

Problem 1.3**5 Points**

a): By symmetry, the solution must be of the form $\rho(\mathbf{x}) = \rho(r) = Q\delta(r - R)f$, with a constant f to be specified by the condition

$$Q = \int \rho(r)4\pi r^2 dr = \int Q\delta(r - R)f4\pi r^2 dr = Qf4\pi R^2. \quad (1)$$

Thus, $\boxed{\rho(r) = \frac{Q\delta(r-R)}{4\pi R^2}}$.

b): By symmetry, the solution must be of the form $\rho(\mathbf{x}) = \rho(r) = \lambda\delta(r - b)f$, with a constant f to be specified by the condition

$$\lambda = \int \rho(r)2\pi r dr = \int \lambda\delta(r - b)f2\pi r dr = 2\lambda f\pi b. \quad (2)$$

Thus, $\boxed{\rho(r) = \frac{\lambda\delta(r-b)}{2\pi b}}$.

c): By symmetry, the solution must be of the form $\rho(\mathbf{x}) = \rho(r, z) = \delta(z)\Theta(R - r)f(r)$. There, Θ is the step function, and $f(r)$ a function specified by a normalization condition that describes how much charge is supposed to be on a ring with radius r and radial thickness dr (for $r < R$):

$$Q \frac{2\pi r dr}{\pi R^2} = \int_z \rho(r, z)2\pi r dz dr = \int_z \delta(z)f(r)2\pi r dz dr = f(r)2\pi r dr. \quad (3)$$

Thus, $\boxed{\rho(r, z) = \frac{Q\delta(z)\Theta(R-r)}{\pi R^2}}$.

d): By symmetry, the solution must be of the form $\rho(\mathbf{x}) = \rho(r, \theta) = \delta(\cos\theta)\Theta(R - r)f(r)$, where $f(r)$ is a function specified by a normalization condition that describes how much charge is supposed to be on a shell with radius r and radial thickness dr (for $r < R$):

$$Q \frac{2\pi r dr}{\pi R^2} = \int_{\cos\theta, \phi} \rho(r, \theta)r^2 d\phi d\cos\theta dr = 2\pi \int_{\cos\theta} \delta(\cos\theta)f(r)r^2 d\cos\theta dr = 2\pi f(r)r^2 dr. \quad (4)$$

Thus, $\boxed{\rho(r, \cos\theta) = \frac{Q\delta(\cos\theta)\Theta(R-r)}{\pi R^2 r}}$.

Problem 1.5**5 Points**

Method 1: Consider first the case $r > 0$. Then, the expression

$$\rho_-(r) = -\epsilon_0 \Delta \Phi(r) = -\frac{q}{4\pi} \Delta \frac{\exp(-\alpha r)}{r} \left(1 + \frac{\alpha r}{2}\right) \quad (5)$$

is well defined, and yields

$$\rho_-(r) = -\frac{q}{4\pi} \frac{1}{r} \partial_r^2 \exp(-\alpha r) \left(1 + \frac{\alpha r}{2}\right) = -\frac{q\alpha^3}{8\pi} \exp(-\alpha r). \quad (6)$$

For $r \rightarrow 0$, it is $\Phi(r) \rightarrow \frac{q}{4\pi\epsilon_0 r}$ and $\rho_+(r) = -\frac{q}{4\pi} \Delta \frac{1}{r} = q\delta^3(\mathbf{r}) = q\frac{\delta(r)}{4\pi r^2}$.

Obviously, $\rho_-(r)$ accounts for the electron and $\rho_+(r)$ for the proton, and the total charge density is the sum

$$\rho = q\left(\delta^3(\mathbf{r}) - \frac{\alpha^3}{8\pi} \exp(-\alpha r)\right) = \frac{q}{8\pi} \left(\frac{2\delta(r)}{r^2} - \alpha^3 \exp(-\alpha r)\right).$$

Method 2: Write $\frac{\exp(-\alpha r)}{r} \left(1 + \frac{\alpha r}{2}\right)$ as a product of functions $f(r) = \frac{1}{r}$ and $g(r) = \exp(-\alpha r) \left(1 + \frac{\alpha r}{2}\right)$, and use the product rule $\Delta(fg) = f\Delta g + g\Delta f + 2\nabla f \cdot \nabla g$. All derivatives are found to be well behaved at $r = 0$, except Δf , which equals $-4\pi\delta^3(\mathbf{r})$. Considering that, the calculation confirms the result of method 1.

