Physics 505 | Electricity and Magnetism | Fall 2003

Prof. G. Raithel

Midterm exam

Problem 1 **15 Points**

a): Insert the closure relation

$$\delta(x-x') = \sum_m \frac{2}{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{m\pi x'}{L}\right)$$

and the expansion

$$G(\mathbf{x},\mathbf{x}') = \sum_m A_m(y|x',y') \sin\left(\frac{m\pi x}{L}\right)$$

into the given equation for the Green's function and execute the $\partial^2/\partial x^2$-part of the Laplace operator to obtain

$$\sum_m \left[\frac{d^2}{dy^2} A_m(y|x',y') - \frac{m^2\pi^2}{L^2} A_m(y|x',y')\right] \sin\left(\frac{m\pi x}{L}\right) = \sum_m \left[-\frac{8\pi}{L}\delta(y-y') \sin\left(\frac{m\pi x'}{L}\right)\right] \sin\left(\frac{m\pi x}{L}\right)$$

The resultant equation for the reduced Green's function,

$$g_m(y,y') = \frac{A_m(y|x',y')}{-\frac{8\pi}{L}\sin\left(\frac{m\pi x'}{L}\right)} \quad ,$$

is

$$\left\{\frac{d^2}{dy^2} - \frac{m^2\pi^2}{L^2}\right\} g_m(y,y') = \delta(y,y')$$

Since the solutions of the homogeneous equation are exponentials, and since the boundary conditions are $g_m(y,y') = 0$ for y or y' equal to 0 or L, the reduced Green's function is of the form

$$g_m(y,y') = C \sinh\left(\frac{m\pi y_<}{L}\right) \sinh\left(\frac{m\pi(L-y_>)}{L}\right)$$

where $y_< = \min(y,y')$ and $y_> = \max(y,y')$. To find the constant C, we integrate around the δ-function and find, for infinitesimal ϵ,

$$
\begin{aligned}
\left.\frac{d}{dy} g_m(y,y')\right|_{y=y'+\epsilon} - \left.\frac{d}{dy} g_m(y,y')\right|_{y=y'-\epsilon} &= 1 \\
C\left[\sinh\left(\frac{m\pi y'}{L}\right)\cosh\left(\frac{m\pi(L-y')}{L}\right)\left(\frac{-m\pi}{L}\right) - \cosh\left(\frac{m\pi y'}{L}\right)\sinh\left(\frac{m\pi(L-y')}{L}\right)\left(\frac{m\pi}{L}\right)\right] &= 1 \\
C\left[-\sinh\left(m\pi\right)\frac{m\pi}{L}\right] &= 1 \\
C = -\frac{L}{m\pi\sinh\left(m\pi\right)}
\end{aligned}
$$

where we have used $\sinh(x)\cosh(y) + \cosh(x)\sinh(y) = \sinh(x+y)$. Inserting the results, it is

$$
g_m(y,y') = -\frac{L}{m\pi\sinh\left(m\pi\right)}\sinh\left(\frac{m\pi y_<}{L}\right)\sinh\left(\frac{m\pi(L-y_>)}{L}\right)
$$

$$
A_m = \frac{8}{m\sinh(m\pi)}\sinh\left(\frac{m\pi y_<}{L}\right)\sinh\left(\frac{m\pi(L-y_>)}{L}\right)\sin\left(\frac{m\pi x'}{L}\right)
$$

$$
G(\mathbf{x},\mathbf{x}') = \sum_{m=1}^{\infty}\frac{8}{m\sinh(m\pi)}\sinh\left(\frac{m\pi y_<}{L}\right)\sinh\left(\frac{m\pi(L-y_>)}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{m\pi x'}{L}\right)
$$

b): The normalized eigenfunctions of

$$
\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \lambda_{mn}\right\}\psi_{mn}(x,y) = 0
$$

are $\psi_{mn}(x,y) = \frac{2}{L}\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right)$, and the corresponding eigenvalues are $\lambda_{mn} = \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{L}\right)^2$. Using Eq. 3.160 of the textbook, we find

$$
G(\mathbf{x},\mathbf{x}') = 4\pi\sum_{mn}\frac{\psi^*_{mn}(x',y')\psi_{mn}(x,y)}{\lambda_{mn}} = \frac{16}{\pi}\sum_{m,n=1}^{\infty}\frac{\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right)\sin\left(\frac{m\pi x'}{L}\right)\sin\left(\frac{n\pi y'}{L}\right)}{m^2+n^2}
$$

c): For zero charge density and given potential on the boundary, the potential is

$$
\Phi(x,y) = -\frac{1}{4\pi}\int_{\partial S} V(x',y')\frac{\partial G(x,y,x',y')}{\partial n'}dl'
$$

