Problem Set 3

Problem 2.7 **5 Points**

a): Green's function:

Using cartesian coordinates $\mathbf{x} = (x, y, z)$, it is

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} - \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}} \quad . \tag{1}$$

Note the required symmetry in $\mathbf{x}$ and $\mathbf{x}'$. A coordinate-free form is

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} - \frac{1}{|\mathbf{x} - \mathbf{x}' + 2\hat{\mathbf{z}}(\hat{\mathbf{z}} \cdot \mathbf{x}')|} \quad . \tag{2}$$

b): It is

$$\Phi(\mathbf{x}) = \frac{1}{4\pi} \int_{x'=-\infty}^{x'=\infty} \int_{y'=-\infty}^{y'=\infty} \frac{\partial}{\partial z'} G(\mathbf{x}, \mathbf{x}')|_{z=0} V(\mathbf{x}') dx'\, dy' \quad . \tag{3}$$

Using $\frac{\partial}{\partial z'} G(\mathbf{x}, \mathbf{x}')|_{z'=0} = \frac{2z}{\sqrt{(x-x')^2 + (y-y')^2 + z^2}^3}$ and re-writing into cylindrical coordinates,

$$x = \rho\cos\phi \quad x' = \rho'\cos\phi' \quad y = \rho\sin\phi \quad y' = \rho'\sin\phi' \tag{4}$$

yields

$$\begin{aligned} \frac{\partial}{\partial z'} G(\mathbf{x}, \mathbf{x}')|_{z=0} &= \frac{2z}{\sqrt{z^2 + \rho^2 + \rho'^2 - 2\rho\rho'(\cos\phi\cos\phi' + \sin\phi\sin\phi')}^3} \\ &= \frac{2z}{\sqrt{z^2 + \rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi - \phi')}^3} \quad . \end{aligned} \tag{5}$$

The potential thus is

$$\Phi(\rho, \phi, z) = \frac{Vz}{2\pi} \int_{\phi'=0}^{2\pi} \int_{\rho'=0}^{a} \frac{\rho' d\rho' d\phi'}{\sqrt{z^2 + \rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi - \phi')}^3} \quad . \tag{6}$$

c): On the z-axis it is $\rho = 0$, and

$$\Phi(0,*,z) = Vz\int_{\rho'=0}^{a} \frac{\rho' d\rho' d\phi'}{\sqrt{z^2+\rho'^2}^3} = -Vz\left[\frac{1}{\sqrt{z^2+\rho'^2}}\right]_0^a = V(1-\frac{z}{\sqrt{z^2+a^2}}) \quad . \tag{7}$$

d): We write

$$\Phi(\rho,\phi,z) = \frac{Vz}{2\pi(\rho^2+z^2)^{3/2}}\int_{\phi'=0}^{2\pi}\int_{\rho'=0}^{a}\left(1+\frac{\rho'^2-2\rho\rho'\cos(\phi-\phi')}{\rho^2+z^2}\right)^{-3/2}\rho' d\rho' d\phi' \quad . \tag{8}$$

and expand

$$(1+\epsilon)^{-3/2} \quad = \quad \underbrace{1}_{A} \quad + \quad \underbrace{-\frac{3}{2}\epsilon}_{B} \quad + \quad \underbrace{\frac{15}{8}\epsilon^2}_{C} \quad -... \qquad , \tag{9}$$

where $\epsilon = \frac{\rho'^2-2\rho\rho'\cos(\phi-\phi')}{\rho^2+z^2} \ll 1$. Integration term by term yields

$$A = \int_{\phi'=0}^{2\pi}\int_{\rho'=0}^{a}\rho' d\rho' d\phi' = \pi a^2 \tag{10}$$

$$B = -\frac{3}{2}\int_{\phi'=0}^{2\pi}\int_{\rho'=0}^{a}\frac{\rho'^2-2\rho\rho'\cos(\phi-\phi')}{\rho^2+z^2}\rho' d\rho' d\phi' = -\frac{3\pi a^4}{4(\rho^2+z^2)} \quad , \tag{11}$$

where the cos-term integrates to zero.

