Physics 505 **Fall 2005**

Homework Assignment #11 — Due Thursday, December 8

Textbook problems: Ch. 7: 7.3, 7.5, 7.8, 7.16

7.3 Two plane semi-infinite slabs of the same uniform, isotropic, nonpermeable, lossless dielectric with index ofrefraction n are parallel and separated by an air gap ($n = 1$) of width d. A plane electromagnetic wave of frequency ω is indicent on the gap from one of the slabs with angle of indicence i. For linear polarization *both* parallel to *and* perpendicular to the plane of incidence,

a) calculate the ratio of power transmitted into the second slab to the incident power and the ratio of reflected to incident power;

b) for i greater than the critical angle for total internal reflection, sketch the ratio of transmitted power to incident power as a function of d measured in units of wavelength in the gap.

7.5 A plane polarized electromagnetic wave $\vec{E} = \vec{E}_i e^{i\vec{k}\cdot\vec{x}-i\omega t}$ is incident normally on a flat uniform sheet of an *excellent* conductor ($\sigma \gg \omega\epsilon_0$) having thickness D. Assuming that in space and in the conducting sheet $\mu/\mu_0 = \epsilon/\epsilon_0 = 1$, discuss the reflection and transmission of the incident wave.

a) Show that the amplitudes of the reflected and transmitted waves, correct to the first order in $(\epsilon_0\omega/\sigma)^{1/2}$, are:

$$\frac{E_r}{E_i} = \frac{-(1-e^{-2\lambda})}{(1-e^{-2\lambda}) + \gamma(1+e^{-2\lambda})}$$
$$\frac{E_t}{E_i} = \frac{2\gamma e^{-\lambda}}{(1-e^{-2\lambda}) + \gamma(1+e^{-2\lambda})}$$

where

$$\gamma = \sqrt{\frac{2\epsilon_0\omega}{\sigma}}(1-i) = \frac{\omega\delta}{c}(1-i)$$
$$\lambda = (1-i)D/\delta$$

and $\delta = \sqrt{2/\omega\mu\sigma}$ is the penetration depth.

b) Verify that for zero thickness and infinite thickness you obtain the proper limiting results.

c) Show that, except for sheets of very small thickness, the transmission coefficient is

$$T = \frac{8(\Re\gamma)^2 e^{-2D/\delta}}{1 - 2e^{-2D/\delta}\cos(2D/\delta) + e^{-4D/\delta}}$$

Sketch $\log T$ as a function of (D/δ), assuming $\Re\gamma = 10^{-2}$. Define “very small thickness.”

7.8 A monochromatic plane wave of frequency ω is indicdent normally on a stack of layers of various thicknesses t_j and lossless indices of refraction n_j. Inside the stack, the wave has both forward and backward moving components. The change in the wave through any interface and also from one side of a layer to the other can be described by means of 2×2 transfer matrices. If the electric field is written as

$$E = E_+ e^{ikx} + E_- e^{-ikx}$$

in each layer, the transfer matrix equation $E' = TE$ is explicitly

$$\begin{pmatrix} E'_+ \\ E'_- \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} E_+ \\ E_- \end{pmatrix}$$

a) Show that the transfer matrix for propagation inside, but across, a layer of index of refraction n_j and thickness t_j is

$$T_{\text{layer}}(n_j, t_j) = \begin{pmatrix} e^{ik_j t_j} & 0 \\ 0 & e^{-ik_j t_j} \end{pmatrix} = I \cos(k_j t_j) + i\sigma_3 \sin(k_j t_j)$$

where $k_j = n_j \omega / c$, I is the unit matrix, and σ_k are the Pauli spin matrices of quantum mechanics. Show that the inverse matrix is T^*.

b) Show that the transfer matrix to cross an interface from n_1 $(x < x_0)$ to n_2 $(x > x_0)$ is

$$T_{\text{interface}}(2,1) = \frac{1}{2} \begin{pmatrix} n+1 & -(n-1) \\ -(n-1) & n+1 \end{pmatrix} = I\frac{(n+1)}{2} - \sigma_1 \frac{(n-1)}{2}$$

where $n = n_1/n_2$.

c) SHow that for a complete stack, the incident, reflected, and transmitted waves are related by

$$E_{\text{trans}} = \frac{\det(T)}{t_{22}} E_{\text{inc}}, \qquad E_{\text{refl}} = -\frac{t_{21}}{t_{22}} E_{\text{inc}}$$

where t_{ij} are the elements of T, the product of the forward-going transfer matrices, including from the material filling space on the incident side into the first layer and from the last layer into the medium filling the space on the transmitted side.

7.16 Plane waves propagate in a homogeneous, nonpermeable, but *anisotropic* dielectric. The dielectric is characterized by a tensor ϵ_{ij}, but if coordinate axes are chosen as the principle axes, the components of displacement along these axes are related to the electric-field components by $D_i = \epsilon_i E_i$ $(i = 1, 2, 3)$, where ϵ_i are the eigenvalues of the matrix ϵ_{ij}.

a) Show that plane waves with frequency ω and wave vector $\vec{k}$ must satisfy

$$\vec{k} \times (\vec{k} \times \vec{E}) + \mu_0 \omega^2 \vec{D} = 0$$

b) Show that for a given wave vector $\vec{k} = k\hat{n}$ there are two distinct modes of propagation with different phase velocities $v = \omega/k$ that satisfy the Fresnel equation

$$\sum_{i=1}^{3} \frac{n_i^2}{v^2 - v_i^2} = 0$$

where $v_i = 1/\sqrt{\mu_0 \epsilon_i}$ is called a principal velocity, and n_i is the component of $\hat{n}$ along the ith principal axis.

c) Show that $\vec{D}_a \cdot \vec{D}_b = 0$, where $\vec{D}_a$, $\vec{D}_b$ are the displacements associated with the two modes of propagation.