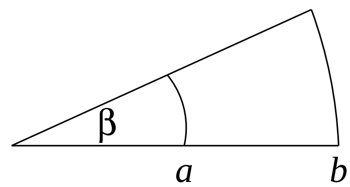
Physics 505 **Fall 2005**

Practice Midterm — Solutions

The midterm will be a 120 minute open book, open notes exam. Do all three problems.

1. A two-dimensional problem is defined by a semi-circular wedge with $0 \le \phi \le \beta$ and $a \le \rho \le b$.



a) For the Dirichlet problem, it is possible to expand the Green's function as

$$G(\rho,\phi;\rho',\phi') = \sum_{m=1}^{\infty} g_m(\rho,\rho') \sin\left(\frac{m\pi\phi}{\beta}\right) \sin\left(\frac{m\pi\phi'}{\beta}\right)$$

Write down the appropriate differential equation that $g_m(\rho,\rho')$ must satisfy.

Note that the expansion in terms of $\sin(m\pi\phi/\beta)$ is designed to satisfy Dirichlet boundary counditions on the straight segments of the wedge. The Green's function equation we wish to solve is

$$\nabla^2_{x'} G(\rho,\phi;\rho',\phi') = -4\pi\delta^2(\vec{x}-\vec{x}\,') = -\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')$$

By completeness, we have

$$\sum_{m=1}^{\infty} \sin\left(\frac{m\pi\phi}{\beta}\right) \sin\left(\frac{m\pi\phi'}{\beta}\right) = \frac{\beta}{2}\delta(\phi-\phi')$$

Hence

$$\nabla^2_{x'} G(\rho,\phi;\rho',\phi') = -\frac{8\pi}{\beta\rho}\delta(\rho-\rho') \sum_{m=1}^{\infty} \sin\left(\frac{m\pi\phi}{\beta}\right) \sin\left(\frac{m\pi\phi'}{\beta}\right) \qquad (1)$$

Using the polar coordinate expression for the Laplacian, we find

$$\nabla^2_{x'} G(\rho,\phi;\rho',\phi') = \sum_{m=1}^{\infty} \left[\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\rho'\frac{\partial}{\partial\rho'} - \frac{1}{\rho'^2}\left(\frac{m\pi}{\beta}\right)^2\right] g_m(\rho,\rho') \times \sin\left(\frac{m\pi\phi}{\beta}\right) \sin\left(\frac{m\pi\phi'}{\beta}\right)$$

Comparing this with (1) yields the ODE

$$\left[\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\rho'\frac{\partial}{\partial\rho'} - \frac{1}{\rho'^2}\left(\frac{m\pi}{\beta}\right)^2\right] g_m(\rho,\rho') = -\frac{8\pi}{\beta\rho}\delta(\rho-\rho')$$

which may be converted into Sturm-Liouville form by multiplying by ρ'

$$\left[\frac{\partial}{\partial\rho'}\rho'\frac{\partial}{\partial\rho'} - \frac{1}{\rho'}\left(\frac{m\pi}{\beta}\right)^2\right] g_m(\rho,\rho') = -\frac{8\pi}{\beta}\delta(\rho-\rho')$$

b) Solve the Green's function equation for $g_m(\rho,\rho')$ subject to Dirichlet boundary conditions and write down the result for $G(\rho,\phi;\rho',\phi')$.

The Dirichlet boundary conditions are that $g_m(\rho,\rho')$ vanish when $\rho' = a$ or b, namely $g_m(\rho,a) = g_m(\rho,b) = 0$. For these homogeneous boundary conditions, the Green's function takes the form

$$g_m(\rho,\rho') = -\frac{8\pi}{\beta A}u(\rho_<)v(\rho_>)$$

where $u(\rho')$ and $v(\rho')$ are solutions to the homogeneous equation satisfying boundary conditions $u(a) = v(b) = 0$, and A is related to the Wronskian by $W(u,v) = -A/\rho'$. Noting that the solution to the homogeneous radial equation has the form

$$g_m(\rho,\rho') \sim \rho^{\pm m\pi/\beta}$$

it is easy to write down the appropriate $u(\rho')$ and $v(\rho')$

$$u = \rho'^{\frac{m\pi}{\beta}}\left(1-\left(\frac{a}{\rho'}\right)^{\frac{2m\pi}{\beta}}\right) \qquad v = \rho'^{\frac{m\pi}{\beta}}\left(1-\left(\frac{b}{\rho'}\right)^{\frac{2m\pi}{\beta}}\right)$$

