Physics 505 **Fall 2005**

Homework Assignment #10 — Solutions

Textbook problems: Ch. 6: 6.3, 6.4, 6.14, 6.18

6.3 The homogeneous diffusion equation (5.160) for the vector potential for quasi-static fields in unbounded conducting media has a solution to the initial value problem of the form

$$\vec{A}(\vec{x},t) = \int d^3x' G(\vec{x}-\vec{x}\,',t)\vec{A}(\vec{x}\,',0)$$

where $\vec{A}(\vec{x}\,',0)$ describes the initial field configuration and G is an appropriate kernel.

a) Solve the initial value problem by use of a three-dimensional Fourier transform in space for $\vec{A}(\vec{x},t)$. With the usual assumption on intrchange of orders of integration, show that the Green function has the Fourier representation

$$G(\vec{x}-\vec{x}\,',t) = \frac{1}{(2\pi)^3}\int d^3k\, e^{-k^2t/\mu\sigma}e^{i\vec{k}\cdot(\vec{x}-\vec{x}\,')}$$

and it is assumed that $t>0$.

We define the Fourier transform by

$$\vec{A}(\vec{x},t) = \frac{1}{(2\pi)^3}\int \vec{A}(\vec{k},t)e^{-\vec{k}\cdot\vec{x}}\,d^3k$$

In this case, the diffusion equation $\nabla^2\vec{A} = \mu\sigma\partial\vec{A}/\partial t$ becomes

$$|i\vec{k}|^2\vec{A} = \mu\sigma\frac{\partial}{\partial t}\vec{A} \qquad \Rightarrow \qquad \frac{\partial}{\partial t}\vec{A} = -\frac{k^2}{\mu\sigma}\vec{A}$$

This equation is first order in time, and is easily solved to yield

$$\vec{A}(\vec{k},t) = \vec{A}(\vec{k},0)e^{-k^2t/\mu\sigma} \tag{1}$$

Note that we have written the solution in terms of initial conditions specified as $\vec{A}(\vec{k},0)$ at time $t=0$. This is essentially the answer in momentum space. All we have to do is to invert the transform. The inverse transform gives

$$\vec{A}(\vec{x},t) = \frac{1}{(2\pi)^3}\int \vec{A}(\vec{k},0)e^{-k^2t/\mu\sigma}e^{i\vec{k}\cdot\vec{x}}\,d^3k$$

where

$$\vec{A}(\vec{k},0) = \int \vec{A}(\vec{x}\,',0)e^{-i\vec{k}\cdot\vec{x}\,'}\,d^3x'$$

The result is

$$\vec{A}(\vec{x},0) = \frac{1}{(2\pi)^3}\int\int \vec{A}(\vec{x}\,',0)e^{-k^2t/\mu\sigma}e^{i\vec{k}\cdot(\vec{x}-\vec{x}\,')}\,d^3k\,d^3x'$$
$$= \int G(\vec{x}-\vec{x}\,',t)\vec{A}(\vec{x}\,',0)\,d^3x'$$

with the Greens function

$$G(\vec{x}-\vec{x}\,',t) = \frac{1}{(2\pi)^3}\int e^{-k^2t/\mu\sigma}e^{i\vec{k}\cdot(\vec{x}-\vec{x}\,')}\,d^3k$$

Alternatively, we could have noted directly from (1) that the solution in momentum space is a product of $e^{-k^2t/\mu\sigma}$ with $\vec{A}(\vec{k},0)$. As a result, the ordinary space solution is a convolution as indicated.

b) By introducing a Fourier decomposition in both space and time, and performing the frequency integral in the complex ω plane to recover the result of part *a*), show that $G(\vec{x}-\vec{x}\,',t)$ is the diffusion Green function that satisfies the inhomogeneous equation

$$\frac{\partial G}{\partial t} - \frac{1}{\mu\sigma}\nabla^2 G = \delta^{(3)}(\vec{x}-\vec{x}\,')\delta(t)$$

and vanishes for $t<0$.

Introducing the Fourier transform

$$G(\vec{x},t) = \frac{1}{(2\pi)^4}\int G(\vec{k},\omega)e^{i(\vec{k}\cdot\vec{x}-\omega t)}\,d^3k\,d\omega$$

the above inhomogeneous equation becomes

$$[(-i\omega)^2 - |i\vec{k}|^2/\mu\sigma]G = e^{-i\vec{k}\cdot\vec{x}'}$$

which gives the Greens' function

$$G(\vec{k},\omega) = \frac{e^{-i\vec{k}\cdot\vec{x}'}}{k^2/\mu\sigma - i\omega}$$

We may invert the transform by first performing the ω integral. We have

$$G(\vec{k},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\vec{k},\omega)e^{-i\omega t}\,d\omega = \frac{ie^{-i\vec{k}\cdot\vec{x}'}}{2\pi}\int \frac{e^{-i\omega t}}{\omega + ik^2/\mu\sigma}\,d\omega$$

