamma_{nm} z) \quad \text{where } \gamma_{nm} = \frac{\pi}{a}\sqrt{n^2+m^2} \tag{9}$$

Thus, on the upper and lower surface it is

$$\begin{aligned} & V \int_0^a \int_0^a \sin(\frac{n'\pi x}{a}) \sin(\frac{m'\pi y}{a})\, dxdy = \\ = & \sum_{n,m=1}^{\infty} A_{nm} \cosh(\gamma_{nm}\frac{a}{2}) \int_0^a \int_0^a \sin(\frac{n\pi x}{a}) \sin(\frac{m\pi y}{a}) \sin(\frac{n'\pi x}{a}) \sin(\frac{m'\pi y}{a}) dxdy \end{aligned} \tag{10}$$

and

$$\begin{aligned} V\frac{a^2}{\pi^2 n' m'} \left[\cos\frac{n'\pi x}{a}\right]_0^a \left[\cos\frac{m'\pi y}{a}\right]_0^a &= A_{n'm'} \cosh(\gamma_{n'm'}\frac{a}{2})\frac{a^2}{4} \\ A_{nm} &= \begin{cases} \frac{16V}{\pi^2 nm \cosh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)} & , \text{ n and m both odd} \\ 0 & , \text{ otherwise} \end{cases} \end{aligned} \tag{11}$$

and, after transformation into the frame specified in the problem,

$$\Phi(x,y,z) = \sum_{n,m \text{ odd}}^{\infty} \frac{16V}{\pi^2 nm \cosh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)} \sin(\frac{n\pi x}{a}) \sin(\frac{m\pi y}{a}) \cosh\left(\frac{\pi}{a}\sqrt{n^2+m^2}(z-\frac{a}{2})\right) \tag{12}$$

b): At the center of the cube

$$\begin{aligned} \Phi(\frac{a}{2},\frac{a}{2},\frac{a}{2}) &= \sum_{n,m \text{ odd}}^{\infty} \frac{16V}{\pi^2 nm \cosh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)} \sin(\frac{n\pi}{2}) \sin(\frac{m\pi}{2}) \\ &= \sum_{i,j=0}^{\infty} \frac{16V(-1)^{i+j}}{\pi^2 (2i+1)(2j+1) \cosh\left(\frac{\pi}{2}\sqrt{(2i+1)^2+(2j+1)^2}\right)} \end{aligned} \tag{13}$$

To achieve an accuracy of three significant digits, it is:

(i,j)	$\frac{(-1)^{i+j}}{\pi^2(2i+1)(2j+1)\cosh\left(\frac{\pi}{2}\sqrt{(2i+1)^2+(2j+1)^2}\right)}$	Multiplicity	
(0,0)	+0.21438	$\times 1$	required
(0,1)	-0.00464	$\times 2$	required
(1,1)	0.00028	$\times 1$	required
(2,0)	0.00013	$\times 2$	required
(2,1)	-0.000007	$\times 2$	not required
(3,0)	-0.000004	$\times 2$	not required
...			not required

and $\Phi(\frac{a}{2},\frac{a}{2},\frac{a}{2}) = \frac{16V}{\pi^2} \times 0.20564 = 0.333371V$. According to Problem 2.28, the exact result must be $\Phi(\frac{a}{2},\frac{a}{2},\frac{a}{2}) = \frac{1}{3}V$. We see that only four terms of the expansion approach the exact result with a relative accuracy of order 10^{-4}.

c): On the upper face, it is $\sigma = \epsilon_0 E_\perp = -\epsilon_0 \frac{\partial \Phi}{\partial n} = +\epsilon_0 \frac{\partial \Phi}{\partial z}|_{z=a/2}$, *i.e.*

$$\sigma(x,y,z=\frac{a}{2}) = \frac{16\epsilon_0 V}{a\pi} \sum_{n,m\,\mathrm{odd}}^{\infty} \frac{1}{nm} \sin(\frac{n\pi x}{a}) \sin(\frac{m\pi y}{a}) \tanh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)\sqrt{n^2+m^2} \tag{14}$$

Problem 2.24 **5 Points**

The functions $\left\{U_m(\phi) = A_m \sin\left(\frac{m\pi\phi}{\beta}\right)\right\}$ with integer m and arbitrary non-zero constants A_m form a complete orthogonal set on the interval $0 \le \phi \le \beta$ with Dirichlet BC. Note the analogy of these functions with the complete set of eigenfunctions of a quantum particle in an infinitely deep square well.

