Physics 505 Electricity and Magnetism Fall 2003

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Problem Set 10

Maximal score: 30 Points

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1. Jackson, Problem 6.8 **6 Points**

Strategy. We first calculate the volume and surface charge densities that correspond to the electric polarization of the sphere. Multiplying the charge densities with the local velocity, we then obtain the volume and surface current density. The current densities will be equivalent to that of a (static) magnetization $\mathbf{M}$. The corresponding magnetostatic potential is obtained by consideration of the magnetic charge density that corresponds to $\mathbf{M}$.

Polarization (Eq. 4.57 in Jackson):

$$\mathbf{P} = 3\epsilon_0 \frac{\epsilon_r - 1}{\epsilon_r + 2} E_0 \hat{\mathbf{x}} =: \alpha E_0 \hat{\mathbf{x}}$$

Volume charge density: $-\nabla \cdot \mathbf{P} = 0$. Volume current density $= 0$.

Surface charge density: $\sigma = \hat{\mathbf{n}} \cdot \mathbf{P} = \hat{\mathbf{r}} \cdot \mathbf{P}$. The surface current density,

$$\mathbf{K} = \sigma(\mathbf{x})\mathbf{v}(\mathbf{x}) = (\hat{\mathbf{r}} \cdot \mathbf{P})\omega\hat{\mathbf{z}} \times r\hat{\mathbf{r}} = \mathbf{M} \times \hat{\mathbf{r}} \quad ,$$

equals that of an effective magnetization

$$\mathbf{M}(\mathbf{x}) = (\mathbf{r} \cdot \mathbf{P})\omega\hat{\mathbf{z}} = \alpha E_0 \omega (\mathbf{r} \cdot \hat{\mathbf{x}})\hat{\mathbf{z}} = \alpha E_0 \omega x \hat{\mathbf{z}}$$

The magnetic charges that correspond to $\mathbf{M}$ are

$$\begin{aligned}
\rho_m &= -\nabla \cdot \mathbf{M} = 0 \\
\sigma_m &= \hat{\mathbf{n}} \cdot \mathbf{M} = \hat{\mathbf{r}} \cdot \mathbf{M} = \alpha E_0 \omega x (\hat{\mathbf{z}} \cdot \hat{\mathbf{r}}) = \alpha E_0 \omega x \cos\theta \\
&= \alpha E_0 \omega a \sin\theta \cos\theta \cos\phi \\
&= -\alpha E_0 \omega a \sqrt{\frac{8\pi}{15}} \frac{1}{2} \left[Y_{21} + Y_{21}^*\right] \\
&=: \beta \left[Y_{21} + Y_{21}^*\right] = \beta \left[Y_{21} - Y_{2-1}\right] \qquad (1)
\end{aligned}$$

Thus, the magnetic potential

$$
\begin{aligned}
\Phi_m(\mathbf{x}) &= \frac{1}{4\pi}\int \frac{\sigma(\mathbf{x}')da'}{|\mathbf{x}-\mathbf{x}'|} \\
&= \frac{a^2\beta}{4\pi}\sum_{l,m}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\phi)\int Y_{lm}^*(\theta',\phi')\left[Y_{21}(\theta',\phi')-Y_{2-1}(\theta',\phi')\right]d\cos\theta' d\phi' \\
&= \frac{a^2\beta}{4\pi}\frac{4\pi}{5}\frac{r_<^2}{r_>^3}\left[Y_{21}(\theta,\phi)+Y_{21}^*(\theta,\phi)\right] \\
&= \frac{a^2\beta}{4\pi}\frac{4\pi}{5}\frac{r_<^2}{r_>^3}\left[-2\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\cos\phi\right] \\
&= \frac{\alpha E_0\omega a^3}{5}\frac{r_<^2}{r_>^3}\sin\theta\cos\theta\cos\phi \\
&= \frac{\alpha E_0\omega a}{5}\frac{r_<^2}{r_>^3}\frac{xz}{r^2} = \frac{\alpha E_0\omega a}{5}xz\begin{cases}\frac{a^5}{r^5} & , \quad a<r \\ 1 & , \quad a\geq r\end{cases} \\
&= \frac{3}{5}\epsilon_0\frac{\epsilon_r-1}{\epsilon_r+2}E_0\omega xz\left(\frac{a}{r_>}\right)^5 \quad \text{q.e.d.}
\end{aligned}
$$