This may be performed by contour integration. For $t>0$, Jordan's lemma tells us to close the contour in the lower-half plane. As a result, we pick up the residue at $\omega = -ik^2/\mu\sigma$. The result is

$$G(\vec{k},t) = (-2\pi i)\frac{ie^{-i\vec{k}\cdot\vec{x}'}}{2\pi}e^{-k^2t/\mu\sigma} = e^{-k^2t/\mu\sigma}e^{-i\vec{k}\cdot\vec{x}\,'}$$

On the other hand, for $t < 0$, we close the contour in the upper-half plane and end up with $G = 0$ as there are no enclosed poles. Finally, writing out the momentum space Fourier transform gives

$$G(\vec{x} - \vec{x}\,', t) = \frac{\Theta(t)}{(2\pi)^3} \int e^{-k^2 t/\mu\sigma} e^{i\vec{k}\cdot(\vec{x}-\vec{x}\,')}\, d^3k \tag{2}$$

c) Show that if σ is uniform throughout all space, the Green function is

$$G(\vec{x}, t; \vec{x}\,', 0) = \Theta(t) \left(\frac{\mu\sigma}{4\pi t}\right)^{3/2} \exp\left(\frac{-\mu\sigma|\vec{x} - \vec{x}\,'|^2}{4t}\right)$$

Actually, we must take both μ and σ to be uniform in all space. In this case, the momentum integration in (2) may be performed by completing the square

$$\begin{aligned} G(\vec{x} - \vec{x}\,', t) &= \frac{\Theta(t)}{(2\pi)^3} e^{-\mu\sigma|\vec{x}-\vec{x}\,'|^2/4t} \int e^{-t|\vec{k} - i\mu\sigma(\vec{x}-\vec{x}\,')/2t|^2/\mu\sigma}\, d^3k \\ &= \frac{\Theta(t)}{(2\pi)^3} \left(\frac{\pi\mu\sigma}{t}\right)^{3/2} e^{-\mu\sigma|\vec{x}-\vec{x}\,'|^2/4t} \\ &= \Theta(t) \left(\frac{\mu\sigma}{4\pi t}\right)^{3/2} e^{-\mu\sigma|\vec{x}-\vec{x}\,'|^2/4t} \end{aligned} \tag{3}$$

d) Suppose that at time $t' = 0$, the initial vector potential $\vec{A}(\vec{x}\,', 0)$ is nonvanishing only in a localized region of linear extent d around the origin. The time dependence of the fields is observed at a point P far from the origin, i.e., $|\vec{x}\,| = r \gg d$. Show that there are three regimes of time, $0 < t \le T_1$, $T_1 \le t \le T_2$, and $t \gg T_2$. Give plausible definitions of T_1 and T_2, and describe qualitatively the time dependence at P. Show that in the last regime, the vector potential is proportional to the volume integral of $\vec{A}(\vec{x}\,', 0)$ times $t^{-3/2}$, assuming that integral exists. Relate your discussion to those of Section 5.18.B and Problems 5.35 and 5.36.

For a local 'disturbance' near the origin, physical intuition tells us that it will take some time before the observer at point P will manage to observe it. For a diffusion problem, this time is essentially the timescale for diffusion, set by the diffusion coefficient $D = 1/\mu\sigma$, where the diffusion equation is of the form $\partial\rho/\partial t = \vec{\nabla}\cdot(D\vec{\nabla}\rho)$. To be specific, the field at point P is given by the convolution

$$\begin{aligned} \vec{A}(\vec{x}, t) &= \int G(\vec{x} - \vec{x}\,', t)\vec{A}(\vec{x}\,', 0)\, d^3x' \\ &= \Theta(t) \left(\frac{\mu\sigma}{4\pi t}\right)^{3/2} \int \vec{A}(\vec{x}\,', 0) e^{-\mu\sigma|\vec{x}-\vec{x}\,'|^2/4t}\, d^3x' \end{aligned}$$

Assuming $|\vec{x}\,| = r \gg d$, we may approximate the integral by simply taking $|\vec{x} - \vec{x}\,'|^2 \approx d^2$. This gives

$$\vec{A}(r, t) \approx \Theta(t) \left(\frac{\mu\sigma d}{4\pi t}\right)^{3/2} e^{-\mu\sigma r^2/4t} \langle \vec{A}\,\rangle_{t=0} \tag{4}$$

where $\langle \vec{A} \rangle$ is the spatial average of $\vec{A}$ in the nonvanishing region. Defining $\tau_1 = \mu\sigma r^2/4$ and $\tau_2 = \mu\sigma d/4\pi$, the behavior of the vector potential is then

$$\vec{A}(r,t) \approx \Theta(t) \left(\frac{\tau_2}{t}\right)^{3/2} e^{-\tau_1/t}$$

At short times, $t \ll \tau_1$, the exponential is highly suppressed, and there is no signal at point P. After time τ_1, however, the exponential becomes of order $\mathcal{O}(1)$. The initial vector potential has now diffused to P. However, because of the $(\tau_2/t)^{3/2}$ prefactor, the signal dies away at long times. For a rough estimate, we take

$$T_1 = \tau_1 = \mu\sigma r^2/4 \qquad T_2 = 2T_1$$

Then for $t < T_1$ the vector potential at point P is essentially zero. Between T_1 and T_2, the vector potential is non-zero, and at long times after T_2, everything has diffused away. Finally, noting that the volume integral of $\vec{A}(\vec{x}\,', 0)$ is simply $d^3\langle \vec{A} \rangle$, the expression in (4) demonstrates that at late times (when the exponential is essentially unity) the vector potential is indeed proportional to this volume integral times $t^{-3/2}$.