$$C = \frac{15}{8}\int_{\phi'=0}^{2\pi}\int_{\rho'=0}^{a}\frac{\rho'^4-4\rho\rho'^3\cos(\phi-\phi')+4\rho^2\rho'^2\cos^2(\phi-\phi')}{(\rho^2+z^2)^2}\rho' d\rho' d\phi' = \frac{5\pi a^6}{8(\rho^2+z^2)^2}+\frac{15\pi\rho^2 a^4}{8(\rho^2+z^2)^2} \quad , \tag{12}$$

where we have used that $\int_{\phi'=0}^{2\pi}\cos(\phi-\phi')d\phi' = 0$ and $\int_{\phi'=0}^{2\pi}\cos^2(\phi-\phi')d\phi' = \pi$. Collecting the terms into Eq. 8, it is found

$$\Phi(\rho,z) = \frac{Va^2z}{2(\rho^2+z^2)^{3/2}}\left(1-\frac{3a^2}{4(\rho^2+z^2)}+\frac{5a^4}{8(\rho^2+z^2)^2}+\frac{15a^2\rho^2}{8(\rho^2+z^2)^2}+...\right) \quad \text{, q.e.d.} \tag{13}$$

On-axis, the expression reduces to

$$\Phi(\rho=0,z) = \frac{Va^2}{2z^2}\left(1-\frac{3a^2}{4z^2}+\frac{5a^4}{8z^4}...\right) \quad , \tag{14}$$

while the expansion of the result of part c) for large z is

$$\Phi(0,z) = V(1-\frac{z}{\sqrt{z^2+a^2}}) = V\left(1-\left[1+\left(\frac{a}{z}\right)^2\right]^{-1/2}\right) \approx \frac{Va^2}{2z^2}\left(1-\frac{3a^2}{4z^2}+\frac{5a^4}{8z^4}\right) \quad . \tag{15}$$

The results in Eq. 14 and Eq. 15 agree, as expected.

Problem 2.8 **5 Points**

a): It is to be shown that the equipotential surfaces of two parallel line charges of equal magnitude and opposite polarity are cylinders. Using the variables identified in the figure, we claim that for any potential V there exists a cylinder with radius r and axis location identified by x such that for all values of γ the potential on the cylinder is V. We prove the claim by finding a unique solution for x and r.

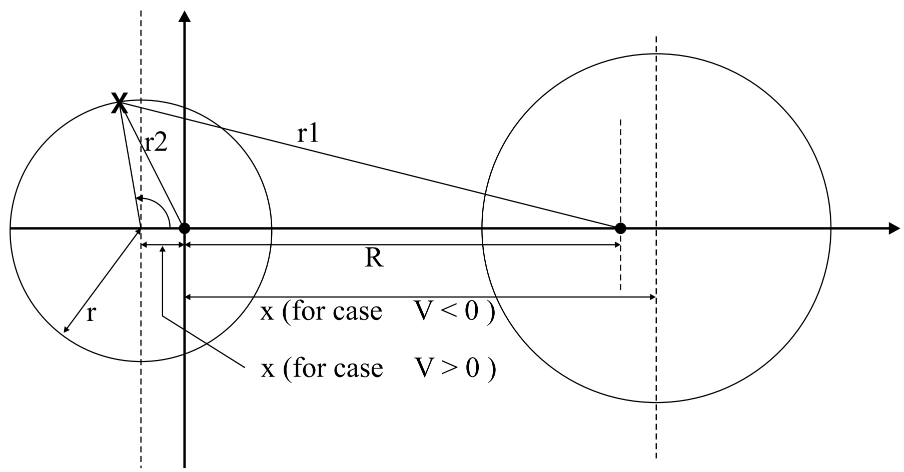

Figure 1: Equipotential surfaces of two parallel line charges of equal magnitude and opposite polarity.