You may verify that the total charge $\int \rho(\mathbf{r}) d^3r = 0$, which must be the case.

Problem 1.5**5 Points**

How to calculate the capacitance of a conductor pair:

- Assume charges $\pm Q$ on conductors.
- Find $\mathbf{E}(\mathbf{x})$ using a method of your choice (often Gauss's law).
- Calculate $V = -\int_1^2 \mathbf{E}(\mathbf{x}) \cdot d\mathbf{l}$ between the two conductors.
- $C = |\frac{Q}{V}|$.

a): The electric field is $E = \frac{\sigma}{\epsilon_0} = \frac{Q}{A\epsilon_0}$. The voltage V that results from the simple line integral is $V = \frac{Qd}{A\epsilon_0}$. Thus, the capacitance is

$$\boxed{C = \frac{\epsilon_0 A}{d}}.$$

b): The electric field is $\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}}$, and

$$V = -\int_1^2 \mathbf{E}(\mathbf{x}) \cdot d\mathbf{l} = -\int_{r=a}^{r=b} \frac{Q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} dr = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{b} - \frac{1}{a} \right), \quad (7)$$

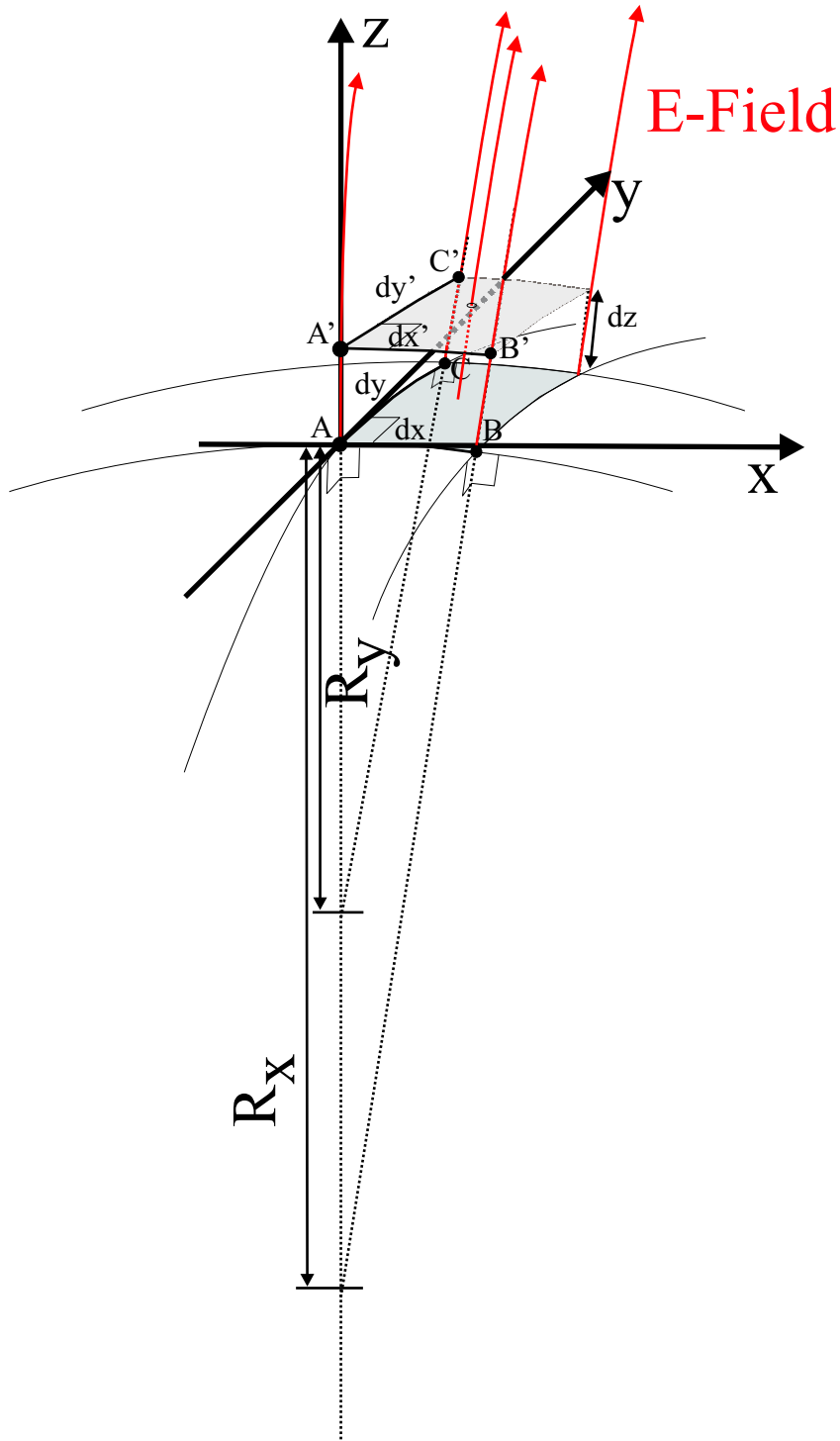
Thus, $\boxed{C = 4\pi\epsilon_0 \frac{ab}{b-a}}$.