6.4 A uniformly magnetized and conducting sphere of radius R and total magnetic moment $m = 4\pi M R^3/3$ rotates about its magnetization axis with angular speed ω. In the steady state no current flows in the conductor. The motion is nonrelativistic; the sphere has no excess charge on it.

a) By considering Ohm's law in the moving conductor, show that the motion induces an electric field and a uniform volume charge density in the conductor, $\rho = -m\omega/\pi c^2 R^3$.

We assume the sphere is magnetized and spinning along the $\hat{z}$ axis. Since the magnetic moment is $\vec{m} = \vec{M}\,V$ where $V = \frac{4}{3}\pi R^3$ is the volume of the sphere, we see that the magnetization is simply $\vec{M} = M\hat{z}$. As demonstrated earlier, a uniformly magnetized sphere has a uniform magnetic induction $\vec{B} = \frac{2}{3}\mu_0\vec{M}$ in its interior. In terms of m, this is

$$\vec{B} = \tfrac{2}{3}\mu_0 \left(\frac{3}{4\pi R^3} m\hat{z}\right) = \frac{\mu_0 m}{2\pi R^3}\hat{z}$$

We now observe that the electric field $\vec{E}'$ in the rotating frame of the sphere may be related to lab quantities by $\vec{E}' = \vec{E} + \vec{v} \times \vec{B}$. Ohm's law in the rotating reference frame is then $\vec{J} = \sigma\vec{E}' = \sigma(\vec{E} + \vec{v} \times \vec{B})$. Since no current flows in the steady state ($\vec{J} = 0$), this motion must induce an electric field $\vec{E} = -\vec{v} \times \vec{B}$. Using $\vec{\omega} = \omega\hat{z}$ and $\vec{v} = \omega \times \vec{r}$, we obtain

$$\vec{E} = -(\vec{\omega} \times \vec{r}) \times \vec{B} = -\frac{\mu_0 m\omega}{2\pi R^3}(\hat{z} \times \vec{r}) \times \hat{z} = -\frac{\mu_0 m\omega}{2\pi R^3}(\vec{r} - \hat{z}(\hat{z} \cdot \vec{r}))$$

The vector structure is essentially a projection of $\vec{r}$ into the horizontal plane perpendicular to the $\hat{z}$ axis. In cylindrical coordinates, this indicates that

$$E_\rho = -\frac{\mu_0 m\omega\rho}{2\pi R^3} \tag{5}$$

It is then a simple matter of applying Gauss' law to recover the volume charge density. However, before we do so, we note that this is a cylindrically symmetric electric field (pointed horizontally inward towards the $\hat{z}$ axis). It may at first be somewhat surprising that a sphere will give a cylindrical electric field. However, rotation about an axis is actually a cylindrical process. So from this point of view, the electric field is quite natural.

Using $\rho = \epsilon_0 \vec{\nabla} \cdot \vec{E}$ we obtain a uniform volume charge density

$$\rho = \epsilon_0 \frac{\partial E_\rho}{\partial \rho} = -\frac{\mu_0 \epsilon_0 m\omega}{2\pi R^3} = -\frac{m\omega}{2\pi c^2 R^3}$$

It is important to note that, while the charge density is uniform inside the sphere, the electric field is *not* radial. The discrepancy between a uniform spherical charge distribution and the cylindrical electric field must then arise due to a surface charge. This then provides a hint as to how to approach the remainder of this problem.

b) Because the sphere is electrically neutral, there is no monopole electric field outside. Use symmetry arguments to show that the lowest possible electric multipolarity is quadrupole. Show that only a quadrupole field exists outside and that the quadrupole moment tensor has nonvanishing components, $Q_{33} = -4m\omega R^2/3c^2$, $Q_{11} = Q_{22} = -Q_{33}/2$.

No charge resides outside the sphere. As a result, the exterior field may be described through the multipole expansion. As indicated, charge neutrality guarantees the vanishing of the monopole ($l = 0$) moment. Furthermore, the odd moments vanish due to symmetry of the electric field (5) under the parity transformation $z \to -z$. (That is of course the internal field; however we may see that the external field must necessarily respect the symmetry of the internal one.) Thus a simple symmetry argument demonstrates that the lowest possible multipole is the quadrupole ($l = 2$). Symmetry along will not preclude higher even moments. However an explicit calculation will.