First we note that for symmetry the axis of the cylinder can only be located on a straight line through the two line charges. Since the potential at a distance ρ from a line charge λ is $-\frac{\lambda}{2\pi\epsilon_0}\ln(\rho)$, at the location identified in the figure by γ and r the potential is $\Phi = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{r_2}{r_1}$, and

$$\frac{r_2}{r_1} = \exp\left(\frac{2\pi\epsilon_0 V}{\lambda}\right) =: \alpha \quad , \tag{16}$$

and using the law of cosines it is

$$\alpha^2 = \frac{(R+x)^2 + r^2 - 2(R+x)r\cos\gamma}{x^2 + r^2 - 2xr\cos\gamma} \quad . \tag{17}$$

For that equation to hold for all γ, the ratio of the coefficients of $\cos\gamma$ and the ratio of the terms without $\cos\gamma$ must both be equal, and

$$\begin{aligned} \alpha^2 &= \frac{(R+x)^2 + r^2}{x^2 + r^2} \Leftrightarrow r^2 = \frac{\alpha^2 x^2 - (x-R)^2}{1-\alpha^2} \\ \alpha^2 &= \frac{2(R+x)r}{2xr} \quad . \end{aligned} \tag{18}$$

The second equation yields $\boxed{x = \frac{R}{\alpha^2-1}}$, which, when inserted into the first equation, yields $\boxed{r = \left|\frac{\alpha R}{\alpha^2-1}\right|}$ with $\boxed{\alpha = \exp\left(\frac{2\pi\epsilon_0 V}{\lambda}\right)}$.

It is noted that for $V > 0$ the value of x is positive. For $V \to \infty$ it is $r \to 0$ and $x \to 0$.

For $V < 0$ the value of x is negative and $< -R$. For $V \to -\infty$ it is $r \to 0$ and $x \to -R$. The figure shows both a case of negative and positive V. These findings are important in the next part of the problem.

b): To determine the capacitance, we need to place two line charges with opposite polarities such that two cylinders with the specified radii a and b and center-to-center separation d are equipotential surfaces of the system. The voltage difference between these equipotential surfaces will then allow us to calculate the capacitance. First we note that for two cylinders outside of each other, *i.e.* $d > a + b$, we are seeking a solution of the type shown in the figure: one circle with positive potential V_+ around the line charge $\lambda > 0$, and one one circle with negative potential V_- around the line charge $-\lambda$. From the given answer we suspect that it will be useful to calculate $(d^2 - a^2 - b^2)/(2ab)$. Defining

$$\alpha_+ = \exp\left(\frac{2\pi\epsilon_0 V_+}{\lambda}\right) \quad \text{and} \quad \alpha_- = \exp\left(\frac{2\pi\epsilon_0 V_-}{\lambda}\right) \tag{19}$$

and using

$$\begin{aligned} a &= \frac{\alpha_+ R}{\alpha_+^2 - 1} > 0 \\ b &= -\frac{\alpha_- R}{\alpha_-^2 - 1} > 0 \\ d &= x_+ - x_- = \frac{\alpha_+ R}{\alpha_+^2 - 1} - \frac{\alpha_- R}{\alpha_-^2 - 1} \end{aligned} \tag{20}$$

it is found that

$$\begin{aligned} d^2 - a^2 - b^2 &= -R^2 \frac{\alpha_+^2 + \alpha_-^2}{(\alpha_+^2 - 1)(\alpha_-^2 - 1)} > 0 \\ 2ab &= -2R^2 \frac{\alpha_+ \alpha_-}{(\alpha_+^2 - 1)(\alpha_-^2 - 1)} > 0 \\ \frac{d^2 - a^2 - b^2}{2ab} &= \frac{\alpha_+^2 + \alpha_-^2}{2\,\alpha_+ \alpha_-} = \frac{1}{2}\left(\frac{\alpha_+}{\alpha_-} + \frac{\alpha_-}{\alpha_+}\right) = \cosh\left(\frac{2\pi\epsilon_0 (V_+ - V_-)}{\lambda}\right) \end{aligned} \tag{21}$$

