Physics 505 Electricity and Magnetism Fall 2003

Prof. G. Raithel

**Problem Set 2**

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**Problem 1.17** **5 Points**

**a)**: There are $n$ conductors with surfaces $S_i$, potentials $V_i$ and charges $Q_i$ $(i = 1, ..., n)$. Assume that there is only one conductor with non-zero potential. Call the index of that conductor $k$, its potential $V$, and its charge $Q$.

$$\begin{aligned}
\epsilon_0 \int_V |\nabla\Phi|^2 d^3x &= \text{(by Green I)} \\
-\epsilon_0 \int_V \Phi\nabla^2\Phi d^3x + \epsilon_0 \int_{\partial V} \Phi \frac{\partial}{\partial n}\Phi da &= \\
0 + \epsilon_0 \sum_{i=1}^{n} V_i \int_{S_i} \frac{\partial}{\partial n}\Phi da &= \\
\sum_{i=1}^{n} V_i \int_{S_i} \sigma da = V_k \int_{S_k} \sigma da &= VQ
\end{aligned}$$

Setting V=1, we have $QV = CV^2 = C$ with capacitance $C$, and therefore $\boxed{C = \epsilon_0 \int_V |\nabla\Phi|^2 d^3x \quad \text{q.e.d.}}$.

**b)**: Write $\Psi = \Phi + \delta\psi$, with $\Phi$ being the exact solution for the potential, and $\delta\psi$ being the difference between the test function $\Psi$ and $\Phi$. Note that $\delta\psi = 0$ on all surfaces $S_i$.

$$\begin{array}{ccccccccccccc}
 & & & & & & & & \epsilon_0 \int_V |\nabla\Psi|^2 d^3x & = & & & \\
 & & \epsilon_0 \int_V |\nabla\Phi|^2 d^3x & + & 2\epsilon_0 \int_V \nabla\Phi\cdot\nabla\delta\psi d^3x & + & \epsilon_0 \int_V |\nabla\delta\psi|^2 d^3x & = & & | & \text{by a) and Green I} \\
C & - & \underbrace{2\epsilon_0 \int_V (\Delta\Phi)\delta\psi d^3x}_{} & + & \underbrace{2\int_{\partial V} \delta\psi \frac{\partial}{\partial n}\Phi da}_{} & + & \underbrace{\epsilon_0 \int_V |\nabla\delta\psi|^2 d^3x}_{} & = & & | & \text{since } \epsilon \geq 0 \\
C & + & \underbrace{0}_{} & + & \underbrace{0}_{} & + & \underbrace{\epsilon}_{} & \geq & C & & &
\end{array} \tag{1}$$

Thus, for all $\Psi$ satisfying the boundary conditions it is $\boxed{C \leq \epsilon_0 \int_V |\nabla\Psi|^2 d^3x \quad \text{q.e.d.}}$.

**Problem 1.22** **5 Points**

For a well-behaved function $f(x,y)$, the $k$-th order Taylor expansion around $(x_0,y_0)$ is

$$f(x,y) = \sum_{m=0,n=0}^{m+n=k} \frac{1}{m!n!} \partial_x^{(m)} \partial_y^{(n)} f\,|_{(x_0,y_0)} (x-x_0)^m (y-y_0)^n \tag{2}$$

**a)**: Cross average. Since either $(x-x_0)$ or $(y-y_0)$ is zero (while the other one $\pm h$), the only non-vanishing terms are ones in which either $n=0$ or $m=0$. Using $F_x = \frac{\partial}{\partial x} F$, $F_{2x} = \frac{\partial^2}{\partial x^2} F$ etc, it is

