Physics 505 **Fall 2005**

Homework Assignment #10 — Due Thursday, December 1

Textbook problems: Ch. 6: 6.3, 6.4, 6.14, 6.18

6.3 The homogeneous diffusion equation (5.160) for the vector potential for quasi-static fields in unbounded conducting media has a solution to the initial value problem of the form

$$\vec{A}(\vec{x},t) = \int d^3x' G(\vec{x}-\vec{x}\,',t)\vec{A}(\vec{x}\,',0)$$

where $\vec{A}(\vec{x}\,',0)$ describes the initial field configuration and G is an appropriate kernel.

a) Solve the initial value problem by use of a three-dimensional Fourier transform in space for $\vec{A}(\vec{x},t)$. With the usual assumption on intrchange of orders of integration, show that the Green function has the Fourier representation

$$G(\vec{x}-\vec{x}\,',t) = \frac{1}{(2\pi)^3}\int d^3k\, e^{-k^2t/\mu\sigma} e^{i\vec{k}\cdot(\vec{x}-\vec{x}\,')}$$

and it is assumed that $t > 0$.

b) By introducing a Fourier decomposition in both space and time, and performing the frequency integral in the complex ω plane to recover the result of part *a*), show that $G(\vec{x}-\vec{x}\,',t)$ is the diffusion Green function that satisfies the inhomogeneous equation

$$\frac{\partial G}{\partial t} - \frac{1}{\mu\sigma}\nabla^2 G = \delta^{(3)}(\vec{x}-\vec{x}\,')\delta(t)$$

and vanishes for $t < 0$.

c) Show that if σ is uniform throughout all space, the Green function is

$$G(\vec{x},t;\vec{x}\,',0) = \Theta(t)\left(\frac{\mu\sigma}{4\pi t}\right)^{3/2}\exp\left(\frac{-\mu\sigma|\vec{x}-\vec{x}\,'|^2}{4t}\right)$$

d) Suppose that at time $t' = 0$, the initial vector potential $\vec{A}(\vec{x}\,',0)$ is nonvanishing only in a localized region of linear extent d around the origin. The time dependence of the fields is observed at a point P far from the origin, i.e., $|\vec{x}\,| = r \gg d$. Show that there are three regimes of time, $0 < t \le T_1$, $T_1 \le t \le T_2$, and $t \gg T_2$. Give plausible definitions of T_1 and T_2, and describe qualitatively the time dependence at P. Show that in the last regime, the vector potential is proportional to the volume integral of $\vec{A}(\vec{x}\,',0)$ times $t^{-3/2}$, assuming that integral exists. Relate your discussion to those of Section 5.18.B and Problems 5.35 and 5.36.

6.4 A uniformly magnetized and conducting sphere of radius R and total magnetic moment $m = 4\pi M R^3/3$ rotates about its magnetization axis with angular speed ω. In the steady state no current flows in the conductor. The motion is nonrelativistic; the sphere has no excess charge on it.

a) By considering Ohm's law in the moving conductor, show that the motion induces an electric field and a uniform volume charge density in the conductor, $\rho = -m\omega/\pi c^2 R^3$.

b) Because the sphere is electrically neutral, there is no monopole electric field outside. Use symmetry arguments to show that the lowest possible electric multipolarity is quadrupole. Show that only a quadrupole field exists outside and that the quadrupole moment tensor has nonvanishing components, $Q_{33} = -4m\omega R^2/3c^2$, $Q_{11} = Q_{22} = -Q_{33}/2$.

c) By considering the radial electric fields inside and outside the sphere, show that the necessary surface-chrage density $\sigma(\theta)$ is

$$\sigma(\theta) = \frac{1}{4\pi R^2}\frac{4m\omega}{3c^2}\left[1 - \frac{5}{2}P_2(\cos\theta)\right]$$

d) The rotating sphere serves as a unipolar induction device if a stationary circuit is attached by a slip ring to the pole and a sliding contact to the equator. Show that the line integral of the electric field from the equator contact to the pole contact by any path) is $\mathcal{E} = \mu_0 m\omega/4\pi R$.

6.14 An ideal circular parallel plate capacitor of radius a and plate separation $d \ll a$ is connected to a current source by axial leads, as shown in the sketch. The current in the wire is $I(t) = I_0 \cos\omega t$.

a) Calculate the electric and magnetic fields between the plates to second order in powers of the frequency (or wave number), neglecting the effects of fringing fields.

b) Calculate the volume integrals of w_e and w_m that enter the definition of the reactance X, (6.140), to second order in ω. Show that in terms of the input current I_i, defined by $I_i = -i\omega Q$, where Q is the *total charge* on one plate, these energies are

$$\int w_e\, d^3x = \frac{1}{4\pi\epsilon_0}\frac{|I_i|^2 d}{\omega^2 a^2}, \qquad \int w_m\, d^3x = \frac{\mu_0}{4\pi}\frac{|I_i|^2 d}{8}\left(1 + \frac{\omega^2 a^2}{12c^2}\right)$$

c) Show that the equivalent series circuit has $C \approx \pi\epsilon_0 a^2/d$, $L \approx \mu_0 d/8\pi$, and that an estimate for the resonant frequency of the system is $\omega_{\text{res}} \approx 2\sqrt{2}c/a$. Compare with the first root of $J_0(x)$.

6.18 Consider the Dirac expression

$$\vec{A}(\vec{x}\,) = \frac{g}{4\pi}\int_L \frac{d\vec{l}\,' \times (\vec{x} - \vec{x}\,')}{|\vec{x} - \vec{x}\,'|^3}$$

for the vector potential of a magnetic monopole and its associated string L. Suppose for definiteness that the monopole is located at the origin and the string along the negative z axis.

a) Calculate $\vec{A}$ explicitly and show that in spherical coordinates it has components

$$A_r = 0, \qquad A_\theta = 0, \qquad A_\phi = \frac{g(1 - \cos\theta)}{4\pi r \sin\theta} = \left(\frac{g}{4\pi r}\right)\tan\frac{\theta}{2}$$

b) Verify that $\vec{B} = \vec{\nabla} \times \vec{A}$ is the Coulomb-like field of a point charge, except perhaps at $\theta = \pi$.

c) With the $\vec{B}$ determined in part *b*), evaluate the total magnetic flux passing through the circular loop of radius $R\sin\theta$ shown in the figure. Consider $\theta < \pi/2$ and $\theta > \pi/2$ spearately, but always calculate the upward flux.

d) From $\oint \vec{A} \cdot d\vec{L}$ around the loop, determine the total magnetic flux through the loop. Compare the result with that found in part *c*). Show that they are equal for $0 < \theta < \pi/2$, but have a *constant* difference for $\pi/2 < \theta < \pi$. Interpret this difference.