Physics 505 **Fall 2005**

Homework Assignment #11 — Solutions

Textbook problems: Ch. 7: 7.3, 7.5, 7.8, 7.16

7.3 Two plane semi-infinite slabs of the same uniform, isotropic, nonpermeable, lossless dielectric with index of refraction n are parallel and separated by an air gap ($n = 1$) of width d. A plane electromagnetic wave of frequency ω is indicent on the gap from one of the slabs with angle of indicence i. For linear polarization *both* parallel to *and* perpendicular to the plane of incidence,

a) calculate the ratio of power transmitted into the second slab to the incident power and the ratio of reflected to incident power;

We introduce (complex) electric field vectors of the form $\vec{E}_i e^{i\vec{k}\cdot\vec{x}}$ and $\vec{E}_r e^{-i\vec{k}\cdot\vec{x}}$ on the incident side, $\vec{E}_+ e^{i\vec{k}_0\cdot\vec{x}}$ and $\vec{E}_- e^{-i\vec{k}_0\cdot\vec{x}}$ in the air gap, and $\vec{E}_t e^{i\vec{k}\cdot(\vec{x}-\vec{d})}$ on the transmitted side. (We have removed an unimportant phase from the transmitted side by shifting $\vec{x}$ by the vector $\vec{d}$ pointing from the incident to the transmitted side of the air gap. If i is the incident angle, then the angle r from the normal in the air gap is given by Snell's law, $n\sin i = \sin r$, and the transmitted angle is also i (because it is the same dielectric). We see that

$$\cos r = \sqrt{1-\sin^2 r} = \sqrt{1-n^2\sin^2 i}$$

and that $\cos r$ is purely imaginary in the event that i is greater than the critical angle for total internal reflection. To obtain E_t and E_r in terms of E_i, we may match the parallel components of $\vec{E}$ as well as the parallel components of $\vec{H}$. We consider two cases.

For $\vec{E}$ perpendicular to the plane of incidence, the matching becomes

	first interface	second interface
$E^{\parallel}:$	$E_i + E_r = E_+ + E_-,$	$E_+e^{i\phi} + E_-e^{-i\phi} = E_t$
$H^{\parallel}:$	$n(E_i - E_r)\cos i = (E_+ - E_-)\cos r,$	$(E_+e^{i\phi} - E_-e^{-i\phi})\cos r = nE_t\cos i$

where we have introduced the phase

$$\phi = \vec{k}_0\cdot\vec{d} = k_0 d\cos r = \frac{\omega d\cos r}{c}$$

The matching conditions at the first interface may be written as

$$\begin{aligned} E_+ &= \tfrac{1}{2}E_i(1+\alpha) + \tfrac{1}{2}E_r(1-\alpha) \\ E_- &= \tfrac{1}{2}E_i(1-\alpha) + \tfrac{1}{2}E_r(1+\alpha) \end{aligned} \tag{1}$$

where we have defined

$$\alpha = \frac{n\cos i}{\cos r} = \frac{n\cos i}{\sqrt{1-n^2\sin^2 i}}$$

Similarly, the matching conditions at the second interface yield

$$\begin{aligned} E_+ &= \tfrac{1}{2}e^{-i\phi}E_t(1+\alpha) \\ E_- &= \tfrac{1}{2}e^{i\phi}E_t(1-\alpha) \end{aligned} \tag{2}$$

Equating (1) and (2) allows us to solve for the ratios

$$\begin{aligned} \frac{E_t}{E_i} &= \frac{4\alpha}{(1+\alpha)^2e^{-i\phi}-(1-\alpha)^2e^{i\phi}} = \frac{2\alpha}{2\alpha\cos\phi - i(1+\alpha^2)\sin\phi} \\ \frac{E_r}{E_i} &= \frac{(1-\alpha^2)(e^{i\phi}-e^{-i\phi})}{(1+\alpha)^2e^{-i\phi}-(1-\alpha)^2e^{i\phi}} = \frac{i(1-\alpha^2)\sin\phi}{2\alpha\cos\phi - i(1+\alpha^2)\sin\phi} \end{aligned} \tag{3}$$

where

$$\alpha = \frac{n\cos i}{\sqrt{1-n^2\sin^2 i}}, \qquad \phi = \frac{\omega d\cos r}{c} = \frac{\omega d\sqrt{1-n^2\sin^2 i}}{c}$$