$$\begin{aligned}
F(h,0) &= F(0,0) + hF_x + \frac{h^2}{2}F_{2x} + \frac{h^3}{6}F_{3x} + \frac{h^4}{24}F_{4x} + \frac{h^5}{120}F_{5x} + h^6 C_6 \\
F(-h,0) &= F(0,0) - hF_x + \frac{h^2}{2}F_{2x} - \frac{h^3}{6}F_{3x} + \frac{h^4}{24}F_{4x} - \frac{h^5}{120}F_{5x} + h^6 C_6 \\
F(0,h) &= F(0,0) + hF_y + \frac{h^2}{2}F_{2y} + \frac{h^3}{6}F_{3y} + \frac{h^4}{24}F_{4y} + \frac{h^5}{120}F_{5y} + h^6 C_6 \\
F(0,-h) &= F(0,0) - hF_y + \frac{h^2}{2}F_{2y} - \frac{h^3}{6}F_{3y} + \frac{h^4}{24}F_{4y} - \frac{h^5}{120}F_{5y} + h^6 C_6
\end{aligned}$$

and the cross sum evidently is

$$S_c = 4F(0,0) + h^2(F_{2x}+F_{2y}) + \frac{h^4}{12}(F_{4x}+F_{4y}) + \mathcal{O}(h^6) = 4F(0,0) + h^2\Delta F + \frac{h^4}{12}(F_{4x}+F_{4y}) + \mathcal{O}(h^6) \quad \text{q.e.d.} \tag{3}$$

**b)**: Square sum. It is easily seen that due to cancellations in the square sum only terms in Eq. 2 contribute with both $m$ and $n$ even. Thus, it is:

$$F(\pm h, \pm h) = F(0,0) + \frac{h^2}{2}(F_{2x}+F_{2y}) + \frac{h^4}{24}(F_{4x}+F_{4y}+6F_{2x2y}) + C_6 h^6 + \dots, \tag{4}$$

where the ... stand for terms that cancel when performing the square sum, and $C_6$ for a 6-th order coefficient. Thus,

$$S_s = 4F(0,0) + 2h^2(F_{2x}+F_{2y}) + \frac{h^4}{6}(F_{4x}+F_{4y}) + h^4 F_{2x2y} + \mathcal{O}(h^6) \quad . \tag{5}$$

Noting that $\Delta(\Delta F) = F_{4x} + 2F_{2x2y} + F_{4y}$, we see that $F_{2x2y} = \frac{1}{2}\left(\Delta(\Delta F) - F_{4x} - F_{4y}\right)$, and the square sum becomes

$$S_s = 4F(0,0) + 2h^2\Delta F + \frac{h^4}{2}\Delta(\Delta F) + \left(\frac{h^4}{6} - \frac{h^4}{2}\right)(F_{4x}+F_{4y}) + \mathcal{O}(h^6) \tag{6}$$

$$S_s = 4F(0,0) + 2h^2\Delta F + \frac{h^4}{2}\Delta(\Delta F) - \frac{h^4}{3}(F_{4x}+F_{4y}) + \mathcal{O}(h^6) \quad \text{q.e.d.} \tag{7}$$

Note: An improved average (=sum/4) can be defined as follows:

$$\bar{S} = \frac{1}{5}S_s + \frac{1}{20}S_c = F(0,0) + \frac{3}{10}h^2\Delta F + \frac{1}{40}h^4\Delta(\Delta F) + \mathcal{O}(h^6) \quad . \tag{8}$$

For functions $F$ that are solutions of the Laplace equation all correction terms up to and excluding the $\mathcal{O}(h^6)$ vanish. In a charged space with charge density $\rho(\mathbf{x})$, the expressions

$$\Delta F = -\frac{\rho}{\epsilon_0} \quad \text{and} \quad \Delta(\Delta F) \approx -\frac{4\rho_c}{\epsilon_0 h^2} + \frac{4\rho}{\epsilon_0 h^2} \quad , \tag{9}$$

with $\rho_c$ being the cross *average* of the charge density, can be inserted into Eq. 8, yielding an equation that can be resolved for $F(0,0)$. This procedure and a consideration of the error in the second equation of Eq. 9 lead to Equation 1.82 in the textbook.