c): The electric field is, in cylindrical coordinates, $\mathbf{E} = \frac{Q}{2\pi\epsilon_0 L r} \hat{\mathbf{r}}$, and

$$V = -\int_1^2 \mathbf{E}(\mathbf{x}) \cdot d\mathbf{l} = -\int_{r=a}^{r=b} \frac{Q}{2\pi\epsilon_0 L r} \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} dr = \frac{Q}{2\pi\epsilon_0 L} \ln\left(\frac{b}{a}\right), \quad (8)$$

Thus, $\boxed{C = 2\pi\epsilon_0 L \frac{1}{\ln(\frac{b}{a})}}$, and the capacitance per length is $2\pi\epsilon_0 \frac{1}{\ln(\frac{b}{a})}$.

d): 6.39 mm and 113 km, respectively. That means, for practical purposes the capacitance per length of a coax cable cannot be made much smaller than a few $10^{-11} F/m$.



Problem 1.11, Method 1 (enlightning but a little cumbersome):

The point of interest is located at the origin of a cartesian coordinate system the xy -plane of which defines the tangent plane to the curved surface (at the point of interest). Further, the directions of the x - and y -axes are such that **locally** the curved surface can be described as the set of points with coordinates $\mathbf{S}(x, y) = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - (\frac{1}{2R_x}x^2 + \frac{1}{2R_y}y^2)\hat{\mathbf{z}}$. It has been discussed in class that it is always possible to find orthogonal coordinate axes on the tangent plane that allow such a parametrization of the curved surface. For the situation depicted in the figure, it would be $R_x > 0$ and $R_y > 0$. Due to the orthogonality of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, a surface element with side lengths dx and dy on the curved surface has the area $da = dx dy$.

At a location $\mathbf{S}(x, y)$ on the curved surface, a normal vector to the plane is $\frac{\partial \mathbf{S}}{\partial x} \times \frac{\partial \mathbf{S}}{\partial y} = \frac{x}{R_x}\hat{\mathbf{x}} + \frac{y}{R_y}\hat{\mathbf{y}} + \hat{\mathbf{z}}$. Using this fact, it can be seen that an area element $da' = dx' dy'$ located at a height dz above da , constructed as shown in the figure, will have the property that the “sidewalls” of a pillbox with da' and da as upper and lower surfaces are orthogonal to the curved surface. Since the electric field, depicted in red in the figure, also is orthogonal to the surface, the electric flux through the sidewalls is zero. Due to Gauss’s law, the fluxes through da and da' must be equal and opposite. Considering that the field is (anti)parallel to both vectors \mathbf{da} and \mathbf{da}' , it follows that $E'da' = E da$, with E and E' being the field magnitudes on da and da' , respectively.