So long as i is below the critical angle, both α and ϕ are real. In this case, the transmission and reflection coefficients are

$$\begin{aligned} T &= \left|\frac{E_t}{E_i}\right|^2 = \frac{4\alpha^2}{4\alpha^2\cos^2\phi + (1+\alpha^2)^2\sin^2\phi} = \frac{4\alpha^2}{4\alpha^2 + (1-\alpha^2)^2\sin^2\phi} \\ R &= \left|\frac{E_r}{E_i}\right|^2 = \frac{(1-\alpha^2)^2\sin^2\phi}{4\alpha^2\cos^2\phi + (1+\alpha^2)^2\sin^2\phi} = \frac{(1-\alpha^2)^2\sin^2\phi}{4\alpha^2 + (1-\alpha^2)^2\sin^2\phi} \end{aligned} \tag{4}$$

Note that $T+R=1$, as expected. However, this exhibits a classic interference behavior, where T oscillates between $(2\alpha/(1+\alpha^2))^2$ and 1 as the number of wavelengths in the gap vary.

For $\vec{E}$ parallel to the plane of incidence, we find instead the matching conditions

$$\begin{array}{lll} & \text{first interface} & \text{second interface} \\ E^{\parallel}: & (E_i - E_r)\cos i = (E_+ - E_-)\cos r, & (E_+e^{i\phi} - E_-e^{-i\phi})\cos r = E_t\cos i \\ H^{\parallel}: & n(E_i+E_r) = (E_+ + E_-), & E_+e^{i\phi} - E_-e^{-i\phi} = nE_t \end{array}$$

These equations have the same structure as the perpendicular case, but with the index of refraction entering somewhat differently. We find the matching conditions

$$\begin{aligned} n^{-1}E_+ &= \tfrac{1}{2}E_i(1+\beta) + \tfrac{1}{2}E_r(1-\beta) \\ n^{-1}E_- &= \tfrac{1}{2}E_i(1-\beta) + \tfrac{1}{2}E_r(1+\beta) \end{aligned}$$

and

$$n^{-1}E_+ = \tfrac{1}{2}e^{-i\phi}E_t(1+\beta)$$
$$n^{-1}E_- = \tfrac{1}{2}e^{i\phi}E_t(1-\beta)$$

where this time

$$\beta = \frac{\cos i}{n\cos r} = \frac{\cos i}{n\sqrt{1-n^2\sin^2 i}}$$

These expressions are similar to (1) and (2) above, and hence the transmission and reflection coefficients are given by expressions identical to (4), except with the replacement $\alpha \to \beta$.

b) for i greater than the critical angle for total internal reflection, sketch the ratio of transmitted power to incident power as a function of d measured in units of wavelength in the gap.

To be concrete, consider the case for $\vec{E}$ perpendicular to the plane of incidence. Since i is greater than the critical angle, both α and ϕ will be purely imaginary. Whatever values they are, define

$$\alpha = i\gamma, \qquad \phi = i\xi$$

Then the ratios E_t/E_i and E_r/E_i in (3) become

$$\frac{E_t}{E_i} = \frac{2i\gamma}{2i\gamma\cosh\xi + (1-\gamma^2)\sinh\xi}$$
$$\frac{E_r}{E_i} = \frac{-(1+\gamma^2)\sinh\xi}{2i\gamma\cosh\xi + (1-\gamma^2)\sinh\xi}$$

so that

$$T = \left|\frac{E_t}{E_i}\right|^2 = \frac{4\gamma^2}{4\gamma^2 + (1+\gamma^2)\sinh^2\xi}$$
$$R = \left|\frac{E_r}{E_i}\right|^2 = \frac{(1+\gamma^2)^2\sinh^2\xi}{4\gamma^2 + (1+\gamma^2)\sinh^2\xi}$$

In this case, there is no oscillatory behavior in the transmitted power, but only exponential suppression as the air gap is widened. It is easy to see that $T \to 1$ when $d \to 0$ (corresponding to $\xi \to 0$) and that T falls exponentially to 0 when $d \to \infty$ (which is the same as $\xi \to \infty$).