**Problem 2.1** **5 Points**

A charge $q$ is located at $\mathbf{x}' = d\hat{\mathbf{x}}$, at a distance $d$ from a conducting surface formed by the $x = 0$ plane. In the volume of interest, $x > 0$, the field is that of the original charge and a charge $-q$ at location $\mathbf{x}_1 = -d\hat{\mathbf{x}}$.

**a)**: In the $yz$-plane it is $\mathbf{E} = E_x\hat{\mathbf{x}} = -\frac{2qd}{4\pi\epsilon_0\sqrt{d^2+\rho^2}^3}\hat{\mathbf{x}}$ with $\rho = \sqrt{x^2+y^2}$. Since $\hat{\mathbf{x}}$ coincides with the normal vector of the conducting surface, the charge density is

$$\boxed{\sigma(\rho) = \epsilon_0 E_x = -\frac{qd}{2\pi\sqrt{d^2+\rho^2}^3}}.$$

**b)**: The force is attractive, and is $\boxed{\mathbf{F} = -\frac{q^2}{16\pi\epsilon_0 d^2}\hat{\mathbf{x}}}$.

**c)**: The electrostatic pressure is $\frac{\sigma^2}{2\epsilon_0}\hat{\mathbf{n}}$, where the normal vector of the surface $\hat{\mathbf{n}} = \hat{\mathbf{x}}$. Thus,

$$\mathbf{F} = \hat{\mathbf{x}}\frac{q^2d^2}{8\pi^2\epsilon_0}\int_0^\infty \frac{2\pi\rho\,d\rho}{(d^2+\rho^2)^3} = \hat{\mathbf{x}}\frac{q^2d^2}{4\pi^2\epsilon_0}\left[-\frac{1}{4(d^2+\rho^2)^2}\right]_0^\infty = \hat{\mathbf{x}}\frac{q^2}{16\pi\epsilon_0 d^2}\,, \tag{10}$$

which is the negative of the result of b) (as expected).

**d)**: The work to be done to move the charge from its location to infinity is

$$W = \int_d^\infty F(d)dd = \frac{q^2}{16\pi^2\epsilon_0}\int_0^\infty \frac{1}{d^2}dd = \left[-\frac{q^2}{16\pi^2\epsilon_0 d}\right]_d^\infty = \frac{q^2}{16\pi^2\epsilon_0 d} > 0 \quad . \tag{11}$$

**e)**: The potential energy between charge and image charge is $\boxed{W_{\text{pot}} = -\frac{q^2}{8\pi^2\epsilon_0 d}}$, which is **not** equal to $-W$, as one might naively expect, but equal to $-2W$. The customary interpretation of the factor 2 is that in the image problem the field fills all space and is symmetric about the $x = 0$ plane, while in the real problem the field only fills the half-space $x > 0$. Therefore, the potential energy in the image problem can be expected to be twice that of the image problem. Generally, calculations of electrostatic energy are **not** directly transferrable from image to real problems, because the motion of real charges usually implies a motion of the respective image charge(s). The latter motion matters in the electrostatic energy of the image problem, while it does not in the electrostatic energy of the real problem. In contrast, potentials, forces and fields in the volume of interest are same for the real and the image problem.

**f)**: For $q = -e$ and $d = 10^{-10}m$ it is $\boxed{W = 3.6eV}$. This energy is substantial. Note that it corresponds to the work function of typical metals.

**Problem 2.2** **5 Points**

We consider the problem of a charge $q$ inside a grounded, thin conducting shell with radius $a$. The calculation of the size and the location of the image charge is analogous to the case of a charge outside a grounded conducting sphere covered in the textbook (swap primed and unprimed variables). For a distance $y < a$ of the charge from the center of the sphere, the image charge $q' = -q\frac{a}{y}$ is located at a distance $y' = y\frac{a^2}{y} > a$. The angle between the vector from the center of the sphere to the observation point and the vector from the center of the sphere to either charge is denoted $\gamma$.