Computing the Wronskian yields

$$\begin{aligned} W = uv' - vu' &= \frac{m\pi}{\beta\rho'}\rho'^{\frac{2m}{\beta}}\left(1-\left(\frac{a}{\rho'}\right)^{\frac{2m\pi}{\beta}}\right)\left(1+\left(\frac{b}{\rho'}\right)^{\frac{2m\pi}{\beta}}\right) \\ &\quad - \frac{m\pi}{\beta\rho'}\rho'^{\frac{2m}{\beta}}\left(1-\left(\frac{b}{\rho'}\right)^{\frac{2m\pi}{\beta}}\right)\left(1+\left(\frac{a}{\rho'}\right)^{\frac{2m\pi}{\beta}}\right) \\ &= \frac{1}{\rho'}\left(\frac{2m\pi}{\beta}\right)\left(b^{\frac{2m\pi}{\beta}} - a^{\frac{2m\pi}{\beta}}\right) \end{aligned}$$

As a result

$$\begin{aligned} g_m(\rho,\rho') &= -\frac{4}{m}\frac{(\rho_<\rho_>)^{\frac{m\pi}{\beta}}}{b^{\frac{2m\pi}{\beta}} - a^{\frac{2m\pi}{\beta}}}\left(1-\left(\frac{a}{\rho_<}\right)^{\frac{2m\pi}{\beta}}\right)\left(1-\left(\frac{b}{\rho_>}\right)^{\frac{2m\pi}{\beta}}\right) \\ &= \frac{4}{m}\left(\frac{\rho_<}{\rho_>}\right)^{\frac{m\pi}{\beta}}\frac{\left(1-\left(\frac{a}{\rho_<}\right)^{\frac{2m\pi}{\beta}}\right)\left(1-\left(\frac{\rho_>}{b}\right)^{\frac{2m\pi}{\beta}}\right)}{1-\left(\frac{a}{b}\right)^{\frac{2m\pi}{\beta}}} \end{aligned}$$

Combining this with the angular functions yields the final result

$$G(\rho,\phi;\rho',\phi') = \sum_{m=1}^{\infty} \frac{4}{m}\left(\frac{\rho_<}{\rho_>}\right)^{\frac{m\pi}{\beta}} \frac{\left(1-\left(\frac{a}{\rho_<}\right)^{\frac{2m\pi}{\beta}}\right)\left(1-\left(\frac{\rho_>}{b}\right)^{\frac{2m\pi}{\beta}}\right)}{1-\left(\frac{a}{b}\right)^{\frac{2m\pi}{\beta}}} \times \sin\left(\frac{m\pi\phi}{\beta}\right)\sin\left(\frac{m\pi\phi'}{\beta}\right)$$

Note that this has the expected behavior as either $a \to 0$ or $b \to \infty$.

2. A conducting spherical shell of inner radius a is held at zero potential. The interior of the shell is filled with electric charge of a volume density

$$\rho(\vec{r}) = \rho_0 \left(\frac{a}{r}\right)^2 \sin^2\theta$$

a) Find the potential everywhere inside the shell. To obtain the potential, we make use of the Green's function for the interior of a conducting sphere

$$G(\vec{x},\vec{x}\,') = 4\pi \sum_{l,m} \frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}\left(1-\left(\frac{r_>}{a}\right)^{2l+1}\right) Y_{lm}^*(\Omega')Y_{lm}(\Omega)$$

Actually, because of azimuthal symmetry, we only need the $m=0$ components of the Green's function expansion

$$G(\vec{x},\vec{x}\,') = \sum_l \frac{r_<^l}{r_>^{l+1}}\left(1-\left(\frac{r_>}{a}\right)^{2l+1}\right) P_l(\cos\theta')P_l(\cos\theta) \quad + \quad (m \neq 0)$$