Consider the expansion of a function $f(\phi)$ satisfying Dirichlet BC, $f(\phi) = \sum_{m=1}^{\infty} a_m U_m(\phi)$ with coefficients a_m, multiply both sides with $U_{m'}(\phi)$ and integrate over ϕ:

$$\begin{aligned}\int_0^\beta f(\phi) A_{m'} \sin\left(\frac{m'\pi\phi}{\beta}\right) d\phi &= \sum_{m=1}^{\infty} a_m A_m A_{m'} \int_0^\beta f(\phi) \sin\left(\frac{m\pi\phi}{\beta}\right) \sin\left(\frac{m'\pi\phi}{\beta}\right) d\phi = a_{m'} A_{m'}^2 \frac{\beta}{2} \\ a_m &= \frac{2}{\beta A_m} \int_0^\beta f(\phi') \sin\left(\frac{m\pi\phi'}{\beta}\right) d\phi' \qquad (15)\end{aligned}$$

where, for later convenience, we have switched the primes in the second line. Inserting this expression into the expansion of $f(\phi)$,

$$\begin{aligned}f(\phi) &= \sum_{m=1}^{\infty} \left\{ \frac{2}{\beta A_m} \int_0^\beta f(\phi') \sin\left(\frac{m\pi\phi'}{\beta}\right) d\phi' \right\} A_m \sin\left(\frac{m\pi\phi}{\beta}\right) \\ &= \int_0^\beta \left\{ \sum_{m=1}^{\infty} \frac{2}{\beta} \sin\left(\frac{m\pi\phi'}{\beta}\right) \sin\left(\frac{m\pi\phi}{\beta}\right) \right\} f(\phi')\, d\phi' \qquad \forall\phi\end{aligned}$$

Thus,

$$\sum_{m=1}^{\infty} \frac{2}{\beta} \sin\left(\frac{m\pi\phi'}{\beta}\right) \sin\left(\frac{m\pi\phi}{\beta}\right) = \delta(\phi - \phi') \quad \text{, q.e.d.} \qquad (16)$$

Problem 3.2 **5 Points**

In Problems 3.2 and 3.3, use of Eq. 3.70 of the textbook is recommended.

It is $\rho(\mathbf{x}') = \frac{Q}{4\pi R^2}\delta(r'-R)$. Using Eq. 3.70 of the textbook, with $r_< = r$ and $r_> = r'$ for the interior region, it is

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{x}')\frac{1}{|\mathbf{x}-\mathbf{x}'|}d^3x' \\ \Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\sum_{l,m}\frac{4\pi}{2l+1}\frac{Q}{4\pi R^2}\int \delta(r'-R)\frac{r^l}{r'^{l+1}}Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi')r'^2 dr' d\cos\theta' d\phi' \end{aligned} \tag{17}$$

Sine the charge density does not depend on ϕ, only terms with $m=0$ occur. Using $Y_{l0}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)$ and Eq. 3.28, which integrates to $\int P_l(x)dx = \frac{1}{2l+1}\left(P_{l+1}(x) - P_{l-1}(x)\right)$,

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{Q}{2}\frac{r^l}{R^{l+1}}P_l(\cos\theta)\int_{\cos\theta=-1}^{\cos\alpha} P_l(\cos\theta')d\cos\theta' \\ \Phi(\mathbf{x}) &= \frac{Q}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{R^{l+1}}P_l(\cos\theta)\frac{1}{2l+1}\left[P_{l+1}(x) - P_{l-1}(x)\right]_{x=-1}^{\cos\alpha}\end{aligned} \tag{18}$$

Since $P_{l+1}(-1) = (-1)^{l+1} = (-1)^{l-1} = P_{l-1}(-1)$,

$$\Phi_{\text{interior}}(\mathbf{x}) = \frac{Q}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{R^{l+1}}P_l(\cos\theta)\frac{1}{2l+1}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right] \quad \text{, q.e.d.} \tag{19}$$