Thus,

$$\cosh^{-1}\left(\frac{d^2 - a^2 - b^2}{2ab}\right) = (V_+ - V_-)\frac{2\pi\epsilon_0}{\lambda} = (V_+ - V_-)\frac{2\pi\epsilon_0 L}{Q} = \frac{2\pi\epsilon_0 L}{C} \quad , \tag{22}$$

and the capacitance per length is

$$\frac{C}{L} = \frac{2\pi\epsilon_0}{\cosh^{-1}\left(\frac{d^2 - a^2 - b^2}{2ab}\right)} \quad , \quad \text{q.e.d.} \tag{23}$$

c): For $d^2 \gg a^2 + b^2$, it is $\frac{d^2 - a^2 - b^2}{2ab} \gg 1$. Since $\cosh^{-1} y \approx \ln(2y)$ for $y \gg 1$, it then is

$$\cosh^{-1}\left(\frac{d^2-a^2-b^2}{2ab}\right) \approx \ln\left(\frac{d^2}{ab}\left[1-\frac{a^2+b^2}{d^2}\right]\right) = 2\ln\left(\frac{d}{\sqrt{ab}}\right) + \ln\left(1-\frac{a^2+b^2}{d^2}\right) \approx 2\ln\left(\frac{d}{\sqrt{ab}}\right) - \frac{a^2+b^2}{d^2} \tag{24}$$

and

$$\begin{aligned} \frac{C}{L} &= \frac{2\pi\epsilon_0}{\cosh^{-1}\left(\frac{d^2-a^2-b^2}{2ab}\right)} \approx \frac{2\pi\epsilon_0}{2\ln\left(\frac{d}{\sqrt{ab}}\right) - \frac{a^2+b^2}{d^2}} \\ &\approx \frac{\pi\epsilon_0}{\ln\left(\frac{d}{\sqrt{ab}}\right)} + \frac{\pi\epsilon_0}{2\left(\ln\left(\frac{d}{\sqrt{ab}}\right)\right)^2} \left(\frac{a^2+b^2}{d^2}\right) \end{aligned} \tag{25}$$

The result exhibits the correct behavior for $\frac{a^2+b^2}{d^2} \to 0$, and the lowest-order correction in $\frac{a^2+b^2}{d^2}$.

d): For cylinders inside each other, choose voltages V_+ and V_- of same polarity. Without loss of generality, we can choose them both positive, and repeat the calculation of b):

$$\begin{aligned} a &= \frac{\alpha_+ R}{\alpha_+^2 - 1} > 0 \\ b &= \frac{\alpha_- R}{\alpha_-^2 - 1} > 0 \\ d &= x_+ - x_- = \frac{\alpha_+ R}{\alpha_+^2 - 1} - \frac{\alpha_- R}{\alpha_-^2 - 1} \end{aligned} \tag{26}$$

It is found that

$$\begin{aligned} a^2 + b^2 - d^2 &= R^2 \frac{\alpha_+^2 + \alpha_-^2}{(\alpha_+^2 - 1)(\alpha_-^2 - 1)} > 0 \\ 2ab &= 2R^2 \frac{\alpha_+ \alpha_-}{(\alpha_+^2 - 1)(\alpha_-^2 - 1)} > 0 \\ \frac{a^2 + b^2 - d^2}{2ab} &= \frac{\alpha_+^2 + \alpha_-^2}{2\,\alpha_+ \alpha_-} = \frac{1}{2}\left(\frac{\alpha_+}{\alpha_-} + \frac{\alpha_-}{\alpha_+}\right) = \cosh\left(\frac{2\pi\epsilon_0 (V_+ - V_-)}{\lambda}\right) \end{aligned} \tag{27}$$

and the capacitance per length becomes

$$\frac{C}{L} = \frac{2\pi\epsilon_0}{\cosh^{-1}\left(\frac{a^2+b^2-d^2}{2ab}\right)} \quad . \tag{28}$$

For $d = 0$, it is $\cosh^{-1}\left(\frac{a^2+b^2}{2ab}\right) = \ln\left(\frac{a}{b}\right)$. You can show this by application of cosh on both sides and evaluation of the expression on the right. Thus, for $d = 0$ it is, as expected,

$$\frac{C}{L} = \frac{2\pi\epsilon_0}{\ln\left(\frac{a}{b}\right)} \quad . \tag{29}$$