2. Jackson, Problem 6.15 **6 Points**

The Hall effect describes an electric field linear in the current density j up to a certain order in H. Here, the maximum order in H is specified to be two. Thus, forming all quantities that "look like" vectors, the electric field could be of the form

$$\mathbf{E} = a\mathbf{j} + b\mathbf{H} \times \mathbf{j} + c(\mathbf{H} \cdot \mathbf{H})\mathbf{j} + d(\mathbf{j} \cdot \mathbf{H})\mathbf{H} \quad .$$

A term $\propto \mathbf{H} \times (\mathbf{H} \times \mathbf{j})$ is not needed, because such a term would be a linear combination of the last two terms of the above. Now, to see which terms are in principle possible we check the transformation behavior of the terms.

a): Parity upon spatial inversion: $\mathbf{E}$ is odd (true vector = polar vector).

a-term: allowed, because $\mathbf{j}$ is odd.

b-term: allowed, because the cross product of an even vector ($\mathbf{H}$) with an odd one ($\mathbf{j}$) is odd.

c-term: allowed, because $\mathbf{H} \cdot \mathbf{H}$ is an even scalar, which, when multiplied with an odd vector ($\mathbf{j}$) results in an odd vector.

d-term: allowed, because $\mathbf{H} \cdot \mathbf{j}$ is an odd scalar, which, when multiplied with an even vector ($\mathbf{H}$) results in an odd vector.

Thus, all terms are allowed under spatial inversion. Relabeling of the constants yields the equation in Problem 6.15 part a).

b): Parity upon time reversal: $\mathbf{E}$ is a time-even vector.

a-term: not allowed, because $\mathbf{j}$ is t-odd.

b-term: allowed, because the cross product of a t-odd vector ($\mathbf{H}$) with another t-odd vector ($\mathbf{j}$) is t-even.

c-term: not allowed, because $\mathbf{H} \cdot \mathbf{H}$ is a t-even scalar, which, when multiplied with a t-odd vector ($\mathbf{j}$) results in a t-odd vector.

d-term: not allowed, because $\mathbf{H} \cdot \mathbf{j}$ is a t-odd scalar, which, when multiplied with a t-odd vector ($\mathbf{H}$) results in a t-odd vector.

Thus, only the usual b-term is allowed under time reversal.

Note 1. In analogy with the elaborations on page 272 of Jackson, terms with arbitrary-order time derivatives of $\mathbf{j}$ may be added, some of which would survive. Seemingly, Jackson only had a static situation in mind.

Note 2. The Ohm-type term $\mathbf{E} = a\mathbf{j}$ does not survive, because dissipation is not invariant under time reversal. To accommodate the (phenomenological) Ohm's law anyways, one can define a as a t-odd scalar resistivity....

3. Jackson, Problem 6.18 **6 Points**

a): Use

$$\mathbf{A}(\mathbf{x}) = \frac{g}{4\pi}\int_{z'=-\infty}^{z'=0} \frac{\hat{\mathbf{z}}dz' \times (\mathbf{x}-\hat{\mathbf{z}}z')}{|\mathbf{x}-\hat{\mathbf{z}}z'|^3}$$

Note $\hat{\mathbf{z}}dz' \times (\mathbf{x}-\hat{\mathbf{z}}z') = dz'(-\hat{\mathbf{x}}y+\hat{\mathbf{y}}x) = dz'\, r\sin\theta\hat{\phi}$ and $|\mathbf{x}-\hat{\mathbf{z}}z'| = r^2+z'^2-2rz'\cos\theta$. Thus, $\mathbf{A}$ is azimuthal and

$$\mathbf{A}(\mathbf{x}) = \hat{\phi}\frac{g}{4\pi}r\sin\theta\int_{z'=-\infty}^{z'=0}\frac{dz'}{\sqrt{r^2-2rz'\cos\theta+z'^2}^3} = \hat{\phi}\frac{g}{4\pi}r\sin\theta\left[\frac{1-\cos\theta}{r^2\sin^2\theta}\right] = \hat{\phi}\frac{g}{4\pi r}\frac{1-\cos\theta}{\sin\theta} \text{ q.e.d.}$$

b): $\mathbf{B} = \nabla\times\mathbf{A}$ for azimuthal $\mathbf{A}$ is

$$\mathbf{B}(\mathbf{x}) = \hat{\mathbf{r}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta A_\phi + \hat{\theta}\frac{-1}{r}\frac{\partial}{\partial r}rA_\phi$$