7.5 A plane polarized electromagnetic wave $\vec{E} = \vec{E}_i e^{i\vec{k}\cdot\vec{x}-i\omega t}$ is incident normally on a flat uniform sheet of an *excellent* conductor ($\sigma \gg \omega\epsilon_0$) having thickness D. Assuming that in space and in the conducting sheet $\mu/\mu_0 = \epsilon/\epsilon_0 = 1$, discuss the reflection and transmission of the incident wave.

$a)$ Show that the amplitudes of the reflected and transmitted waves, correct to the first order in $(\epsilon_0\omega/\sigma)^{1/2}$, are:

$$\begin{aligned}\frac{E_r}{E_i} &= \frac{-(1-e^{-2\lambda})}{(1-e^{-2\lambda})+\gamma(1+e^{-2\lambda})}\\ \frac{E_t}{E_i} &= \frac{2\gamma e^{-\lambda}}{(1-e^{-2\lambda})+\gamma(1+e^{-2\lambda})}\end{aligned}$$

where

$$\begin{aligned}\gamma &= \sqrt{\frac{2\epsilon_0\omega}{\sigma}}(1-i) = \frac{\omega\delta}{c}(1-i)\\ \lambda &= (1-i)D/\delta\end{aligned}$$

and $\delta = \sqrt{2/\omega\mu\sigma}$ is the penetration depth.

So long as we treat the conductor as a medium with *complex* dielectric constant

$$\epsilon/\epsilon_0 = 1 + i\frac{\sigma}{\omega\epsilon_0}$$

we may proceed as if everything were a dielectric. Since there are two boundaries, this problem is very much like the above Problem 7.3, except the expressions are even simpler because of the normal incidence. As above, we introduce electric field vectors of the form $\vec{E}_i e^{i\vec{k}\cdot\vec{x}}$ and $\vec{E}_r e^{-i\vec{k}\cdot\vec{x}}$ on the incident side, $\vec{E}_+ e^{i\vec{k}_1\cdot\vec{x}}$ and $\vec{E}_- e^{-i\vec{k}_1\cdot\vec{x}}$ in the conductor, and $\vec{E}_t e^{i\vec{k}\cdot(\vec{x}-\vec{D})}$ on the transmitted side. We use matching for $\vec{E}$ perpendicular to the plane of incidence (which corresponds to a sign convention of having all electric fields pointing in the same direction). In this case, the matching becomes

$$\begin{array}{lll} & \text{first interface} & \text{second interface}\\ E^{\parallel}: & E_i + E_r = E_+ + E_-, & E_+e^{i\phi} + E_-e^{-i\phi} = E_t\\ H^{\parallel}: & (E_i - E_r) = n(E_+ - E_-), & n(E_+e^{i\phi} - E_-e^{-i\phi}) = E_t\end{array}$$

where n is the complex index of refraction

$$n = \sqrt{\frac{\epsilon}{\epsilon_0}} = \sqrt{1 + i\frac{\sigma}{\omega\epsilon_0}} \tag{5}$$

and ϕ is the phase change for going through the dielectric

$$\phi = k_1 D = \frac{\omega n}{c}D = \frac{\omega D}{c}\sqrt{1 + i\frac{\sigma}{\omega\epsilon_0}} \tag{6}$$

Solving for E_t and E_r in terms of E_i, we obtain

$$\begin{aligned}\frac{E_t}{E_i} &= \frac{4/n}{(1+1/n)^2e^{-i\phi} - (1-1/n)^2e^{i\phi}} = \frac{4/ne^{i\phi}}{(1+1/n^2)(1-e^{2i\phi}) + 2/n(1+e^{2i\phi})}\\ \frac{E_r}{E_i} &= \frac{(1-1/n^2)(e^{i\phi}-e^{-i\phi})}{(1+1/n)^2e^{-i\phi} - (1-1/n)^2e^{i\phi}} = \frac{-(1-1/n^2)(1-e^{2i\phi})}{(1+1/n^2)(1-e^{2i\phi}) + 2/n(1+e^{2i\phi})}\end{aligned}$$