Problem 2.9 **5 Points**

a): According to Eq. 2.15 in the textbook, it is $\sigma = 3\epsilon_0 E_0 \cos\theta$. The electrostatic pressure is $\mathbf{P} = \frac{\sigma^2}{2\epsilon_0}\hat{\mathbf{r}}$. Due to symmetry, only the z-components of the resultant force will integrate to $\neq 0$. Integration over one hemisphere yields

$$F_0 = \int dF_z = \int_{\phi=0}^{2\pi}\int_{\cos\theta=0}^{1} \frac{9}{2}\epsilon_0 E_0^2 a^2\ \cos^3\theta\, d\cos\theta d\phi = \epsilon_0\pi\left(\frac{3E_0 a}{2}\right)^2 \quad . \tag{30}$$

The force is repulsive.

b): The additional charge spreads evenly over the full sphere, yielding an additional $\sigma_0 = \frac{Q}{4a^2\pi}$. An additional repulsive force due to additional electrostatic pressure is

$$F_1 = \int_{\phi=0}^{2\pi}\int_{\cos\theta=0}^{1} \frac{\sigma_0^2}{2\epsilon_0}a^2\ \cos\theta\, d\cos\theta d\phi = \frac{\pi\sigma_0^2}{2\epsilon_0} = \frac{Q^2}{32a^2\epsilon_0\pi} \quad . \tag{31}$$

Also, there is an additional force $F_2 = E_0 Q/2$ on half the net charge located on each hemisphere. Noting that F_2 points in the same direction for both hemispheres, while F_0 and F_1 point in opposite directions, the **force required to hold the spheres together** is just $F_0 + F_1$, *i.e.*

$$F = \epsilon_0\pi\left(\frac{3E_0 a}{2}\right)^2 + \frac{Q^2}{32a^2\epsilon_0\pi} \quad . \tag{32}$$

Note that this force does not depend on the sign of Q. The **net force acting on the whole assembly of both hemispheres** is $2F_2 = E_0 Q$, as expected, and depends on the sign of Q.

Problem 2.10 **5 Points**

a): One may expect Eq. 2.14 of the textbook, $\Phi = -E_0(r - \frac{a^3}{r^2})\cos\theta$ to be the solution. This suspicion turns into certainty by reiterating that $\Delta\Phi = 0$ in the volume of interest, and by verification of the boundary conditions. The latter are that on the boss surface $r = a$ and on the plane $\theta = \pi/2$ the potential must be a constant, and that for $r \to \infty$ it must be $\mathbf{E} = -\nabla\Phi = E_0\hat{\mathbf{z}}$. The potential $\Phi = -E_0(r - \frac{a^3}{r^2})\cos\theta$ accords with both boundary conditions.

It is $\sigma = \epsilon_0 \frac{\partial}{\partial n}\Phi$, with $\hat{n}$ being the normal vector from the volume of interest into the conductor. Thus, on the boss it is

$$\sigma = -\epsilon_0 \frac{\partial}{\partial r}\Phi|_{r=a} = \epsilon_0 E_0 (1 + 2\frac{a^3}{r^3})\cos\theta = 3\epsilon_0 E_0 \cos\theta \quad , \tag{33}$$

and on the surface

$$\sigma = -\epsilon_0 \frac{\partial}{\partial z}\Phi|_{\theta=\pi/2} = +\epsilon_0 \frac{\partial}{r\partial\theta}\Phi|_{\theta=\pi/2} = \epsilon_0 E_0 (1 - \frac{a^3}{r^3}) \quad . \tag{34}$$

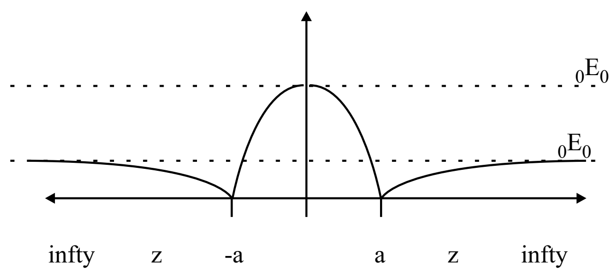

Figure 2: Sketch of σ.