Without knowing the surface charge, we cannot directly calculate the electric multipoles. However, we note that the interior electric field (5) can be integrated to obtain the interior electrostatic potential

$$\Phi(\rho) = -\int \vec{E} \cdot d\vec{\ell} = -\int E_\rho \, d\rho = \Phi_0 + \frac{\mu_0 m\omega\rho^2}{4\pi R^3}$$

Converted back to spherical coordinates, this gives

$$\Phi(r,\theta) = \Phi_0 + \frac{\mu_0 m\omega}{4\pi R^3} r^2 \sin^2\theta = \Phi_0 + \frac{\mu_0 m\omega}{6\pi R^3} r^2 [P_0(\cos\theta) - P_2(\cos\theta)]$$

where we have converted $\sin^2\theta$ into Legendre polynomials. This can be written explicitly as a Legendre expansion

$$\Phi(r,\theta) = \left(\Phi_0 + \frac{\mu_0 m\omega}{6\pi R^3}r^2\right)P_0(\cos\theta) - \frac{\mu_0 m\omega}{6\pi R^3}r^2 P_2(\cos\theta)$$

so that in particular the potential at the surface of the sphere is

$$\Phi(R,\theta) = \left(\Phi_0 + \frac{\mu_0 m\omega}{6\pi R}\right)P_0(\cos\theta) - \frac{\mu_0 m\omega}{6\pi R}P_2(\cos\theta)$$

We may now solve for the exterior potential by treating this as an electrostatic boundary value problem. We recall that, given a sphere with azimuthally symmetric potential $V(\theta) = \sum_l \alpha_l P_l(\cos\theta)$ on the surface, the exterior solution has the form $\Phi(r,\theta) = \sum_l \alpha_l (R/r)^{l+1} P_l(\cos\theta)$. Furthermore, charge neutrality in the present case forces the monopole ($l = 0$) term to vanish. Hence we find that $\Phi_0 = -\mu_0 m\omega/6\pi R$, and that the external potential is

$$\Phi(r,\theta) = -\frac{\mu_0 m\omega R^2}{6\pi r^3}P_2(\cos\theta) \tag{6}$$

Incidentally, we could write an expression valid both in the interior and the exterior as

$$\Phi(r,\theta) = \frac{\mu_0 m\omega}{6\pi R}\left[\left(\frac{r^2}{R^2} - 1\right)\Theta(R-r)P_0(\cos\theta) - R\frac{r_<^2}{r_>^3}P_2(\cos\theta)\right] \tag{7}$$

Note that this potential is only harmonic outside the sphere; inside the sphere the r^2/R^2 term multiplying $P_0(\cos\theta)$ is not of the right $(A_l r^l + B_l r^{-l-1})P_l(\cos\theta)$ form to be harmonic. However, this is present precisely because of the uniform volume charge density, which acts as a $l = 0$ source.

In any case, we are essentially done, as the exterior potential (6) clearly has only a quadrupole term

$$\Phi = -\sqrt{\frac{4\pi}{5}}\frac{\mu_0 m\omega R^2}{6\pi}\frac{Y_{2,0}(\theta,\phi)}{r^3}$$

Comparing this with the multipole expansion

$$\Phi = \frac{1}{4\pi\epsilon_0}\sum_{l,m}\frac{4\pi}{2l+1}q_{l,m}\frac{Y_{l,m}(\theta,\phi)}{r^{l+1}}$$

gives

$$q_{2,0} = -4\pi\epsilon_0\sqrt{\frac{5}{4\pi}}\frac{\mu_0 m\omega R^2}{6\pi} = -\sqrt{\frac{5}{4\pi}}\frac{2m\omega R^2}{3c^2}$$

Converting to cartesian tensors yields

$$Q_{33} = 2\sqrt{\frac{4\pi}{5}}q_{2,0} = -\frac{4m\omega R^2}{3c^2}, \qquad Q_{11} = Q_{22} = -\tfrac{1}{2}Q_{33}$$

c) By considering the radial electric fields inside and outside the sphere, show that the necessary surface-chrage density $\sigma(\theta)$ is

$$\sigma(\theta) = \frac{1}{4\pi R^2} \frac{4m\omega}{3c^2} \left[1 - \frac{5}{2} P_2(\cos\theta) \right]$$

The surface charge may be computed by first obtaining the jump in the normal component of the electric field at the surface of the sphere. Working in spherical components, and taking the gradient of the potential (7), we find

$$\begin{aligned} E_r^{\text{out}} &= -\frac{\mu_0 m\omega R^2}{2\pi r^4} P_2(\cos\theta) \\ E_r^{\text{in}} &= -\frac{\mu_0 m\omega r}{3\pi R^3} [P_0(\cos\theta) - P_2(\cos\theta)] \end{aligned}$$

The surface charge is then computed as

$$\begin{aligned} \sigma = \epsilon_0 (E_r^{\text{out}} - E_r^{\text{in}})\Big|_{r=R} &= -\frac{\mu_0 \epsilon_0 m\omega}{3\pi R^2} [\tfrac{3}{2} P_2(\cos\theta) - (P_0(\cos\theta) - P_2(\cos\theta))] \\ &= \frac{m\omega}{3\pi c^2 R^2} [P_0(\cos\theta) - \tfrac{5}{2} P_2(\cos\theta)] \end{aligned}$$

d) The rotating sphere serves as a unipolar induction device if a stationary circuit is attached by a slip ring to the pole and a sliding contact to the equator. Show that the line integral of the electric field from the equator contact to the pole contact by any path) is $\mathcal{E} = \mu_0 m\omega / 4\pi R$.