which is essentially equivalent to (3), up to redefining $\alpha \to 1/n$. (In fact, this problem can easily be generalized for incidence at an arbitrary angle i by taking $1/n \to \cos i/n\cos r$.) We now take the limit where this is an excellent conductor, $\sigma/\omega\epsilon_0 \gg 1$. In this case, the index of refraction (5) and phase change (6) may be approximated by

$$
\begin{aligned}
n &\approx \sqrt{i\frac{\sigma}{\omega\epsilon_0}} = (1+i)\sqrt{\frac{\sigma}{2\epsilon_0\omega}} = \frac{2}{\gamma} \\
\phi &= \frac{\omega D}{c} n \approx (1+i)\frac{\omega D}{c}\sqrt{\frac{\sigma}{2\epsilon_0\omega}} = (1+i)D\sqrt{\frac{\mu_0\sigma\omega}{2}} = i\lambda
\end{aligned}
$$

For $|\gamma| \ll 1$ (equivalent to $|n| \gg 1$) we drop terms of $\mathcal{O}(1/n^2)$ compared to 1 to arrive at

$$
\begin{aligned}
\frac{E_t}{E_i} &= \frac{2\gamma e^{-\lambda}}{(1-e^{-2\lambda}) + \gamma(1+e^{-2\lambda})} \\
\frac{E_r}{E_i} &= \frac{-(1-e^{-2\lambda})}{(1-e^{-2\lambda}) + \gamma(1+e^{-2\lambda})}
\end{aligned}
\tag{7}
$$

where we have kept the $\mathcal{O}(\gamma)$ term in the denominator which becomes important in the limit $\lambda \to 0$.

b) Verify that for zero thickness and infinite thickness you obtain the proper limiting results.

The zero thickness limit corresponds to $\lambda \to 0$. In this case, it is easy to see from (7) that

$$
\lambda \to 0: \qquad \frac{E_t}{E_i} \to 1, \quad \frac{E_r}{E_i} \to 0
$$

In the infinite thickness limit, we find instead

$$
\lambda \to \infty: \qquad \frac{E_t}{E_i} \to 0, \quad \frac{E_r}{E_i} \to \frac{-1}{1+\gamma}
$$

Note that the reflection coefficient does not go to unity, as some of the power is dissipated in the conductor. A perfect conductor ($\sigma = \infty$) has $\gamma = 0$, so all the power is reflected in the perfect conductor limit.

c) Show that, except for sheets of very small thickness, the transmission coefficient is

$$
T = \frac{8(\Re\gamma)^2 e^{-2D/\delta}}{1 - 2e^{-2D/\delta}\cos(2D/\delta) + e^{-4D/\delta}}
$$

Sketch $\log T$ as a function of (D/δ), assuming $\Re\gamma = 10^{-2}$. Define "very small thickness."

To compute the transmission coefficient from (7), we keep in mind that both γ and λ are complex. As long as we are not in the "very small thickness" limit, the

$\mathcal{O}(\gamma)$ term in the denominator can be ignored. In this case

$$\frac{E_t}{E_i} \approx \frac{2\gamma e^{-\lambda}}{(1-e^{-2\lambda})}$$

so that

$$T = \left|\frac{E_t}{E_i}\right|^2 = \frac{4|\gamma|^2 e^{-2\Re\lambda}}{1-2\Re(e^{-2\lambda})+e^{-4\Re\lambda}}$$

Taking $|\gamma|^2 = 2(\Re\gamma)^2$ as well as $e^{-2\lambda} = e^{2iD/\delta}e^{-2D/\delta}$ then gives

$$T = \frac{8(\Re\gamma)^2 e^{-2D/\delta}}{1-2e^{-2D/\delta}\cos(2D/\delta)+e^{-4D/\delta}}$$

The very small thickness limit corresponds to when the $\mathcal{O}(\gamma)$ term becomes important. This occurs when

$$|1-e^{-2\lambda}| \simeq |\gamma(1+e^{-2\lambda})|$$

Expanding for small λ yields

$$|2\lambda| \simeq |2\gamma| \qquad \Rightarrow \qquad \frac{D}{\delta} \simeq \frac{\omega\delta}{c}$$