For the given case, we only need $\frac{\partial G(x,y,x',y')}{\partial n'}$ on the left side of the square, where it is

$$
\frac{\partial G(x,y,x',y')}{\partial n'} = -\left.\frac{\partial G(x,y,x',y')}{\partial x'}\right|_{x'=0}
$$

Using the result of part b), we have

$$\frac{\partial G(x,y,x',y')}{\partial n'} = -\frac{16}{L} \sum_{m,n=1}^{\infty} \frac{m}{m^2+n^2} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{n\pi y'}{L}\right)$$

Thus,

$$\Phi(x,y) = \frac{4V_0}{L\pi} \sum_{m,n=1}^{\infty} \frac{m}{m^2+n^2} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{L}\right) \int_{y'=0}^{y'=L} \sin\left(\frac{n\pi y'}{L}\right) \sin\left(\frac{3\pi y'}{L}\right) dy'$$

$$\Phi(x,y) = \frac{2V_0}{\pi} \sin\left(\frac{3\pi y}{L}\right) \sum_{m=1}^{\infty} \frac{m}{m^2+9} \sin\left(\frac{m\pi x}{L}\right)$$

Note 1: I have checked that the same answer is obtained when using the result of part a), as expected. However, when using the Green's function from part a) one cannot employ any orthogonality condition when integrating over dy'. As a result, the calculation is lengthy.

Note 2: Due to symmetry, in the result of part a) x and y-variables can be swapped. If the resultant Green's function is used for c), a closed result for $\Phi(x,y)$ is obtained:

$$\Phi(x,y) = \frac{V_0}{\sinh(3\pi)} \sin\left(\frac{3\pi y}{L}\right) \sinh\left(\frac{3\pi(L-x)}{L}\right)$$

Note 3: The result of Note 2 can also be obtained by direct variable separation.

Note 4: Some students replaced both δ-functions in part a) with closure relations and expanded the Green's function as a double sum. This method essentially is an alternate route to arrive at the eigenfunction expansion, which is asked for in part b).

Problem 2 **15 Points**

a): Interior potential ($r < a$): $\Phi_1 = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$

Exterior potential ($a < r < b$): $\Phi_2 = \sum_{l=0}^{\infty} \left[B_l r^l + C_l r^{-l-1}\right] P_l(\cos\theta)$

There are three boundary conditions: the potential must be zero at $r = b$, the radial D-field component must satisfy $D_{r,2} - D_{r,1} = \sigma$ at $r = a$, and the tangential E-field component must satisfy $E_{\theta,2} - E_{\theta,1} = 0$ at $r = a$.

The first boundary condition is satisfied by setting

$$B_l b^l + C_l b^{-l-1} = 0 \quad \Leftrightarrow \quad C_l = -B_l b^{2l+1} \quad .$$

The second boundary condition reads

$$\begin{aligned} D_{r,2} - D_{r,1} &= \sigma(\theta) \\ -\epsilon_2 \frac{\partial}{\partial r}\Phi_2(r,\theta)\bigg|_{r=a} + \epsilon_1 \frac{\partial}{\partial r}\Phi_1(r,\theta)\bigg|_{r=a} &= \sigma_0 P_1(\cos\theta) \\ \sum_{l=0}^{\infty}\left\{\epsilon_1 A_l l a^{l-1} - \epsilon_2 B_l\left[l a^{l-1} + b^{2l+1}(l+1)a^{-l-2}\right]\right\} P_l(\cos\theta) &= \sigma_0 P_1(\cos\theta) \\ \epsilon_1 A_l l a^{l-1} - \epsilon_2 B_l\left[l a^{l-1} + b^{2l+1}(l+1)a^{-l-2}\right] &= \sigma_0 \delta_{1,l} \quad \forall \quad l \end{aligned}$$