Although the charge density is specified in terms of $\sin^2\theta$, this can be converted into Legendre polynomials. Since $\sin^2\theta = 1-\cos^2\theta$, and since $P_l(\cos\theta)$ is of degree $(\cos\theta)^l$, we see that $\sin^2\theta$ has to be a linear combination of P_0 and P_2. It is not too hard to see that

$$\sin^2\theta = \tfrac{2}{3}[P_0(\cos\theta) - P_2(\cos\theta)]$$

We now note that since the surface is held at zero potential the solution in the interior is given by

$$\begin{aligned}\Phi(\vec{x}\,) &= \frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}\,')G(\vec{x},\vec{x}\,')\,d^3x' \\ &= \frac{\rho_0 a^2}{4\pi\epsilon_0}\sum_l P_l(\cos\theta)\int \tfrac{2}{3}[P_0(\cos\theta') - P_2(\cos\theta')] \\ &\qquad \times P_l(\cos\theta')\frac{r_<^l}{r_>^{l+1}}\left(1-\left(\frac{r_>}{a}\right)^{2l+1}\right) dr'\,d\phi'\,d(\cos\theta')\end{aligned}$$

By orthogonality of Legendre polynomials, this becomes

$$\begin{aligned}
\Phi(\vec{x}\,) &= \frac{2\rho_0 a^2}{3\epsilon_0}\left[P_0(\cos\theta)\int_0^a \frac{1}{r_>}\left(1-\left(\frac{r_>}{a}\right)\right)\,dr'\right.\\
&\qquad \left. -\frac{1}{5}P_2(\cos\theta)\int_0^a \frac{r_<^2}{r_>^3}\left(1-\left(\frac{r_>}{a}\right)^5\right)\,dr'\right]\\
&= \frac{2\rho_0 a^2}{3\epsilon_0}\left[P_0(\cos\theta)\int_0^a \left(\frac{1}{r_>}-\frac{1}{a}\right)\,dr'\right.\\
&\qquad \left. -\frac{1}{5}P_2(\cos\theta)\int_0^a r_<^2\left(\frac{1}{r_>^3}-\frac{r_>^2}{a^5}\right)\,dr'\right]\\
&= \frac{2\rho_0 a^2}{3\epsilon_0}\left[P_0(\cos\theta)\left(\left(\frac{1}{r}-\frac{1}{a}\right)\int_0^r dr' + \int_r^a \left(\frac{1}{r'}-\frac{1}{a}\right)\,dr'\right)\right.\\
&\qquad \left. -\frac{1}{5}P_2(\cos\theta)\left(\left(\frac{1}{r^3}-\frac{r^2}{a^5}\right)\int_0^r r'^2\,dr' + r^2\int_r^a \left(\frac{1}{r'^3}-\frac{r'^2}{a^5}\right)\,dr'\right)\right]\\
&= \frac{2\rho_0 a^2}{3\epsilon_0}\left[P_0(\cos\theta)\ln\frac{a}{r}-\frac{1}{6}P_2(\cos\theta)\left(1-\left(\frac{r}{a}\right)^2\right)\right]
\end{aligned}$$

Inserting the expressions for Legendre polynomials, this becomes

$$\Phi(\vec{x}\,') = \frac{2\rho_0 a^2}{3\epsilon_0}\left[\ln\frac{a}{r}-\frac{1}{12}\left(1-\left(\frac{r}{a}\right)^2\right)(3\cos^2\theta-1)\right]$$

b) What is the surface charge density on the inside surface of the shell?