Hence small thicknesses correspond to

$$D < \frac{\omega\delta^2}{c}$$

7.8 A monochromatic plane wave of frequency ω is indicdent normally on a stack of layers of various thicknesses t_j and lossless indices of refraction n_j. Inside the stack, the wave has both forward and backward moving components. The change in the wave through any interface and also from one side of a layer to the other can be described by means of 2×2 transfer matrices. If the electric field is written as

$$E = E_+ e^{ikx} + E_- e^{-ikx}$$

in each layer, the transfer matrix equation $E' = TE$ is explicitly

$$\begin{pmatrix} E'_+ \\ E'_- \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} E_+ \\ E_- \end{pmatrix}$$

a) Show that the transfer matrix for propagation inside, but across, a layer of index of refraction n_j and thickness t_j is

$$T_{\text{layer}}(n_j, t_j) = \begin{pmatrix} e^{ik_j t_j} & 0 \\ 0 & e^{-ik_j t_j} \end{pmatrix} = I\cos(k_j t_j) + i\sigma_3 \sin(k_j t_j)$$

where $k_j = n_j\omega/c$, I is the unit matrix, and σ_k are the Pauli spin matrices of quantum mechanics. Show that the inverse matrix is T^*.

Again, normal incidence makes this problem straightforward. For a right moving plane wave of the form e^{ik_jz} passing through a layer of thickness t_j, one picks up a phase $e^{ik_jt_j}$, while for a left moving wave, one picks up a phase $e^{-ik_jt_j}$. More precisely

$$\begin{aligned} E'_+ &= E_+(z=t_j) = E_+(z=0)e^{ik_jt_j} = E_+e^{ik_jt_j} \\ E'_- &= E_-(z=_j) = E_-(z=0)e^{-ik_jt_j} = E_-e^{-ik_jt_j} \end{aligned}$$

This directly leads to the transfer matrix

$$T_{\text{layer}}(n_j,t_j) = \begin{pmatrix} e^{ik_jt_j} & 0 \\ 0 & e^{-ik_jt_j} \end{pmatrix}$$

where the inverse is obviously the complex conjugate.

b) Show that the transfer matrix to cross an interface from n_1 $(x < x_0)$ to n_2 $(x > x_0)$ is

$$T_{\text{interface}}(2,1) = \frac{1}{2}\begin{pmatrix} n+1 & -(n-1) \\ -(n-1) & n+1 \end{pmatrix} = I\frac{(n+1)}{2} - \sigma_1\frac{(n-1)}{2}$$

where $n = n_1/n_2$.

For the matching across layers, we again take the $\vec{E}$ perpendicular to plane of incidence conventions. This gives simply

$$\begin{aligned} E^{\parallel}: &\qquad E_+ + E_- = E'_+ + E'_- \\ H^{\parallel}: &\qquad n_1(E_+ - E_-) = n_2(E'_+ - E'_-) \end{aligned}$$

which may be solved to give

$$\begin{aligned} E'_+ &= \tfrac{1}{2}E_+(1+n) + \tfrac{1}{2}E_-(1-n) \\ E'_- &= \tfrac{1}{2}E_+(1-n) + \tfrac{1}{2}E_-(1+n) \end{aligned}$$

where $n = n_1/n_2$. This yields the transfer matrix

$$T_{\text{interface}}(2,1) = \frac{1}{2}\begin{pmatrix} n+1 & -(n-1) \\ -(n-1) & n+1 \end{pmatrix}$$

c) Show that for a complete stack, the incident, reflected, and transmitted waves are related by

$$E_{\text{trans}} = \frac{\det(T)}{t_{22}}E_{\text{inc}}, \qquad E_{\text{refl}} = -\frac{t_{21}}{t_{22}}E_{\text{inc}}$$

where t_{ij} are the elements of T, the product of the forward-going transfer matrices, including from the material filling space on the incident side into the first layer and from the last layer into the medium filling the space on the transmitted side.