The third boundary condition,

$$\begin{aligned} E_{\theta,2} - E_{\theta,1} &= 0 \\ -\frac{\partial}{r\partial\theta}\Phi_2(r,\theta)\bigg|_{r=a} + \frac{\partial}{r\partial\theta}\Phi_1(r,\theta)\bigg|_{r=a} &= 0 \\ \sum_{l=1}^{\infty}\left\{A_l a^{l-1} + B_l\left[-a^{l-1} + b^{2l+1}a^{-l-2}\right]\right\}\frac{\partial}{\partial\theta}P_l(\cos\theta) &= 0 \\ A_l a^l + B_l\left[-a^l + b^{2l+1}a^{-l-1}\right] &= 0 \quad \forall \quad l > 0 \end{aligned}$$

is equivalent to $\Phi_2(r,\theta)|_{r=a} = \Phi_1(r,\theta)|_{r=a}$, except in the case $l = 0$, in which the continuity of the tangential field does not provide a condition. Since the potential has to be continuous across surface charges in general, we extend the last equation to include the case $l = 0$, making it entirely equivalent with $\Phi_2(r,\theta)|_{r=a} = \Phi_1(r,\theta)|_{r=a}$:

$$A_l a^l + B_l\left[-a^l + b^{2l+1}a^{-l-1}\right] = 0 \quad \forall \quad l$$

In matrix form, the second and third boundary conditions read

$$\begin{pmatrix} \epsilon_1 l a^{l-1} & -\epsilon_2\left[l a^{l-1} + b^{2l+1}(l+1)a^{-l-2}\right] \\ a^l & \left[-a^l + b^{2l+1}a^{-l-1}\right] \end{pmatrix}\begin{pmatrix} A_l \\ B_l \end{pmatrix} = \begin{pmatrix} \sigma_0\delta_{1,l} \\ 0 \end{pmatrix}$$

The determinant

$$det = la^{2l-1}(\epsilon_2 - \epsilon_1) + b^{2l+1}a^{-2}(l\epsilon_1 + (l+1)\epsilon_2) \neq 0 \quad \forall \quad l \quad .$$

Thus, $A_l = B_l = 0$ unless $l = 1$. In the latter case,

$$\begin{aligned} A_1 &= \frac{\sigma_0(b^3 - a^3)}{a^3(\epsilon_2 - \epsilon_1) + b^3(2\epsilon_2 + \epsilon_1)} \\ B_1 &= \frac{-\sigma_0 a^3}{a^3(\epsilon_2 - \epsilon_1) + b^3(2\epsilon_2 + \epsilon_1)} \end{aligned}$$

The potential thus is

$$\begin{aligned} \Phi_1(r,\theta) &= \frac{\sigma_0(b^3 - a^3)}{a^3(\epsilon_2 - \epsilon_1) + b^3(2\epsilon_2 + \epsilon_1)} r\cos\theta \quad , \quad r < a \\ \Phi_2(r,\theta) &= \frac{-\sigma_0 a^3}{a^3(\epsilon_2 - \epsilon_1) + b^3(2\epsilon_2 + \epsilon_1)} \left[r - \frac{b^3}{r^2} \right] \cos\theta \quad , \quad a < r < b \end{aligned}$$

b): The radial component of the electric field, $E_r = -\frac{\partial}{\partial r}\Phi(r,\theta)$, is

$$\begin{aligned} E_{r,1}(r,\theta) &= -\frac{\sigma_0(b^3 - a^3)}{a^3(\epsilon_2 - \epsilon_1) + b^3(2\epsilon_2 + \epsilon_1)} \cos\theta \quad , \quad r < a \\ E_{r,2}(r,\theta) &= \frac{\sigma_0 a^3}{a^3(\epsilon_2 - \epsilon_1) + b^3(2\epsilon_2 + \epsilon_1)} \left[1 + 2\frac{b^3}{r^3} \right] \cos\theta \quad , \quad a < r < b \end{aligned}$$

The tangential component of the electric field, $E_\theta = -\frac{\partial}{r\partial\theta}\Phi(r,\theta)$, is

$$\begin{aligned} E_{\theta,1}(r,\theta) &= \frac{\sigma_0(b^3 - a^3)}{a^3(\epsilon_2 - \epsilon_1) + b^3(2\epsilon_2 + \epsilon_1)} \sin\theta \quad , \quad r < a \\ E_{\theta,2}(r,\theta) &= \frac{-\sigma_0 a^3}{a^3(\epsilon_2 - \epsilon_1) + b^3(2\epsilon_2 + \epsilon_1)} \left[1 - \frac{b^3}{r^3} \right] \sin\theta \quad , \quad a < r < b \end{aligned}$$