The surface charge density is given by

$$\begin{aligned}
\sigma = \epsilon_0\left.\frac{\partial\Phi}{\partial r}\right|_{r=a} &= \frac{2\rho_0 a^2}{3}\left[-\frac{1}{r}+\frac{1}{6}\frac{r}{a^2}(3\cos^2\theta-1)\right]_{r=a}\\
&= -\frac{2\rho_0 a}{3}\left(1-\tfrac{1}{6}(3\cos^2\theta-1)\right)
\end{aligned}$$

Note that only the $l=0$ term contributes to the total charge induced on the shell. This is simply

$$Q_{\text{shell}} = -\frac{2\rho_0 a}{3}(4\pi a^2) = -\frac{8\pi\rho_0 a^3}{3}$$

This is the negative of the charge contained in the interior

$$Q_{\text{inside}} = \int \rho(\vec{x}\,)\,d^3x = 2\pi\rho_0 a^3\int \sin^2\theta\,d(\cos\theta) = \frac{8\pi\rho_0 a^3}{3}$$

3. A thin disk of radius a lies in the x-y plane with its center at the coordinate origin. The disk is uniformly charged with a surface density σ.

a) Calculate the multipole moments of the charge distribution. Make sure to indicate which moments are non-vanishing.

The volume charge density for the disk can be written as

$$\rho(\vec{x}\,) = \frac{\sigma}{r}\delta(\cos\theta)$$

(provided $r < a$). Note that the factor of $1/r$ ensures uniform surface charge density since

$$d\rho = \rho(\vec{x}\,)\,d^3x = \frac{\sigma}{r}\delta(\cos\theta)r^2\,dr\,d\phi\,d(\cos\theta) = \sigma r\,dr\,d\phi\big|_{\theta=\pi/2}$$

and $r\,dr\,d\phi$ is the standard area element in polar coordinates. The multipole moments are then given by

$$q_{lm} = \int r^l Y_{lm}^*(\Omega) r(\vec{x}\,)\,d^3x = \sigma\int r^{l+1}Y_{lm}(\theta,\phi)\delta(\cos\theta)\,dr\,d\phi\,d(\cos\theta)$$

By azimuthal symmetry, only the $m = 0$ moments are non-vanishing. Integrating the ϕ and θ angles gives

$$q_{l,0} = 2\pi\sigma Y_{l,0}(\frac{\pi}{2},0)\int_0^a r^{l+1}\,dr = 2\pi\sigma\sqrt{\frac{2l+1}{4\pi}}P_l(0)\frac{a^{l+2}}{l+2}$$

(Note that $Y_{l,0}$ is independent of ϕ.) Since the Legendre polynomials are even and odd depending on l, we see that only even l moments are non-vanishing

$$q_{2k,0} = \frac{\sqrt{(4k+1)\pi}P_l(0)}{2k+2}\sigma a^{2k+2} = \frac{(-)^k\sqrt{4k+1}\Gamma(k+\frac{1}{2})}{2(k+1)!}\sigma a^{2k+2}$$

Since the disk is uniformly charged, the total charge is simply $q = \sigma(\pi a^2)$. This allows us to write

$$q_{2k,0} = \frac{(-)^k\sqrt{4k+1}\Gamma(k+\frac{1}{2})}{2\pi(k+1)!}qa^{2k}$$

The first two non-vanishing moments are

$$q_{00} = \sqrt{\frac{1}{4\pi}}q \qquad q_{20} = -\frac{1}{4}\sqrt{\frac{5}{4\pi}}qa^2$$

b) Write down the multipole expansion for the potential in explicit form up to the first two non-vanishing terms.

The multipole expansion yields

$$\begin{aligned}
\Phi(\vec{x}\,) &= \frac{1}{4\pi\epsilon_0}4\pi\left[q_{00}\frac{Y_{00}(\Omega)}{r} + \frac{1}{5}q_{20}\frac{Y_{20}(\Omega)}{r^3} + \cdots\right] \\
&= \frac{q}{4\pi\epsilon_0}4\pi\left[\sqrt{\frac{1}{4\pi}}\frac{1}{r}\sqrt{\frac{1}{4\pi}} - \frac{1}{20}\sqrt{\frac{5}{4\pi}}\frac{a^2}{r^3}\sqrt{\frac{5}{4\pi}}P_2(\cos\theta) + \cdots\right] \\
&= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r} - \frac{1}{4}\frac{a^2}{r^3}P_2(\cos\theta) + \cdots\right] = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r} - \frac{1}{8}\frac{a^2}{r^3}(3\cos^2\theta - 1) + \cdots\right]
\end{aligned}$$