b): The charge on the boss is $2\pi a^2 \int_0^1 3\epsilon_0 E_0 \cos\theta \, d\cos\theta = 3\pi\epsilon_0 E_0 a^2$.

c): The problem is solved in analogy with an image charge problem of two charges located outside a conducting sphere with zero net charge. The original charges q and $-q$ are located at $d\hat{\mathbf{z}}$ and $-d\hat{\mathbf{z}}$, respectively. The image charges $q' = -q\frac{a}{d}$ and $q\frac{a}{d}$ are located at $\frac{a^2}{d}\hat{\mathbf{z}}$ and $-\frac{a^2}{d}\hat{\mathbf{z}}$. The boundary conditions of the image charge problem coincide with the ones in the problem. Using Eq. 2.5 of the textbook and the superposition principle, it is then, on the boss,

$$\sigma = -\frac{q}{4\pi a^2}\frac{a}{d}\left(1 - \frac{a^2}{d^2}\right)\left\{ \frac{1}{\sqrt{1 + \frac{a^2}{d^2} - 2\frac{a}{d}\cos\theta}^{\,3}} - \frac{1}{\sqrt{1 + \frac{a^2}{d^2} + 2\frac{a}{d}\cos\theta}^{\,3}} \right\} \quad . \tag{35}$$

The charge Q on the boss is $2\pi a^2 \int_0^1 \sigma \, d\cos\theta$, and with $z = \cos\theta$, $u = 1 + \frac{a^2}{d^2}$, and $v = 2\frac{a}{d}$ it is

$$
\begin{aligned}
Q &= -2\pi a^2 \frac{q}{4\pi a^2}\frac{a}{d}\left(1-\frac{a^2}{d^2}\right)\left\{\int_0^1 \frac{dz}{\sqrt{u-vz}^3} - \int_0^1 \frac{dz}{\sqrt{u+vz}^3}\right\} \\
&= -\frac{q}{2}\frac{a}{d}\left(1-\frac{a^2}{d^2}\right)\left\{\left[+\frac{2}{v\sqrt{u-vz}}\right]_0^1 - \left[-\frac{2}{v\sqrt{u+vz}}\right]_0^1\right\} \\
&= \frac{q}{2}\frac{a}{d}\left(1-\frac{a^2}{d^2}\right)\frac{d}{a}\left\{\frac{2}{\sqrt{1+\frac{a^2}{d^2}}} - \frac{1}{1+\frac{a}{d}} - \frac{1}{1-\frac{a}{d}}\right\} \\
&= q\left(1-\frac{a^2}{d^2}\right)\left\{\frac{1}{\sqrt{1+\frac{a^2}{d^2}}} - \frac{d^2}{d^2-a^2}\right\} \\
&= q\frac{d^2-a^2}{d^2}\frac{d^2}{d^2-a^2}\left\{\frac{d^2-a^2}{d\sqrt{d^2+a^2}} - 1\right\} \\
&= -q\left\{1-\frac{d^2-a^2}{d\sqrt{d^2+a^2}}\right\} \quad \text{, q.e.d.} \qquad (36)
\end{aligned}
$$

Problem 2.30 **5 Points**

Additional information: You may read Problem 1.21, 1.24, 2.15, 2.16, and use any relevant information given in these problems. In the comparison part of the problem, it is only required to compare the result of the finite-element method with the exact result.