Although the sphere is rotating, both the electric and the magnetic field are static. Hence the line integral of the electric field gives simply the electrostatic potential. In this case

$$\mathcal{E} = \int_{\text{equator}}^{\text{pole}} \vec{E} \cdot d\vec{\ell} = \Phi_{\text{equator}} - \Phi_{\text{pole}} = \Phi(\theta = \tfrac{\pi}{2}) - \Phi(\theta = 0)$$

Using (6) or (7) evaluated on the surface, this becomes

$$\mathcal{E} = -\frac{\mu_0 m\omega}{6\pi R} [P_2(0) - P_2(1)] = \frac{\mu_0 m\omega}{4\pi R}$$

6.14 An ideal circular parallel plate capacitor of radius a and plate separation $d \ll a$ is connected to a current source by axial leads, as shown in the sketch. The current in the wire is $I(t) = I_0 \cos\omega t$.

a) Calculate the electric and magnetic fields between the plates to second order in powers of the frequency (or wave number), neglecting the effects of fringing fields.

We begin with an observation that this problem is cylindrically symmetric. For a static circular parallel plate capacitor, the electric field is in the axial direction (which we take to be $\hat{z}$). If this were to vary in time, it would induce an azimuthal magnetic field. Hence we are concerned with finding the cylindrical coordinate components $E_z(\rho,t)$ and $B_\phi(\rho,t)$, at least if we were to neglect fringing. Note that, in this case, both $\vec{\nabla}\cdot\vec{E}=0$ and $\vec{\nabla}\cdot\vec{B}=0$ are automatically satisfied, and we are left with Faraday's law and the Ampère-Maxwell law.

One method to approach this problem is to expand in powers of frequency. Working with complex quantities, a current $I(t)=I_0e^{-i\omega t}$ results in a charge $q=\int I\,dt=(iI_0/\omega)e^{-i\omega t}$. At lowest order in frequency, we essentially have an electrostatic problem. Thus the charge $q=iI_0/\omega$ gives rise to a surface charge density $\sigma=q/\pi a^2=iI_0/\pi a^2\omega$. The resulting electric field is

$$E_z^{(0)}=-\frac{\sigma}{\epsilon_0}=\frac{-iI_0}{\epsilon_0\pi a^2\omega}$$

(where we have hidden the $e^{-i\omega t}$ behavior). By the Ampère-Maxwell law, this induces a magnetic field

$$\vec{\nabla}\times\vec{B}^{(1)}=\frac{1}{c^2}(-i\omega)\vec{E}^{(0)}=-\frac{\mu_0 I_0}{\pi a^2}\hat{z}$$

Working in cylindrical coordinates, we solve

$$(\vec{\nabla}\times\vec{B}^{(1)})_z=\frac{1}{\rho}\partial_\rho\rho B_\phi^{(1)}=-\frac{\mu_0 I_0}{\pi a^2}$$

to obtain

$$B_\phi^{(1)}=-\frac{\mu_0 I_0\rho}{2\pi a^2}$$

Note that a possible integration constant was dropped to avoid an unphysical $1/\rho$ singularity in the magnetic induction.

Proceeding in a similar fashion, this oscillating magnetic field will induce an electric field by Faraday's law

$$-\partial_\rho E_z^{(2)}=(\vec{\nabla}\times\vec{E}^{(2)})_\phi=i\omega B_\phi^{(1)}=-\frac{i\omega I_0\rho}{2\pi\epsilon_0 a^2c^2}$$

This integrates to

$$E_z^{(2)}=\frac{iI_0\rho^2\omega}{4\pi\epsilon_0 a^2c^2}$$

Note that this time we have chosen to drop a possible constant of integration since any such ρ independent constant can be absorbed in $E_z^{(0)}$ by suitable redefinition of I_0. This does indicate to us, however, that we may need to reexamine the assumed boundary condition that was used to relate $E_z^{(0)}$ to the current specified by I_0. Ignoring this for the moment, we proceed once more with the Ampère-Maxwell law to obtain

$$B_\phi^{(3)}=\frac{\mu_0 I\rho^3\omega^2}{16\pi a^2c^2}$$

Putting the above expressions together gives

$$\begin{aligned} E_z &= -\frac{iI_0}{\pi\epsilon_0 a^2\omega}\left(1 - \frac{\rho^2\omega^2}{4c^2} + \cdots\right) \\ B_\phi &= -\frac{\mu_0 I_0 \rho}{2\pi a^2}\left(1 - \frac{\rho^2\omega^2}{8c^2} + \cdots\right) \end{aligned} \tag{8}$$

and restoring the time dependence gives

$$\begin{aligned} E_z &= -\frac{I_0}{\pi\epsilon_0 a^2\omega}\left(1 - \frac{\rho^2\omega^2}{4c^2} + \cdots\right)\sin\omega t \\ B_\phi &= -\frac{\mu_0 I_0 \rho}{2\pi a^2}\left(1 - \frac{\rho^2\omega^2}{8c^2} + \cdots\right)\cos\omega t \end{aligned}$$