Simple geometry apparent in the figure yields $dx' = dx(1 + \frac{dz}{R_x})$ and $dy' = dy(1 + \frac{dz}{R_y})$. Thus, for infinitesimally small dx , dy and dz it is $da' = (1 + \frac{dz}{R_x} + \frac{dz}{R_y})da$, and

$$E' = E \frac{da}{da'} = E(1 + \frac{dz}{R_x} + \frac{dz}{R_y})^{-1} = E(1 - \frac{dz}{R_x} - \frac{dz}{R_y}) \quad (9)$$

Thus,

$$\frac{1}{E} \frac{E' - E}{dz} = \frac{1}{E} \frac{\partial E}{\partial n} = - \left(\frac{1}{R_x} + \frac{1}{R_y} \right) \quad \text{q.e.d.} \quad (10)$$

Note: for the case depicted in the figure, both principal radii are > 0 , and consequently $\frac{\partial E}{\partial n} < 0$, as expected.

Problem 1.11, Method 2: (“cleanest” proof):

The electric field is (anti)parallel to both vectors \mathbf{da} and \mathbf{da}' of the lower and upper planes in the figure, respectively. It follows from Gauss’s law that $E'da' = E da$, with E and E' being the field magnitudes on da and da' .

To find da , refer to the points defined in the figure and set $\mathbf{A} = \mathbf{S}(0, 0) = 0$, $\mathbf{B} = \mathbf{S}(dx, 0) = dx\hat{\mathbf{x}} - \frac{1}{2R_x}dx^2\hat{\mathbf{z}}$, and $\mathbf{C} = \mathbf{S}(0, dy) = dy\hat{\mathbf{y}} - \frac{1}{2R_y}dy^2\hat{\mathbf{z}}$. Then, in lowest order of dx and dy it is $da = |(\mathbf{B} - \mathbf{A}) \times (\mathbf{C} - \mathbf{A})| = dx dy$.

To find da' , we need to determine the points \mathbf{A}' , \mathbf{B}' , \mathbf{C}' that lie at a distance dz above the points \mathbf{A} , \mathbf{B} , \mathbf{C} . Thereby, “above” means that the lines $\mathbf{A}' - \mathbf{A}$ etc. are parallel to the position-dependent normal vectors $\hat{\mathbf{n}}$ of the curved surface. The normal vectors are $\hat{\mathbf{n}}(x, y) = \frac{\partial \mathbf{S}}{\partial x} \times \frac{\partial \mathbf{S}}{\partial y} = \frac{x}{R_x}\hat{\mathbf{x}} + \frac{y}{R_y}\hat{\mathbf{y}} + \hat{\mathbf{z}}$. Note that $|\hat{\mathbf{n}}(x, y)| = 1$, up to second- and higher-order corrections. It is thus found $\mathbf{A}' = \mathbf{A} + dz\hat{\mathbf{n}}(0, 0) = dz\hat{\mathbf{z}}$, $\mathbf{B}' = (dx + \frac{dx dz}{R_x})\hat{\mathbf{x}} + (dz - \frac{dx^2}{2R_x})\hat{\mathbf{z}}$, and $\mathbf{C}' = (dy + \frac{dy dz}{R_y})\hat{\mathbf{y}} + (dz - \frac{dy^2}{2R_y})\hat{\mathbf{z}}$. Then, in first order of dx and dy it is found that $da' = |(\mathbf{B}' - \mathbf{A}') \times (\mathbf{C}' - \mathbf{A}')| = dx dy \left(1 + \frac{dz}{R_x} + \frac{dz}{R_y}\right)$. It follows