For the $\mathbf{A}$ from part a) the second term vanishes and

$$\mathbf{B}(\mathbf{x}) = \hat{\mathbf{r}}\frac{g}{4\pi r^2} \quad \text{q.e.d.}$$

c): Case $\theta < \pi/2$, upward flux.

$$\int \mathbf{B}\cdot\hat{\mathbf{n}}da = \int_{\cos\theta}^{1}\int_{\phi=0}^{2\pi}\mathbf{B}\cdot\hat{\mathbf{r}}r^2 d\cos\theta d\phi = \frac{g}{4\pi}2\pi\left[\cos\theta\right]_{\cos\theta}^{1} = \frac{g}{2}(1-\cos\theta)$$

Case $\theta > \pi/2$, upward flux

$$\int \mathbf{B}\cdot\hat{\mathbf{n}}da = \int_{-1}^{\cos\theta}\int_{\phi=0}^{2\pi}\mathbf{B}\cdot(-\hat{\mathbf{r}})r^2 d\cos\theta d\phi = -\frac{g}{2}\left[\cos\theta\right]_{-1}^{\cos\theta} = -\frac{g}{2}(\cos\theta+1)$$

This does not count any flux inside the string.

d): For any θ, it is

$$\oint \mathbf{A}\cdot d\mathbf{l} = \int_{\phi=0}^{2\pi}\frac{g}{4\pi r}\frac{1-\cos\theta}{\sin\theta}r\sin\theta d\phi = \frac{g}{2}(1-\cos\theta)$$

In the region $\theta < \pi/2$, this is the same as in part c), while for $\theta > \pi/2$ there is a constant difference of

$$\oint \mathbf{A}\cdot d\mathbf{l} - \int \mathbf{B}\cdot\hat{\mathbf{n}}da = g \quad .$$

Obviously, the difference g is the (upward) magnetic flux through the string, which is included in part d) but has been neglected in part c).

4. Jackson, Problem 7.3 **6 Points** Only the case of E perpendicular to the plane of incidence is to be considered.

The geometry of the problem is shown in the figure. Note that the internal up- and downward traveling waves with respective electric fields E_+ and E_- account for reflections up to infinite order.

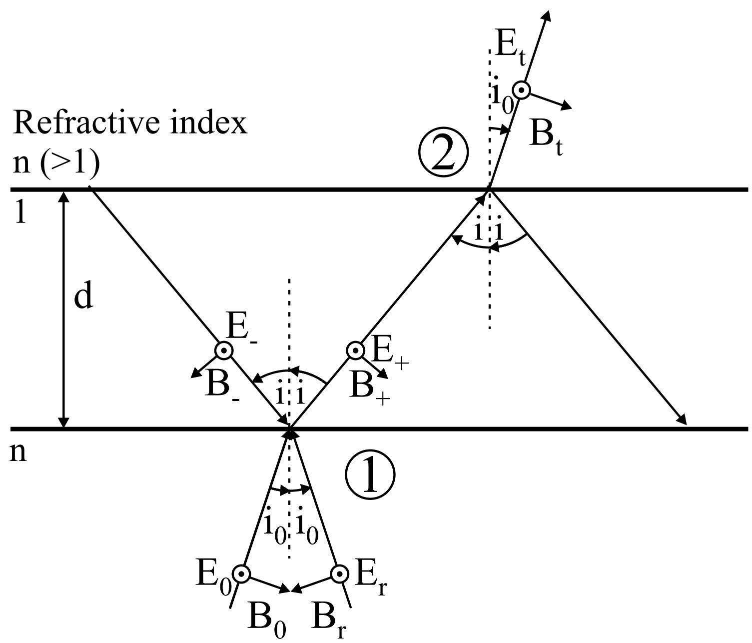

Figure 1: Geometry of the problem.

A sufficient set of equations is obtained as follows. Boundary conditions at point 1:

Equation 1. Tangential E-component:

$$E_+ + E_- = E_0 + E_r$$

Equation 2. Tangential H-component: $\cos(i_0)\,[H_0 - H_r] = \cos(i)\,[H_+ - H_-]$. Since $H = \frac{1}{\mu}B = \frac{\sqrt{\epsilon\mu}}{\mu}E$ and for non-permeable media $\mu = \mu_0$, $H = \frac{n}{c_0\mu_0}E$ with refractive index n and vacuum velocity of light c_0,

$$n\cos(i_0)\,[E_0 - E_r] = \cos(i)\,[E_+ - E_-]$$

Boundary conditions at point 2: Defining $\phi = \frac{kd}{\cos(i)}$, the internal electric-field amplitudes at location 2 are $E_+\exp(\mathrm{i}\phi)$ and $E_-\exp(-\mathrm{i}\phi)$ for the up- and downgoing waves, respectively. Thus,