It ought to be clear that the complete effect of going through several layers is to take a product of transfer matrices. For example

$$E' = TE, \quad \text{where} \quad T = \cdots T(4,3)T(n_3,t_3)T(3,2)T(n_2,t_2)T(2,1)$$

The transmitted and reflected electric fields are obtained by solving

$$\begin{pmatrix} E_t \\ 0 \end{pmatrix} = T \begin{pmatrix} E_i \\ E_r \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} E_i \\ E_r \end{pmatrix}$$

This gives explicitly

$$E_t = t_{11}E_i + t_{12}E_r, \qquad 0 = t_{21}E_i + t_{22}E_r$$

which may be solved to obtain

$$E_r = -\frac{t_{21}}{t_{22}}E_i, \qquad E_t = \frac{t_{11}t_{22} - t_{12}t_{21}}{t_{22}}E_i = \frac{\det(T)}{t_{22}}E_i$$

7.16 Plane waves propagate in a homogeneous, nonpermeable, but *anisotropic* dielectric. The dielectric is characterized by a tensor ϵ_{ij}, but if coordinate axes are chosen as the principle axes, the components of displacement along these axes are related to the electric-field components by $D_i = \epsilon_i E_i$ ($i = 1, 2, 3$), where ϵ_i are the eigenvalues of the matrix ϵ_{ij}.

a) Show that plane waves with frequency ω and wave vector $\vec{k}$ must satisfy

$$\vec{k} \times (\vec{k} \times \vec{E}) + \mu_0 \omega^2 \vec{D} = 0$$

This is in fact the general Maxwell wave equation, and does not depend on the details of the dielectric tensor. This may be derived from the curl equations, using $\vec{\nabla} \to i\vec{k}$ and $\partial/\partial_t \to -i\omega$. In a source-free region, the Ampère-Maxwell and Faraday laws give

$$i\vec{k} \times \vec{H} = -i\omega\vec{D}, \qquad i\vec{k} \times \vec{E} - i\omega\vec{B} = 0$$

Taking $i\vec{k}$ cross Faraday's law, and using $\vec{B} = \mu_0 \vec{H}$ gives

$$i\vec{k} \times (i\vec{k} \times \vec{E}) - i\mu_0\omega(i\vec{k} \times \vec{H}) = 0$$

It is then straightforward to substitute in Ampère's law in the second term to arrive at

$$\vec{k} \times (\vec{k} \times \vec{E}) + \mu_0 \omega^2 \vec{D} = 0$$

b) Show that for a given wave vector $\vec{k} = k\hat{n}$ there are two distinct modes of propagation with different phase velocities $v = \omega/k$ that satisfy the Fresnel equation

$$\sum_{i=1}^{3} \frac{n_i^2}{v^2 - v_i^2} = 0$$

where $v_i = 1/\sqrt{\mu_0 \epsilon_i}$ is called a principal velocity, and n_i is the component of $\hat{n}$ along the ith principal axis.

Letting $\vec{k} = k\hat{n}$, and using the BAC–CAB rule, we find

$$\hat{n}(\hat{n} \cdot \vec{E}) - \vec{E} + \mu_0 v^2 \vec{D} = 0$$

By working with the principle axes, this equation may be entirely written in terms of $\vec{E}$. Introducing the real symmetric matrices

$$A_{ij} = n_i n_j - \delta_{ij}, \qquad W_{ij} = \delta_{ij}\mu_0\epsilon_j = \delta_{ij}/v_j^2$$

we arrive at a generalized eigenvalue problem

$$\mathbf{A}\vec{E} = -v^2 \mathbf{W}\vec{E} \qquad \text{or} \qquad (\mathbf{A} + v^2\mathbf{W})\vec{E} = 0 \tag{8}$$

The velocities of propagation are then the eigenvalues of this problem, and may be obtained by solving the secular equation

$$\begin{aligned} 0 = \det(\mathbf{A} + v^2\mathbf{W}) &= v^6 \frac{n_1^2 + n_2^2 + n_3^2}{v_1^2 v_2^2 v_3^2} - v^4 \left(\frac{n_2^2 + n_3^2}{v_2^2 v_3^2} + \frac{n_1^2 + n_3^2}{v_1^2 v_3^2} + \frac{n_1^2 + n_2^2}{v_1^2 v_2^2} \right) \\ &\quad + v^2 \left(\frac{n_1^2}{v_1^2} + \frac{n_2^2}{v_2^2} + \frac{n_3^2}{v_3^2} \right) \\ &= \frac{v^2}{v_1^2 v_2^2 v_3^2} \Big[n_1^2 (v^2 - v_2^2)(v^2 - v_3^2) + n_2^2 (v^2 - v_1^2)(v^2 - v_3^2) \\ &\qquad + n_3^2 (v^2 - v_1^2)(v^2 - v_2^2) \Big] \end{aligned}$$