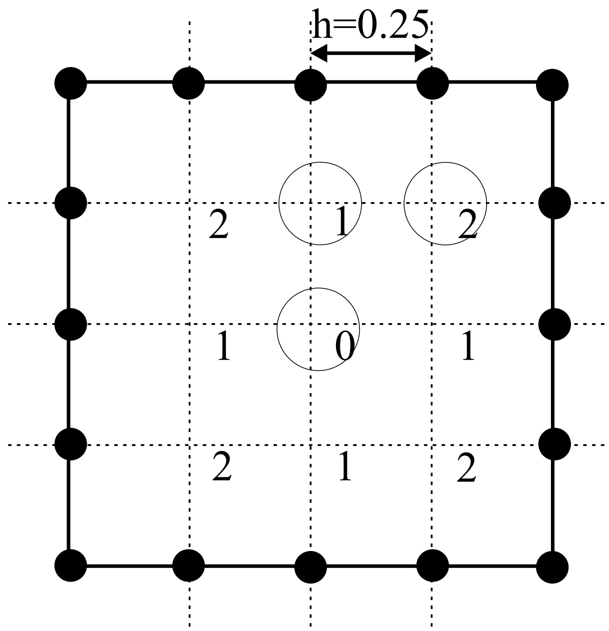

Figure 3: Finite element method for 2D square problem.

For symmetry, it is sufficient to write down linear equations for the three encircled grid points of the figure,

$$\begin{array}{lcccl} \sum_k \Psi_k \underbrace{\int_V \nabla\phi_i \cdot \nabla\phi_k \, dxdy}_{A_{ki}} & = & \frac{\rho_i}{\epsilon_0} \underbrace{\int_V \phi_i dxdy}_{h^2} & & i = 0,1,2 \\ \sum_k \Psi_k \; A_{ki} & = & \frac{\rho_i}{\epsilon_0} \; h^2 & & i = 0,1,2 \end{array} \tag{37}$$

The first equation is Eq. 2.80 in the textbook, except that for simplicity we use only one index k instead of k and l, and we use i instead of i and j. The A_{ki} are 8/3, -1/3 or 0 according to the rules explained in the textbook (see, for instance, Problem 2.29). The sum over k includes, in principle, the boundary nodes. However, since the boundary is on zero potential, in the present problem the sums do not explicitly show boundary points. Also, when writing down the sums we employ the fact that due to symmetry the potentials on many nodes are equal (see figure). Further, ρ_i is constant and equals ρ. We find

$$\begin{array}{rcrcrcl} \frac{8}{3}\Psi_0 & - & \frac{4}{3}\Psi_1 & - & \frac{4}{3}\Psi_2 & = & h^2\frac{\rho}{\epsilon_0} \\ -\frac{1}{3}\Psi_0 & + & \frac{6}{3}\Psi_1 & - & \frac{2}{3}\Psi_2 & = & h^2\frac{\rho}{\epsilon_0} \\ -\frac{1}{3}\Psi_0 & - & \frac{2}{3}\Psi_1 & + & \frac{8}{3}\Psi_2 & = & h^2\frac{\rho}{\epsilon_0} \end{array} \Leftrightarrow \begin{array}{rcrcrcl} 8\Psi_0 & - & 4\Psi_1 & - & 4\Psi_2 & = & a \\ -\Psi_0 & + & 6\Psi_1 & - & 2\Psi_2 & = & a \\ -\Psi_0 & - & 2\Psi_1 & + & 8\Psi_2 & = & a \end{array} \tag{38}$$

where $a = \frac{3h^2\rho}{\epsilon_0}$. The solution is $\Psi_0 = \frac{29}{70}a$, $\Psi_1 = \frac{9}{28}a$ and $\Psi_2 = \frac{9}{35}a$. For a charge density of unity and $h = 0.25$, we find the following numerical values listed under F.E.M.:

$\times$	F.E.M.	Exact
$4\pi\epsilon_0\Psi_0$	0.9761	0.9258
$4\pi\epsilon_0\Psi_1$	0.7573	0.7205
$4\pi\epsilon_0\Psi_2$	0.6059	0.5691

The exact values are taken from Problem 1.24c). Considering the roughness of the grid, the accuracy achieved by the finite element method (F.E.M.) is good. In contrast to the iteration methods, which require many iterations to converge to a final result, the F.E.M. yields its final result after only a single calculation.

In larger problems, the complexity of the F.E.M. calculation rapidly increases with the number of grid points, while iteration methods remain very simple.

Total 25 Points

Please report typos.