Explicit integration of the dubious case $l=0$ shows that our result is correct if we define $P_{-1}(\cos\alpha) = -1$. In the exterior region, choose $r_< = r'$ and $r_> = r$ to find

$$\Phi_{\text{exterior}}(\mathbf{x}) = \frac{Q}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{R^l}{r^{l+1}}P_l(\cos\theta)\frac{1}{2l+1}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right] \quad . \tag{20}$$

b): The interior radial field $E_r = -\frac{\partial}{\partial r}\Phi$ is

$$E_r = \frac{-Q}{8\pi\epsilon_0}\sum_{l=1}^{\infty}\frac{r^{l-1}}{R^{l+1}}P_l(\cos\theta)\frac{l}{2l+1}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right] \quad . \tag{21}$$

By symmetry, the field near the origin must point in z-direction. Therefore, the field at the origin is

$$\begin{aligned}\mathbf{E}(0) &= \hat{\mathbf{z}}\lim_{r\to 0} E_r(r,\theta=0) \\ &= \hat{\mathbf{z}}\frac{-Q}{8\pi\epsilon_0}\sum_{l=1}^{\infty}\frac{r^{l-1}}{R^{l+1}}\frac{l}{2l+1}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right]|_{r\to 0}\end{aligned}$$

$$
\begin{aligned}
&= \hat{\mathbf{z}}\frac{-Q}{8\pi\epsilon_0}\frac{1}{R^2}\frac{1}{3}\left[P_2(\cos\alpha) - P_0(\cos\alpha)\right] \\
&= \hat{\mathbf{z}}\frac{-Q}{8\pi\epsilon_0}\frac{1}{R^2}\frac{1}{3}\left[\frac{1}{2}(3\cos^2\alpha - 1) - 1\right] && (22) \\
&= \hat{\mathbf{z}}\frac{Q}{16\pi\epsilon_0 R^2}\sin^2\alpha && (23)
\end{aligned}
$$

c): Limit of a small cap. For $\alpha \to 0$ it is $\cos\alpha = 1 - \frac{1}{2}\alpha^2$ and, in the case $l \geq 1$,

$$
\begin{aligned}
P_{l+1}(1 - \frac{1}{2}\alpha^2) - P_{l-1}(1 - \frac{1}{2}\alpha^2) &\approx P_{l+1}(1) - \frac{1}{2}\alpha^2 P'_{l+1}(1) - P_{l-1}(1) + \frac{1}{2}\alpha^2 P'_{l-1}(1) \\
= -\frac{1}{2}\alpha^2(P'_{l+1}(1) - P'_{l-1}(1)) &= -\frac{1}{2}\alpha^2(2l+1)P_l(1) \\
&= -\frac{2l+1}{2}\alpha^2 \quad , && (24)
\end{aligned}
$$

where in the last step we have used Eq. 3.28 of the textbook, $(2l+1)P_l = P'_{l+1} - P'_{l-1}$.

In the case $l = 0$ it is $P_{l+1}(1 - \frac{1}{2}\alpha^2) + 1 = 2 - \frac{1}{2}\alpha^2 = 2 - \frac{2l+1}{2}\alpha^2$. Thus, using $r_> = \max(R, r)$ and $r_> = \min(R, r)$, the potential in the interior and the exterior region is

$$
\begin{aligned}
\Phi(\mathbf{x}) &= \frac{Q}{8\pi\epsilon_0}\sum_l \frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)\frac{1}{2l+1}\left(-\frac{2l+1}{2}\alpha^2\right) + \frac{Q}{8\pi\epsilon_0}\frac{1}{r_>}\cdot 2 \\
&= \frac{Q}{4\pi\epsilon_0 r_>} - \frac{Q}{4\pi\epsilon_0}\frac{\alpha^2\pi}{4\pi}\sum_l \frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta) \\
&= \frac{Q}{4\pi\epsilon_0 r_>} - \frac{Q}{4\pi\epsilon_0}\frac{\alpha^2\pi}{4\pi}\frac{1}{|\mathbf{x} - \hat{\mathbf{z}}R|} \quad . && (25)
\end{aligned}
$$

This is the potential of a homogeneously charged sphere (first term) plus the potential of a charge q located at the north pole of the sphere (second term). The charge q equals $-Q$ times the ratio $\frac{\alpha^2\pi}{4\pi}$ between the solid angle of the cap and the solid angle of a full sphere. This behavior in the limit of a small cap is to be expected.