Note that the inner field is a homogeneous field, and the outer field is the superposition of a homogeneous field and a dipole field.

c): The macroscopic polarization is $\mathbf{P}(r,\theta) = (\epsilon_i - \epsilon_0)\mathbf{E}_i(r,\theta)$, with $i = 1$ or $i = 2$.

d): Due the absence of free volume charge densities, the volume polarization charge is zero ($\rho_{\text{pol}} = -\nabla \cdot \mathbf{P} = -(1 - \epsilon_0/\epsilon)\nabla \cdot \mathbf{D} = (\epsilon_0/\epsilon - 1)\rho_{\text{free}} = 0$).

Explicit calculation from the fields is not required, but it could proceed as follows:

$$\rho_{\text{pol}} = -\nabla \cdot \mathbf{P} = -(\epsilon_i - \epsilon_0)\nabla \cdot \mathbf{E}_i = -(\epsilon_i - \epsilon_0) \left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_{r,i}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta E_{\theta,i}) \right] ,$$

with $i = 1$ or $i = 2$. The result is always zero.

On the surface $r = a$, the surface polarization charge density is

$$\begin{aligned}
\sigma_{\mathrm{pol,r=a}} &= P_{r,1} - P_{r,2} = (\epsilon_1 - \epsilon_0)E_{r,1} - (\epsilon_2 - \epsilon_0)E_{r,2} \\
\sigma_{\mathrm{pol,r=a}} &= \sigma_0 \cos\theta \left(\frac{a^3(\epsilon_1 - \epsilon_2) + b^3(3\epsilon_0 - \epsilon_1 - 2\epsilon_2)}{a^3(\epsilon_2 - \epsilon_1) + b^3(2\epsilon_2 + \epsilon_1)} \right) \quad ,
\end{aligned}$$

which equals 0 for $\epsilon_1 = \epsilon_2 = \epsilon_0$ (as required). On the surface $r = b$, the surface polarization charge density is

$$\begin{aligned}
\sigma_{\mathrm{pol,r=b}} &= P_{r,2} - P_{r,conductor} = (\epsilon_2 - \epsilon_0)E_{r,2} \\
\sigma_{\mathrm{pol,r=b}} &= \sigma_0 \cos\theta \left(\frac{3a^3(\epsilon_2 - \epsilon_0)}{a^3(\epsilon_2 - \epsilon_1) + b^3(2\epsilon_2 + \epsilon_1)} \right) \quad ,
\end{aligned}$$

which equals 0 for $\epsilon_2 = \epsilon_0$ (as required). The free induced charge density on the conductor is

$$\begin{aligned}
\sigma_{\mathrm{free,r=b}} &= -D_{r,2} = -\epsilon_2 E_{r,2} \\
\sigma_{\mathrm{free,r=b}} &= \sigma_0 \cos\theta \left(\frac{-3\epsilon_2 a^3}{a^3(\epsilon_2 - \epsilon_1) + b^3(2\epsilon_2 + \epsilon_1)} \right) \quad .
\end{aligned}$$

Note 1: The E- and D-fields are zero in the conductor. Application of the respective versions of Gauss's law then shows that both the total and the free charges need to add up to zero. This result is guaranteed by the $\cos\theta$-dependence of all involved charge densities.

Note 2: In the conductor region, the E-field is zero. Thus, when accounting for both free and polarization charges explicitly, all generated multipole moments must vanish. In Note 1 we have already seen that the monopole moment is zero. Further, due to the $\cos\theta$-dependence of all involved charge densities, the only other multipole moment that could possibly arise in the region $r > b$ is that of a net dipole moment $\mathbf{p}$ in the z-direction. Since p_z would produce a field $\propto \frac{1}{r^3}$ in the conductor, our results must satisfy the condition $p_z = \sum_k p_{k,z} = \sum \frac{4\pi}{3}\sigma_k r_k^3 = 0$, where the σ_k are the pre-factors of the $\cos\theta$-terms of the surface charge densities, and r_k are their radii. In the present case, there are two surfaces with free charge, and two with polarization charge (*i.e.* $k = 1, 2, 3, 4$), and the radii are either a or b. A calculation with the above results shows $p_z = 0$.

Note 3: The case $\epsilon_i = \epsilon_0$ can also be treated with the Green's function given in Eq. 3.125 of the textbook; see also Chapter 3.10.