Equation 3. Tangential E-component:

$$E_+\exp(\mathrm{i}\phi) + E_-\exp(-\mathrm{i}\phi) = E_t$$

Equation 4. Tangential H-component:

$$\cos(i)\left[E_+ \exp(\mathrm{i}\phi) - E_- \exp(-\mathrm{i}\phi)\right] = E_t n \cos(i_0)$$

Solution: The first two equations can be used to eliminate E_+ and E_-,

$$\begin{aligned} E_+ &= \frac{1}{2}\left[E_0 + E_r - \frac{n\cos(i_0)}{\cos(i)}(E_r - E_0)\right] \\ E_- &= \frac{1}{2}\left[E_0 + E_r + \frac{n\cos(i_0)}{\cos(i)}(E_r - E_0)\right] \quad ; \end{aligned}$$

call this Equation 5. Then, Equation 3 minus $\frac{1}{n\cos(i_0)}\times$ Equation 4 yields Equation 6):

$$E_+ \exp(\mathrm{i}\phi)(1 - \frac{1}{\alpha}) + E_- \exp(-\mathrm{i}\phi)(1 + \frac{1}{\alpha}) = 0$$

where $\alpha = \frac{n\cos(i_0)}{\cos(i)}$. Inserting 5 into 6 then yields

$$\frac{E_r}{E_0} = \frac{(\alpha^2 - 1)(\exp(2\mathrm{i}\phi) - 1)}{(\alpha - 1)^2 \exp(2\mathrm{i}\phi) - (\alpha + 1)^2}$$

Equation 3 plus $\frac{1}{n\cos(i_0)}\times$ Equation 4 yields

$$\frac{E_t}{E_0} = \frac{-4\alpha \exp(\mathrm{i}\phi)}{(\alpha - 1)^2 \exp(2\mathrm{i}\phi) - (\alpha + 1)^2}$$

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a): For i_0 less than the critical angle of total internal reflection, $i_0 < \sin^{-1}\left(\frac{1}{n}\right)$, we find the intensity reflection and transmission coefficients, R and T,

$$R = \frac{E_r}{E_0}\left(\frac{E_r}{E_0}\right)^* = \frac{2(\alpha^2 - 1)^2(1 - \cos(2\phi))}{(\alpha + 1)^4 + (\alpha - 1)^4 - 2(\alpha + 1)^2(\alpha - 1)^2 \cos(2\phi)}$$

$$T = \frac{E_t}{E_0}\left(\frac{E_t}{E_0}\right)^* = \frac{16\alpha^2}{(\alpha + 1)^4 + (\alpha - 1)^4 - 2(\alpha + 1)^2(\alpha - 1)^2 \cos(2\phi)}$$

Note that $R + T = 1$ and that both $\phi = \frac{kd}{\cos(i)}$ and $\alpha = \frac{n\cos(i_0)}{\cos(i)}$ are real.

For $i_0 > \sin^{-1}\left(\frac{1}{n}\right)$, write $\cos(i) = \mathrm{i}\sqrt{(n\sin(i_0))^2 - 1}$ (purely imaginary),

$$\beta := \exp(2\mathrm{i}\phi) = \exp\left(\frac{2kd}{\sqrt{(n\sin(i_0))^2 - 1}}\right) \quad \text{(real)},$$

$$\gamma := \frac{\alpha}{-\mathrm{i}} = \frac{n\cos(i_0)}{\sqrt{(n\sin(i_0))^2 - 1}} \quad \text{(real)},$$

to find

$$T = \frac{E_t}{E_0}\left(\frac{E_t}{E_0}\right)^* = \frac{16\gamma^2\beta}{(1-\gamma^2)^2(1-\beta)^2 + 4\gamma^2(1+\beta)^2}$$

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b: The transmitted power in the case $i_0 > \sin^{-1}\left(\frac{1}{n}\right)$ can be sketched by considering the limiting cases

$d \to 0$: Then, $\beta \to 1$ and $T \to 1$.

$d \gg \lambda_0$ (the vacuum wavelength): Then, $\beta \to \infty$ and $T \to \frac{16\gamma^2}{\beta(1+\gamma^2)^2} = \left(\frac{4\gamma}{1+\gamma^2}\right)^2 \exp\left(-\frac{4\pi d}{\lambda_0\sqrt{(n\sin(i_0))^2-1}}\right)$.