$$E' = E \frac{da}{da'} = E \left(1 + \frac{dz}{R_x} + \frac{dz}{R_y}\right)^{-1} = E \left(1 - \frac{dz}{R_x} - \frac{dz}{R_y}\right) \quad (11)$$

and

$$\frac{1}{E} \frac{E' - E}{dz} = \frac{1}{E} \frac{\partial E}{\partial n} = - \left(\frac{1}{R_x} + \frac{1}{R_y} \right) \quad \text{q.e.d.} \quad (12)$$

Problem 1.11, Method 3: (fast, less enlightning):

Using the coordinate system in the figure, at location $\mathbf{S}(x, y)$ on the curved surface the normal vector to the plane is $\frac{\partial \mathbf{S}}{\partial x} \times \frac{\partial \mathbf{S}}{\partial y} = \frac{x}{R_x} \hat{\mathbf{x}} + \frac{y}{R_y} \hat{\mathbf{y}} + \hat{\mathbf{z}}$. Since we are only interested in the local behavior in the vicinity of the origin, $x \ll R_x$ and $y \ll R_y$, and the length of the normal vector is 1, with corrections of order $(x/R_x)^2$ and $(y/R_y)^2$. Thus, the electric field on the surface has, locally, a form $\mathbf{E}(x, y) = E \left(\frac{x}{R_x} \hat{\mathbf{x}} + \frac{y}{R_y} \hat{\mathbf{y}} + \hat{\mathbf{z}} \right)$ with locally constant magnitude E . At the origin, we therefore have

$$\frac{\partial E_x}{\partial x} = \frac{E}{R_x} \quad \text{and} \quad \frac{\partial E_y}{\partial y} = \frac{E}{R_y} \quad . \quad (13)$$

Noting that the field is normal to the curved surface, it also is

$$\frac{\partial E_z}{\partial z} = \frac{\partial E}{\partial z} = \frac{\partial E}{\partial n} \quad (14)$$

From $\nabla \cdot \mathbf{E} = 0$ it then follows

$$\frac{\partial E_z}{\partial z} = \frac{\partial E}{\partial n} = - \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} \right) = -E \left(\frac{1}{R_x} + \frac{1}{R_y} \right) \quad (15)$$

and

$$\frac{1}{E} \frac{\partial E}{\partial n} = - \left(\frac{1}{R_x} + \frac{1}{R_y} \right) \quad \text{q.e.d.} \quad (16)$$

Problem 1.12**5 Points**

We consider a volume V limited by a conducting surface ∂V . For volume and surface charge densities $\rho(\mathbf{x})$ and $\sigma(\mathbf{x})$ the potential is $\Phi(\mathbf{x})$, while for $\rho'(\mathbf{x})$ and $\sigma'(\mathbf{x})$ it is $\Phi'(\mathbf{x})$.

It is $\rho(\mathbf{x}) = -\epsilon_0 \Delta \Phi(\mathbf{x})$, and $\frac{\partial \Phi}{\partial n} = \frac{\sigma}{\epsilon_0}$ (with $\hat{\mathbf{n}}$ pointing from the volume into the conductor). Corresponding equations for the primed quantities apply. The reciprocity theorem follows from these facts and Green's 2nd identity:

$$\begin{aligned} & \int_V \rho \Phi' d^3x + \int_{\partial V} \sigma \Phi' da = \\ & -\epsilon_0 \int_V (\Delta \Phi) \Phi' d^3x + \epsilon_0 \int_{\partial V} \frac{\partial \Phi}{\partial n} \Phi' da = \\ & \text{(by Green's 2nd identity)} \\ & -\epsilon_0 \int_V \Phi (\Delta \Phi') d^3x + \epsilon_0 \int_{\partial V} \Phi \frac{\partial \Phi'}{\partial n} da = \\ & \int_V \rho' \Phi d^3x + \int_{\partial V} \sigma' \Phi da \quad \text{q.e.d.} \end{aligned}$$