Other than a trivial solution, $v = 0$ (which does not correspond to a propagating mode), we find two velocities, v_a and v_b, corresponding to the two roots of the quadratic equation for v^2 in the square brackets. In fact, taking the equation in brackets and dividing out by the product $\Pi_i(v^2 - v_i^2)$ immediately gives the Fresnel equation

$$\sum_i \frac{n_i^2}{v^2 - v_i^2} = 0$$

c) Show that $\vec{D}_a \cdot \vec{D}_b = 0$, where $\vec{D}_a$, $\vec{D}_b$ are the displacements associated with the two modes of propagation.

Here we may use standard linear algebra techniques related to the orthogonality of eigenvectors. Considering first the generalized eigenvalue problem (8), we take distinct eigenvalues v_a and v_b. Then the corresponding eigenvectors satisfy the equations

$$(\mathbf{A} + v_a^2\mathbf{W})\vec{E}_a = 0, \qquad (\mathbf{A} + v_b^2\mathbf{W})\vec{E}_b = 0$$

Left-multiplying the first equation by $\vec{E}_b$ and the second by $\vec{E}_a$ gives

$$\vec{E}_b\mathbf{A}\vec{E}_a + v_a^2\vec{E}_b\mathbf{W}\vec{E}_a = 0, \qquad \vec{E}_a\mathbf{A}\vec{E}_b + v_b^2\vec{E}_a\mathbf{W}\vec{E}_b = 0$$

Since $\mathbf{A}$ and $\mathbf{W}$ are symmetric (real Hermitian), we may transpose the first equation and subtract it from the second. The result is

$$(v_b^2 - v_a^2)\vec{E}_a\mathbf{W}\vec{E}_b = 0$$

which implies $\vec{E}_a\mathbf{W}\vec{E}_b = 0$, since $v_a \neq v_b$ (in the case that $v_a = v_b$, we may instead Gram-Schmidt orthogonalize to make the eigenvectors orthogonal). Finally, since $\mathbf{W}$ is μ_0 times the dielectric matrix $\mathbf{\Sigma} = \mathrm{diag}(\epsilon_1, \epsilon_2, \epsilon_3)$, and since $\vec{D} = \mathbf{\Sigma}\vec{E}$, we may equivalently rewrite this orthogonality (with respect to the 'measure' or 'metric' $\mathbf{W}$) as

$$\vec{E}_a \cdot \vec{D}_b = 0 \qquad \text{or} \qquad \vec{E}_b \cdot \vec{D}_a = 0$$

However, we can in fact learn more than this. Since the matrix $\mathbf{A} = \hat{n} \otimes \hat{n} - I$ is not arbitrary, it satisfies the (almost) projection condition $\mathbf{A}^2 = -\mathbf{A}$. As a result

$$\vec{D}_a \cdot \vec{D}_b = \vec{E}_a\mathbf{\Sigma}^2\vec{E}_b = \frac{1}{\mu_0^2}\vec{E}_a\mathbf{W}^2\vec{E}_b = \frac{1}{\mu_0^2 v_a^2 v_b^2}\vec{E}_a\mathbf{A}^2\vec{E}_b = -\frac{1}{\mu_0^2 v_a^2 v_b^2}\vec{E}_a\mathbf{A}\vec{E}_b$$

But since $\mathbf{A}\vec{E}_b = -v_b^2\mathbf{W}\vec{E}_b$, we obtain

$$\vec{D}_a \cdot \vec{D}_b = \frac{1}{\mu_0^2 v_a^2}\vec{E}_a\mathbf{W}\vec{E}_b = \frac{1}{\mu_0 v_a^2}\vec{E}_a \cdot \vec{D}_b = 0$$

(Note, however, that in general $\vec{E}_a \cdot \vec{E}_b \neq 0$.)