A more formal method for solving this problem is to develop a series expansion in ω of the form

$$E_z = \sum_{n=-1}^{\infty} e_n(\rho)\omega^n, \qquad B_\phi = \sum_{n=0}^{\infty} b_n(\rho)\omega^n$$

and to substitute this into the component Faraday and Ampère-Maxwell equations

$$\frac{1}{\rho}\partial_\rho \rho B_\phi = -\frac{i\omega}{c^2}E_z, \qquad \partial_\rho E_z = -i\omega B_\phi \tag{9}$$

Collecting the resulting powers of ω of course reduces to the same expressions that we solved one at a time to arrive at (8) above. Of course, we could also be more clever, and simply manipulate (9) to obtain the second order equations

$$\frac{1}{\rho}\partial_\rho \rho \partial_\rho E_z = -\frac{\omega^2}{c^2}E_z, \qquad \partial_\rho \frac{1}{\rho}\partial_\rho \rho B_\phi = -\frac{\omega^2}{c^2}B_\phi$$

which may be rewritten as

$$\zeta^2 E_z''(\zeta) + \zeta E_z'(\zeta) + (\zeta^2 - 0)E_z(\zeta) = 0, \qquad \zeta^2 B\phi''(\zeta) + \zeta B_\phi'(\zeta) + (\zeta^2 - 1)B_\phi(\rho) = 0$$

where $\zeta = \omega\rho/c$. These may be recognized as Bessel's equations, with solution

$$E_z(\rho) = E_z^0 J_0(\omega\rho/c), \qquad B_\phi(\rho) = B_\phi^0 J_1(\omega\rho/c)$$

(We have avoided the Neumann functions since they blow up at $\rho = 0$.) Note, however, that the first order equations (9) tie together E_z^0 and B_ϕ^0. Using

$$B_\phi = \frac{i}{\omega}\partial_\rho E_z = \frac{i}{c}\partial_\zeta E_z = \frac{iE_z^0}{c}J_0'(\zeta) = -\frac{iE_z^0}{c}J_1(\zeta)$$

(since $J_0' = -J_1$ by the Bessel recursion relations), we see that $B_\phi^0 = -iE_z^0/c$. Since $J_0(\zeta) \to 1$ for small argument, $\zeta \to 0$, the above above boundary conditions are satisfied by taking

$$E_z = -\frac{I_0}{\pi\epsilon_0 a^2\omega} J_0(\omega\rho/c)\sin\omega t, \qquad B_\phi = -\frac{\mu_0 I_0 c}{\pi a^2 \omega} J_1(\omega\rho/c)\cos\omega t \tag{10}$$

b) Calculate the volume integrals of w_e and w_m that enter the definition of the reactance X, (6.140), to second order in ω. Show that in terms of the input current I_i, defined by $I_i = -i\omega Q$, where Q is the *total charge* on one plate, these energies are

$$\int w_e\, d^3x = \frac{1}{4\pi\epsilon_0}\frac{|I_i|^2 d}{\omega^2 a^2}, \qquad \int w_m\, d^3x = \frac{\mu_0}{4\pi}\frac{|I_i|^2 d}{8}\left(1 + \frac{\omega^2 a^2}{12c^2}\right)$$

Working with the series expansion, (8), the small signal energy densities are given by

$$\begin{aligned} w_e &= \tfrac{1}{4}\vec{E}\cdot\vec{D}^* = \frac{\epsilon_0}{4}|\vec{E}\,|^2 = \frac{|I_0|^2}{4\pi^2\epsilon_0 a^4\omega^2}\left(1 - \frac{\rho^2\omega^2}{2c^2} + \cdots\right) \\ w_m &= \tfrac{1}{4}\vec{B}\cdot\vec{H}^* = \frac{1}{4\mu_0}|\vec{B}\,|^2 = \frac{|I_0|^2\rho^2}{16\pi^2\epsilon_0 a^4 c^2}\left(1 - \frac{\rho^2\omega^2}{4c^2} + \cdots\right) \end{aligned}$$

Integrating over the volume of the capacitor with

$$\int d^3x = 2\pi d\int_0^a \rho d\rho$$

gives

$$\begin{aligned} \int w_e\, d^3x &= \frac{|I_0|^2 d}{4\pi\epsilon_0 a^2\omega^2}\left(1 - \frac{a^2\omega^2}{4c^2} + \cdots\right) \\ \int w_m\, d^3x &= \frac{\mu_0|I_0|^2 d}{32\pi}\left(1 - \frac{a^2\omega^2}{6c^2} + \cdots\right) \end{aligned} \tag{11}$$

These do not seem to agree with the expected results. However, we have used the constant I_0 to describe the current, whereas $I_i = -i\omega Q$, defined by the total charge on one plate, may differ from I_0 due to frequency dependent effects. To compute the total charge Q, we may first use Gauss' law to obtain the surface charge density σ and then integrate σ over the plate. For E_z given in (8), this gives simply

$$\sigma = \epsilon_0(-E_z) = \frac{iI_0}{\pi a^2\omega}\left(1 - \frac{\rho^2\omega^2}{4c^2} + \cdots\right)$$