Limit of a large cap. In this case, $\alpha = \pi - \epsilon$ with $\epsilon \to 0$. Then, $\cos\alpha \approx \frac{\epsilon^2}{2} - 1$, and for $l \geq 1$ it is

$$
\begin{aligned}
P_{l+1}(\frac{1}{2}\epsilon^2 - 1) - P_{l-1}(\frac{1}{2}\epsilon^2 - 1) &= (-1)^{l+1}\left[P_{l+1}(1 - \frac{1}{2}\epsilon^2) - P_{l-1}(1 - \frac{1}{2}\epsilon^2)\right] \\
&\approx (-1)^{l+1}\left(-\frac{\epsilon^2}{2}\right)\left(P'_{l+1}(1) - P'_{l-1}(1)\right) \\
&= (-1)^l\,\frac{2l+1}{2}\,P_l(1)\epsilon^2 = (-1)^l\,\frac{2l+1}{2}\,\epsilon^2 \quad , && (26)
\end{aligned}
$$

where Eq. 3.28 of the textbook has been used. In the special case $l = 0$, it also is $P_{l+1}(\frac{1}{2}\epsilon^2 - 1) - P_{l-1}(\frac{1}{2}\epsilon^2 - 1) = P_1(\frac{1}{2}\epsilon^2 - 1) + 1 = \frac{1}{2}\epsilon^2 = (-1)^l\,\frac{2l+1}{2}\epsilon^2$. Thus, the potential in the interior and the exterior region is

$$
\begin{aligned}
\Phi(\mathbf{x}) &= \frac{Q}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)\,(-1)^l\,\frac{2l+1}{2}\,\epsilon^2 \\
&= \frac{Q}{4\pi\epsilon_0}\,\frac{\epsilon^2\pi}{4\pi}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\pi-\cos\theta) \\
&= \frac{Q}{4\pi\epsilon_0}\,\frac{\epsilon^2\pi}{4\pi}\,\frac{1}{|\mathbf{x}+\hat{\mathbf{z}}R|} \quad .
\end{aligned} \tag{27}
$$

This is the potential of a charge q located at the south pole of the sphere. The charge q equals Q times the ratio $\frac{\epsilon^2\pi}{4\pi}$ between the solid angle of the charged area and the solid angle of a full sphere. This behavior in the limit of a large cap is to be expected.

In both cases of small and large cap, the **electric field at the origin** is $\mathbf{E}(0) = \hat{\mathbf{z}}\frac{Q\alpha^2}{16\pi\epsilon_0 R^2}$ and $\mathbf{E}(0) = \hat{\mathbf{z}}\frac{Q\epsilon^2}{16\pi\epsilon_0 R^2}$, respectively. This result is consistent with the interpretations of the potentials.

Problem 3.3 **5 Points**

Hint: A closed expression exists for $\int_0^R \frac{r^{2n+1}}{\sqrt{R^2-r^2}}\,dr$ (you can find it with Mathematica, for instance).

a): Exterior potential ($r > R$). The surface charge density is $\sigma(r) = \frac{\lambda}{\sqrt{R^2-r^2}}$ for $r < R$ and zero otherwise. The volume charge density must be of the form $\rho(\mathbf{x}) = f(r)\delta(\cos\theta)$. The function $f(r)$ is determined by considering the charge in a shell of radius r and thickness dr:

$$dq = \sigma(r)2\pi r dr = \int_{\phi=0}^{2\pi} d\phi \int_{\cos\theta=-1}^{1} d\cos\theta r^2 dr f(r)\delta(\cos\theta) = 2\pi r^2 f(r) dr \tag{28}$$

and thus $\rho(r) = \frac{\sigma(r)}{r}\,\delta(\cos\theta)$. To find λ, we calculate the potential at the origin and equate the result to V:

$$V = \Phi(0) = \frac{1}{4\pi\epsilon_0}\int_0^R \frac{\lambda}{r\sqrt{R^2-r^2}} 2\pi r dr = \frac{\lambda}{2\epsilon_0}\left[\sin^{-1}\left(\frac{r}{R}\right)\right]_0^R = \frac{\lambda\pi}{4\epsilon_0} \tag{29}$$