**a)**: The potential at a location $\mathbf{x}$ characterized by a radial coordinate $x$ and angle $\gamma$ is

$$\Phi(x,\gamma) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{x^2+y^2-2xy\,\cos\gamma}} - \frac{a}{y\sqrt{x^2+\frac{a^4}{y^2}-2x\frac{a^2}{y}\,\cos\gamma}}\right) \tag{12}$$

**b)**: The charge density is the negative of Eq. 2.5 in the textbook,

$$\sigma(\cos\gamma) = \epsilon_0\frac{\partial}{\partial x}\Phi(x,\gamma)|_{x=a} = \frac{q}{4\pi a^2}\frac{a}{y}\frac{1-\frac{a^2}{y^2}}{\sqrt{1+\frac{a^2}{y^2}-2\frac{a}{y}\,\cos\gamma}^3} \tag{13}$$

**c)**: The force is radially outward and given by

$$\mathbf{F} = \hat{\mathbf{y}}\frac{q^2}{4\pi\epsilon_0}\frac{a}{y}\frac{1}{(y-\frac{a^2}{y})^2} = \hat{\mathbf{y}}\frac{q^2}{4\pi\epsilon_0}\frac{ay}{(a^2-y^2)^2} \tag{14}$$

**d)**: In the following, we identify quantities obtained for the case of a grounded sphere with a subscript I.

**Sphere on potential $V$:** Add all charge densities and the potentials of the solutions of the following problems: I=grounded sphere with charge $q$ inside. II=sphere on potential $V$ and no charge inside. The solution of case II is a constant potential $V$ in the shell and its entire interior, because conductors with charge-free internal cavities are equipotential volumes. Outside the shell, the potential of case II drops as $\frac{1}{x}$. The sum of the charge densities of case I and II, and the sum of the potentials, $\Phi = \Phi_I + \Phi_{II}$, satisfy the boundary conditions. That is: the sums produce the correct internal charge distribution and the correct potential on the boundary, respectively. Also, due to the superposition principle, the sum potential and the sum charge distribution are a solution of the Poisson equation. Due to the uniqueness theorem, this must be the only solution for the given surface potential $V$ and the given charge distribution in the volume of interest.

In the conductor and its interior cavity the potential is $\Phi = \Phi_I + \Phi_{II}$, *i.e.* $\boxed{\Phi = \Phi_I + V}$. The charge density induced on the inner surface is $\boxed{\sigma = \sigma_I}$, since in case II there are no surface charges on the inner surface at all. The internal electric fields derived from $\Phi$ and $\Phi_I$ are the same, and thus the forces on the charge $q$ are the same, $\boxed{\mathbf{F} = \mathbf{F}_I}$.

**Sphere with total charge $Q$:** Again, we think of two solutions and form their sum. Case I is as before. In case I, the total surface charge induced on the inner surface of the shell is $-q$ (think of a Gaussian surface between inner and outer surface of the conducting shell). Also, in case I there is no charge on the outer

surface of the shell, because the shell is grounded and, lacking any further information, the exterior potential must be assumed to be zero as well. Case II is a shell with a total charge $Q' = Q + q$ and no charge inside. The charge $Q'$ evenly distributes on the outer surface of the shell; there is no surface charge on the inner surface of the shell. The potential of case II is $V = \frac{Q+q}{4\pi\epsilon_0 a}$.

The solution for the case of a shell with total charge $Q$ and internal charge $q$ is given by summing the potential and the charge distributions of case I and II. Inside the shell it is $\boxed{\Phi = \Phi_I + V = \Phi_I + \frac{Q+q}{4\pi\epsilon_0 a}}$. The charge density induced on the inner surface is $\boxed{\sigma = \sigma_I}$, since in case II there are no surface charges in the inner surface. (The problem doesn't ask for the charge density on the outer surface; it would be $\sigma_{\text{outer}} = \frac{Q+q}{4\pi a^2}$.) The forces on the charge $q$ are the same, $\boxed{\mathbf{F} = \mathbf{F}_I}$.