(where the extra sign comes from taking the charge on the top plate) so that

$$Q = 2\pi\int_0^a \sigma(\rho)\,\rho d\rho = \frac{2iI_0}{a^2\omega}\left(\frac{a^2}{2} - \frac{a^4\omega^4}{16c^2} + \cdots\right) = \frac{iI_0}{\omega}\left(1 - \frac{a^2\omega^2}{8c^2} + \cdots\right)$$

This yields the relation between I_0 and I_i

$$I_i = -i\omega Q = I_0\left(1 - \frac{a^2\omega^2}{8c^2} + \cdots\right)$$

so that

$$|I_i|^2 = |I_0|^2\left(1 - \frac{a^2\omega^2}{4c^2} + \cdots\right)$$

Substituting this into (11) finally results in

$$\int w_e\, d^3x = \frac{|I_i|^2 d}{4\pi\epsilon_0 a^2\omega^2}, \qquad \int w_m\, d^3x = \frac{\mu_0 |I_i|^2 d}{32\pi}\left(1 + \frac{a^2\omega^2}{12c^2}\right)$$

at least to the order that we are working at.

c) Show that the equivalent series circuit has $C \approx \pi\epsilon_0 a^2/d$, $L \approx \mu_0 d/8\pi$, and that an estimate for the resonant frequency of the system is $\omega_{\mathrm{res}} \approx 2\sqrt{2}c/a$. Compare with the first root of $J_0(x)$.

The reactance is given by

$$\begin{aligned} X &\approx \frac{4\omega}{|I|^2}\int (w_m - w_e)\, d^3x \approx \frac{4\omega}{|I|^2}\left(\frac{\mu_0|I|^2 d}{32\pi} - \frac{|I|^2 d}{4\pi\epsilon_0 a^2\omega^2}\right) \\ &= \frac{\mu_0 d}{8\pi}\omega - \left(\frac{\pi\epsilon_0 a^2}{d}\omega\right)^{-1} \end{aligned}$$

Since the reactance of an inductor is $X = L\omega$ and a capacitor is $X = 1/(C\omega)$, we see that the equivalent series circuit has $C \approx \pi\epsilon_0 a^2/d$ and $L \approx \mu_0 d/8\pi$ as indicated. The resonant frequency of a LC circuit is then

$$\omega_{\mathrm{res}} = \frac{1}{\sqrt{LC}} \approx \sqrt{\frac{8}{\mu_0\epsilon_0 a^2}} = \frac{2\sqrt{2}c}{a}$$

The reason we may compare this with the first zero of the Bessel function $J_0(x)$ is that the 'exact' expression for the electric field in (10) involves $J_0(\omega\rho/c)$. The assumption of no fringing outside the capacitor demands that the electric field vanish at $\rho = a$. This occurs when $\omega a/c = x_{0,1}$ where $x_{0,1} \approx 2.4$ is the first zero of J_0. The value we have computed, $2\sqrt{2} \approx 2.8$, differs from $x_{0,1}$ by more than 15%. Nevertheless, it is at least of the correct magnitude.

6.18 Consider the Dirac expression

$$\vec{A}(\vec{x}\,) = \frac{g}{4\pi}\int_L \frac{d\vec{l}\,' \times (\vec{x} - \vec{x}\,')}{|\vec{x} - \vec{x}\,'|^3}$$

for the vector potential of a magnetic monopole and its associated string L. Suppose for definiteness that the monopole is located at the origin and the string along the negative z axis.

a) Calculate $\vec{A}$ explicitly and show that in spherical coordinates it has components

$$A_r = 0, \qquad A_\theta = 0, \qquad A_\phi = \frac{g(1-\cos\theta)}{4\pi r \sin\theta} = \left(\frac{g}{4\pi r}\right)\tan\frac{\theta}{2}$$

Taking the Dirac string along the negative $\hat{z}$ axis, we write $\vec{x}\,' = z'\hat{z}$ and $d\vec{\ell}' = \hat{z}dz'$. Hence Dirac's expression is

$$\begin{aligned}\vec{A}(\vec{x}\,) &= \frac{g}{4\pi}\int_{-\infty}^{0} dz'\frac{\hat{z}\times(\vec{x}-z'\hat{z})}{|\vec{x}-z'\hat{z}|^3}\\ &= \frac{g}{4\pi}\int_{-\infty}^{0} dz'\frac{\hat{z}\times\vec{x}}{[\rho^2+(z-z')^2]^{3/2}}\\ &= \frac{g}{4\pi}(\hat{z}\times\vec{x})\int_{-\infty}^{-z}\frac{du}{(\rho^2+u^2)^{3/2}}\end{aligned}$$