5. Jackson, Problem 7.4 **6 Points**

a): Use $\epsilon = \epsilon_b + \mathrm{i}\frac{\sigma}{\omega}$ (Eq. 7.57) and Eq. 7.39 for $i = 0$:

$$\frac{E_0''}{E_0} = \frac{n - n'}{n + n'} = \frac{1 - n'}{1 + n'} \quad \text{with} \quad n' = \sqrt{\frac{1}{\epsilon_0}\left(\epsilon_b + \mathrm{i}\frac{\sigma}{\omega}\right)}$$

Define r and ϕ via $n' = \sqrt{r}\exp(\mathrm{i}\phi/2)$. Then,

$$\begin{aligned} r &= \frac{1}{\epsilon_0}\sqrt{\epsilon_b^2 + \left(\frac{\sigma}{\omega}\right)^2} \\ \phi &= \tan^{-1}\left(\frac{\sigma}{\omega\epsilon_b}\right) \end{aligned}$$

Note that $\epsilon_b > 0$ and $\sigma \geq 0$; thus, $0 \leq \phi \leq \frac{\pi}{2}$. Then,

$$\frac{E_0''}{E_0} = \frac{1 - n'}{1 + n'} = \frac{1 - r - 2\mathrm{i}\sqrt{r}\sin\left(\frac{\phi}{2}\right)}{1 + r + 2\sqrt{r}\cos\left(\frac{\phi}{2}\right)}$$

Define a and δ via the amplitude reflectivity $\frac{E_0''}{E_0} = a\exp(\mathrm{i}\delta)$, where

$$\begin{aligned} a &= \frac{\sqrt{(1-r)^2 + 4r\sin^2\left(\frac{\phi}{2}\right)}}{1 + r + 2\sqrt{r}\cos\left(\frac{\phi}{2}\right)} \\ \delta &= \begin{cases} \tan^{-1}\left(\frac{-2\sqrt{r}\sin\left(\frac{\phi}{2}\right)}{1-r}\right) & , \quad 1 - r > 0 \\ \pi + \tan^{-1}\left(\frac{-2\sqrt{r}\sin\left(\frac{\phi}{2}\right)}{1-r}\right) & , \quad 1 - r < 0 \end{cases} \end{aligned}$$

b): Good conductor. In this case, $\sigma \rightarrow \infty$ and ϵ_b finite. Then, $\phi \rightarrow \frac{\pi}{2}$ and $r \rightarrow \frac{\sigma}{\epsilon_0\omega} \gg 1$, and $1 - r < 0$. Take the corresponding result from part a) to find

$$\begin{aligned} \delta &\rightarrow \pi \\ a &\rightarrow 1 - \sqrt{2\frac{\epsilon_0\omega}{\sigma}} \end{aligned}$$

The significance of $\delta = \pi$ is that the electric field undergoes a phase jump of π upon refection. Since $\frac{\epsilon_0\omega}{\sigma} \ll 1$, the intensity reflectivity $R = a^2 \rightarrow 1 - 2\sqrt{2\frac{\epsilon_0\omega}{\sigma}}$.

Also, the skin depth $s = \sqrt{\frac{2}{\mu\sigma\omega}}$, which for $\mu = \mu_0$ and vacuum velocity of light $c_0 = \frac{1}{\sqrt{\mu_0\epsilon_0}}$ is $s = c_0\sqrt{\frac{2\epsilon_0}{\sigma\omega}}$.

Thus, $\sqrt{\frac{2\epsilon_0\omega}{\sigma}} = \frac{s\omega}{c_0}$ and

$$R \to 1 - 2\frac{s\omega}{c_0} \quad \text{q.e.d.}$$

Poor conductor. In this case, $\sigma \to 0$. Then, $\phi = 0$, and

$$\begin{aligned} a &\to \frac{|1-r|}{1+r+2\sqrt{r}} = \frac{|\sqrt{r}-1|}{\sqrt{r}+1} = \frac{|n'-1|}{n'+1} \\ \delta &\to \begin{cases} 0 & , \quad n' = \sqrt{r} < 1 \\ \pi & , \quad n' = \sqrt{r} > 1 \end{cases} \end{aligned}$$

We thus recover the "normal" result Eq. 7.39 for reflection at an optically less dense (1st line) and an optically denser (2nd line) dielectric medium.