Thus, $\lambda = \frac{4\epsilon_0 V}{\pi}$, and $\boxed{\rho(r) = \frac{4\epsilon_0 V}{\pi r\sqrt{R^2-r^2}}\,\delta(\cos\theta)}$. Using Eq. 3.70 of the textbook for the case $r > R$ we have $r_> = r$ and $r_< = r'$, and we find

$$\begin{aligned}
\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\frac{4\epsilon_0 V}{\pi}\sum_{l,m}\frac{4\pi}{2l+1}Y_{lm}(\theta,\phi)\frac{1}{r^{l+1}}\int\int\int r'^l Y^*_{lm}(\theta',\phi')\frac{1}{r'\sqrt{R^2-r'^2}}\,\delta(\cos\theta')r'^2 d\cos\theta' d\phi' dr' \\
&= \frac{V}{\pi^2}\sum_l P_l(\cos\theta)\frac{2\pi}{r^{l+1}}\int\int r'^{l+1}P_l(\cos\theta')\frac{1}{\sqrt{R^2-r'^2}}\,\delta(\cos\theta')d\cos\theta' dr' \\
&= \frac{2V}{\pi}\sum_l \frac{1}{r^{l+1}}P_l(\cos\theta)P_l(0)\int_0^R \frac{r'^{l+1}}{\sqrt{R^2-r'^2}}\,dr' \\
&= \frac{2V}{\pi}\sum_{n=0}^{\infty}\frac{1}{r^{2n+1}}P_{2n}(\cos\theta)\frac{(-1)^n(2n-1)!!}{2^n n!}\int_0^R \frac{r'^{l+1}}{\sqrt{R^2-r'^2}}\,dr' \quad .
\end{aligned}$$

Using integral tables or software, it is found that $\int_0^R \frac{r^{l+1}}{\sqrt{R^2-r^2}}\,dr = R^{2n+1}\frac{n!2^n}{(2n+1)!!}$, and thus

$$\Phi_{\mathrm{r>R}}(\mathbf{x}) = \frac{2V}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{R}{r}\right)^{2n+1} P_{2n}(\cos\theta) \quad \text{, q.e.d.} \tag{30}$$

b): Interior potential ($r < R$). On the surface $r = R$ the expansions for $r > R$ and $r < R$ must agree, *i.e.* the respective coefficient functions $B_l r^{-l-1}$ and $A_l r^l$ of the $P_l(\cos\theta)$ must be equal for $r = R$ and for all l. Thus, the interior coefficients $A_l = B_l R^{-2l-1}$. Here, $l = 2n$ and $A_{2n} = B_{2n}R^{-4n-1} = R^{2n+1}R^{-4n-1} = R^{-2n}$. Thus,

$$\Phi_{\mathrm{r<R}}(\mathbf{x}) = \frac{2V}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{r}{R}\right)^{2n} P_{2n}(\cos\theta) \quad . \tag{31}$$

With $r_< = \min(r, R)$ and $r_> = \max r, R$ the potential in all space can be written as

$$\Phi_{\text{everywhere}}(\mathbf{x}) = \frac{2VR}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \frac{r_<^{2n}}{r_>^{2n+1}} P_{2n}(\cos\theta) \quad . \tag{32}$$

c): The capacitance is $C = \frac{Q}{V}$. From part a), we know that $V = \frac{\lambda\pi}{4\epsilon_0}$. The total charge Q on the disk is obtained as

$$Q = \int_0^R \frac{\lambda}{\sqrt{R^2 - r^2}} 2\pi r dr = 2\pi\lambda \left[-\sqrt{R^2 - r^2} \right]_0^R = 2\pi\lambda R \quad . \tag{33}$$

Thus, $C = \frac{Q}{V} = 2\pi\lambda R \frac{4\epsilon_0}{\lambda\pi}$, and $\boxed{C = 8R\epsilon_0}$.