This integral is easily performed by trig substitution. The result is

$$\vec{A}(\vec{x}\,) = \frac{g}{4\pi}\frac{\hat{z}\times\vec{x}}{\rho^2}\left(1-\frac{z}{r}\right)$$

where $\rho^2 = x^2+y^2$ and $r^2 = x^2+y^2+z^2$. Noting that $\hat{z}\times\vec{x} = \rho\hat{\phi} = r\sin\theta\hat{\phi}$, and converting to spherical coordinates, we obtain

$$\vec{A}(\vec{x}\,) = \frac{g}{4\pi}\frac{r-z}{r^2\sin\theta}\hat{\phi} = \frac{g}{4\pi}\frac{1-\cos\theta}{r\sin\theta}\hat{\phi}$$

b) Verify that $\vec{B} = \vec{\nabla}\times\vec{A}$ is the Coulomb-like field of a point charge, except perhaps at $\theta = \pi$.

Note that the vector potential blows up on the negative $\hat{z}$ axis. (The positive $\hat{z}$ axis is safe, as a Taylor or l'Hopital expansion near $\theta = 0$ will demonstrate.) Away from this point, we have

$$\begin{aligned}\vec{B} = \vec{\nabla}\times\vec{A} &= \hat{r}\frac{1}{r\sin\theta}\partial_\theta(\sin\theta A_\phi) - \hat{\theta}\frac{1}{r}\partial_r(rA_\phi)\\ &= \hat{r}\frac{1}{r\sin\theta}\partial_\theta\left(\frac{g}{4\pi}\frac{1-\cos\theta}{r}\right) = \hat{r}\left(\frac{g}{4\pi r^2}\right)\end{aligned}$$

which is the expected field of a magnetic monopole.

c) With the $\vec{B}$ determined in part b), evaluate the total magnetic flux passing through the circular loop of radius $R\sin\theta$ shown in the figure. Consider $\theta < \pi/2$ and $\theta > \pi/2$ spearately, but always calculate the upward flux.

Assuming $\vec{B} = g\hat{r}/4\pi r^2$ everywhere, the flux through a circular loop of radius $R\sin\theta$ is

$$\begin{aligned}
\Phi &= \int \vec{B}\cdot\hat{n}\,da = \int B_z\,da = \frac{g}{4\pi}\int \frac{z}{(\rho^2+z^2)^{3/2}}\,\rho d\rho\,d\phi \\
&= \frac{gz}{4}\int_0^{(R\sin\theta)^2} \frac{du}{(u+z^2)^{3/2}} = -\frac{gz}{2}\frac{1}{\sqrt{u+z^2}}\bigg|_0^{(R\sin\theta)^2} \\
&= \frac{gR\cos\theta}{2}\left(\frac{1}{R|\cos\theta|} - \frac{1}{R}\right) = \frac{g}{2}\big(\mathrm{sgn}(\cos\theta) - \cos\theta\big)
\end{aligned}$$

where we have used $z = R\cos\theta$. For $\theta < \pi/2$ (the top hemisphere) we find $\Phi_{\text{top}} = \frac{g}{2}(1-\cos\theta)$, while for $\theta > \pi/2$ we find $\Phi_{\text{bottom}} = \frac{g}{2}(-1-\cos\theta)$. Note that the (upward) flux so calculated is discontinuous as we pass through the plane of the monopole.

d) From $\oint \vec{A}\cdot d\vec{L}$ around the loop, determine the total magnetic flux through the loop. Compare the result with that found in part *c*). Show that they are equal for $0 < \theta < \pi/2$, but have a *constant* difference for $\pi/2 < \theta < \pi$. Interpret this difference.

By Stokes' theorem, the line integral of the vector potential gives the magnetic flux. We find

$$\oint \vec{A}\cdot d\vec{\ell} = \int 0^{2\pi} A_\phi(R,\theta)\,R\sin\theta\,d\phi = \frac{g}{4\pi}\frac{1-\cos\theta}{R\sin\theta}(2\pi R\sin\theta) = \frac{g}{2}(1-\cos\theta)$$

Thus

$$\oint \vec{A}\cdot d\vec{\ell} = \Phi_{\text{top}} = \Phi_{\text{bottom}} + g$$

What has happened in this case is that the computation of part *c*) did not take into account the flux of the Dirac string. For a positively charged monopole, the Dirac string carries upward magnetic flux. So the total flux of the monopole plus string is really $\Phi_{\text{bottom}} + g$. This is fully accounted for by taking the line integral of the vector potential (which is after all the vector potential due to the Dirac string).

Of course, an 'honest' magnetic monopole will have a magnetic field $\vec{B} = g\hat{r}/4\pi r^2$ *everywhere* in space. In this case, the flux calculation of part *c*) is the 'correct' one. Every calculation involving the vector potential must then be treated with care, and in particular the location of the Dirac string may have to be moved by gauge transformation when working with $\vec{A}$ in the southern hemisphere. In the modern language of differential geometry (fiber bundles), we have to introduce separate coordinate patches for the northern and southern hemisphere, with an overlap region around the equator. We then define differentiable transition functions (essentially gauge transformations) connecting the different sections of the bundle in the overlap region. The Dirac string can then be avoided by working with the fiber bundle itself.