Electric quadrupole

5 Points

An electric quadrupole consists of two dipoles, a dipole \mathbf{p} located at $\mathbf{r} = \mathbf{b}/2$ and a dipole $-\mathbf{p}$ located at $\mathbf{r} = -\mathbf{b}/2$. The orientation of \mathbf{p} relative to \mathbf{b} can be arbitrary. Find the potential at a location \mathbf{x} in the limit that $b \rightarrow 0$ with pb remaining constant and both \mathbf{p} and \mathbf{b} maintaining their direction. Write the potential in the form

$$V = \sum_{ij} Q_{ij} x_i x_j / x^5, \quad (17)$$

and express the elements of the quadrupole tensor, Q_{ij} , in terms of \mathbf{p} and \mathbf{b} .

You may use the fact that the potential of an idealized dipole at location \mathbf{x}' is given by

$$\Phi_{\text{dip}}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \quad (18)$$

Solution:

The potential is

$$\Phi(\mathbf{x}) = \frac{\mathbf{p}}{4\pi\epsilon_0} \cdot \left(\frac{(\mathbf{x} - \mathbf{b}/2)}{|\mathbf{x} - \mathbf{b}/2|^3} - \frac{(\mathbf{x} + \mathbf{b}/2)}{|\mathbf{x} + \mathbf{b}/2|^3} \right) \quad (19)$$

Using the expansion

$$\begin{aligned} \frac{\mathbf{x} + \mathbf{a}}{|\mathbf{x} + \mathbf{a}|^3} &= \sum_i \frac{x_i + a_i}{|\mathbf{x} + \mathbf{a}|^3} \hat{\mathbf{x}}_i \approx \sum_i \left(\frac{x_i}{|\mathbf{x}|^3} + \mathbf{a} \cdot \nabla \frac{x_i}{|\mathbf{x}|^3} \right) \hat{\mathbf{x}}_i \\ &= \frac{\mathbf{x}}{|\mathbf{x}|^3} + \sum_{i,j} a_j \left(\frac{\partial}{\partial x_j} \frac{x_i}{|\mathbf{x}|^3} \right) \hat{\mathbf{x}}_i = \frac{\mathbf{x}}{|\mathbf{x}|^3} + \sum_{i,j} a_j \frac{x^2 \delta_{ij} - 3x_i x_j}{x^5} \hat{\mathbf{x}}_i \end{aligned} \quad (20)$$

with $\mathbf{a} = \pm \mathbf{b}/2$ we can expand $\Phi(\mathbf{x})$ as

$$\begin{aligned} \Phi(\mathbf{x}) &\approx -\frac{\mathbf{p}}{4\pi\epsilon_0} \cdot \left(\sum_{i,j} b_j \frac{x^2 \delta_{ij} - 3x_i x_j}{x^5} \hat{\mathbf{x}}_i \right) = \frac{1}{4\pi\epsilon_0} \sum_{i,j} p_i b_j \frac{3x_i x_j - x^2 \delta_{ij}}{x^5} \\ &= \frac{1}{4\pi\epsilon_0} \left(\left\{ \sum_{i,j} p_i b_j \frac{3x_i x_j}{x^5} \right\} - \frac{x^2}{x^5} \mathbf{p} \cdot \mathbf{b} \right) = \frac{1}{4\pi\epsilon_0} \sum_{i,j} \frac{x_i x_j}{x^5} \{3p_i b_j - \mathbf{p} \cdot \mathbf{b} \delta_{ij}\} \\ &= \sum_{i,j} Q_{ij} \frac{x_i x_j}{x^5} \end{aligned} \quad (21)$$

with quadrupole coefficients $Q_{ij} = \frac{1}{4\pi\epsilon_0} (3p_i b_j - \mathbf{p} \cdot \mathbf{b} \delta_{ij})$.

Total 30 Points