**Convergence speed of relaxation methods.** **5 Points**

We consider a two-dimensional square region with three sides on zero potential, and one side on unit potential $V = 1$. Use a square grid of 21x21 potential values, $\{V(i,j), i = 0,1,2...20, j = 0,1,2...20\}$. The grid includes the boundaries, and all internal points are initialized to zero.

**a)**: We use the Jacobian iteration of "cross averages", as defined in Eq. 1.80a of the textbook, and consider how the potential at the mid point, $V(10,10)$, changes from one iteration to the next. A piece of Fortran code that does this is:

```
C flag1=1 will indicate that the desired accuracy level has been reached.
C n is a counter for the iteration index
      flag1=0
      n=0

C Initialize potential array
      DO i=0,20
      DO j=0,20
      potold(i,j)=0d0
      potnew(i,j)=0d0
      ENDDO
      ENDDO
      vold=potold(10,10)

C Initialize one side of potential array to 1
      DO i=0,20
      potold(20,i)=1d0
      potnew(20,i)=1d0
      ENDDO

C Begin iteration loop
      DO WHILE ((n.LT.2000).AND.(flag1.EQ.0))

C Calculate new potentials
        DO i=1,19
        DO j=1,19
        r1=(potold(i-1,j) + potold(i,j-1) + potold(i+1,j) + potold(i,j+1))
        potnew(i,j)=r1/4d0
        ENDDO
        ENDDO
        vnew=potnew(10,10)

C Check for termination condition
        IF (ABS(vold.LT.1d-10)) vold=1d-10
        IF ((ABS((vnew-vold)/vold).LT.1d-5).AND.(n.GT.15).AND.(flag1.EQ.0))THEN
        WRITE (*,'(A,X,I4,A,F10.7)') 'Hit accuracy of 10^-5 at iteration:',n,'  V=',vnew
        flag1=1
        ENDIF

C Replace old potentials by new potential
        DO i=1,19
        DO j=1,19
        potold(i,j)=potnew(i,j)
        ENDDO
        ENDDO
        vold=vnew
```

```
C Increment iteration counter
      n=n+1

C End iteration loop
      ENDDO
```

During the first few iterations, the potential at the center does not change yet. To ensure that the potential at the center has begun to change, in the termination condition we check for $n > 15$.

| | |
|---|---|
| Result: Reached accuracy of $10^{-5}$ at iteration: 613 | pot(10,10)=0.24979 |

Note that the exact potential at the center is 1/4.

**b)**: The computation is repeated using the Gauss-Seidel iteration of "cross averages". The changes in the program are that instead of two only one potential array is used, and that the potential values are replaced at the time when they are calculated. Think about why in the case of the Gauss-Seidel iteration there is a slight dependence of the convergence speed on which potential side is initialized to 1. Typical results, obtained with initializing different sides to $V = 1$, are:

| | |
|---|---|
| Result 1: Reached accuracy of $10^{-5}$ at iteration: 340 | pot(10,10)=0.24990 |
| Result 2: Reached accuracy of $10^{-5}$ at iteration: 331 | pot(10,10)=0.24990 |

**c)**: The convergence of the Gauss-Seidel iteration is accelerated using the hyper-relaxation method. While the plain Gauss-Seidel iteration uses the cross-average replacement

$$\mathrm{pot}(i,j) = \frac{1}{4}\left(\mathrm{pot}(i+1,j) + \mathrm{pot}(i-1,j) + \mathrm{pot}(i,j+1) + \mathrm{pot}(i,j-1)\right) \quad , \tag{15}$$

the hyper-relaxation method with hyper-relaxation parameter $p$ operates as follows:

$$\begin{aligned} r1 &= \frac{1}{4}\left(\mathrm{pot}(i+1,j) + \mathrm{pot}(i-1,j) + \mathrm{pot}(i,j+1) + \mathrm{pot}(i,j-1)\right) \\ \mathrm{pot}(i,j) &= \mathrm{pot}(i,j) + p\left(r1 - \mathrm{pot}(i,j)\right) \quad , \end{aligned} \tag{16}$$

where $r1$ is an auxiliary variable. Using the hyper-relaxation method defined in Eq. 16, the following results are obtained:

| | | |
|---|---|---|
| p=1.00 | Accuracy $10^{-5}$ at iteration 340 | pot(10,10)=0.24990 |
| p=1.10 | Accuracy $10^{-5}$ at iteration 285 | pot(10,10)=0.24992 |
| p=1.20 | Accuracy $10^{-5}$ at iteration 237 | pot(10,10)=0.24993 |
| p=1.30 | Accuracy $10^{-5}$ at iteration 196 | pot(10,10)=0.24994 |
| p=1.40 | Accuracy $10^{-5}$ at iteration 159 | pot(10,10)=0.24995 |
| p=1.50 | Accuracy $10^{-5}$ at iteration 126 | pot(10,10)=0.24997 |
| p=1.60 | Accuracy $10^{-5}$ at iteration 094 | pot(10,10)=0.24997 |
| p=1.70 | Accuracy $10^{-5}$ at iteration 061 | pot(10,10)=0.24998 |
| p=1.80 | Accuracy $10^{-5}$ at iteration 046 | pot(10,10)=0.25003 |
| p=1.90 | Accuracy $10^{-5}$ at iteration 102 | pot(10,10)=0.25000 |
| p=2.00 | Accuracy $10^{-5}$ at iteration 016 | pot(10,10)=0.50000 |

Considering the potential value obtained for $p = 2.0$ and inspecting the whole potential grid obtained for that $p$, it is seen that for $p = 2.0$ the method fails, and that the small number of iterations listed for $p = 2.0$ is a fluke.

**The following addendum to part c) of the problem is not required to obtain full score.**

Using the maximum relative change of the potential over the whole grid as an improved figure of merit, we obtain

| | | |
|---|---|---|
| p=1.00 | Accuracy $10^{-5}$ at iteration 381 | pot(10,10)=0.24996 |
| p=1.10 | Accuracy $10^{-5}$ at iteration 320 | pot(10,10)=0.24997 |
| p=1.20 | Accuracy $10^{-5}$ at iteration 268 | pot(10,10)=0.24997 |
| p=1.30 | Accuracy $10^{-5}$ at iteration 222 | pot(10,10)=0.24998 |
| p=1.40 | Accuracy $10^{-5}$ at iteration 182 | pot(10,10)=0.24998 |
| p=1.50 | Accuracy $10^{-5}$ at iteration 145 | pot(10,10)=0.24999 |
| p=1.60 | Accuracy $10^{-5}$ at iteration 111 | pot(10,10)=0.24999 |
| p=1.70 | Accuracy $10^{-5}$ at iteration 075 | pot(10,10)=0.24999 |
| p=1.80 | Accuracy $10^{-5}$ at iteration 082 | pot(10,10)=0.25000 |
| p=1.90 | Accuracy $10^{-5}$ at iteration 161 | pot(10,10)=0.25000 |

In this list, $p = 2.00$ does not show up any more because, due to oscillations, no convergence is achieved.

**d)**: Due to the immediate updating of the potential values, the Gauss-Seidel iteration converges about a factor two faster than the Jacobi iteration. Also, 50% in memory is saved (which may matter in large problems). The hyper-relaxation method requires - for the best choice of the hyper-relaxation parameter $p = 1.8$ - about a factor seven fewer iterations than the plain Gauss-Seidel method, and at most 50% more operations in one potential-value computation, leading to an overall gain of a factor four to five in efficiency. One could, of course, time the various methods against each other to obtain direct speed comparisons.

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**Total 25 Points**