Physics 505 **Fall 2005**

Homework Assignment #9 — Solutions

Textbook problems: Ch. 5: 5.20, 5.22, 5.26
Ch. 6: 6.1

5.20 *a*) Starting from the force equation (5.12) and the fact that a magnetization $\vec{M}$ inside a volume V bounded by a surface S is equivalent to a volume current density $\vec{J}_m = (\vec{\nabla} \times \vec{M})$ and a surface current density $(\vec{M} \times \vec{n}\,)$, show tha in the absence of macroscopic conduction currents the total magnetic force on the body can be written

$$\vec{F} = -\int_V (\vec{\nabla} \cdot \vec{M})\vec{B}_e \, d^3x + \int_S (\vec{M} \cdot \vec{n}\,)\vec{B}_e \, da$$

where $\vec{B}_e$ is the applied magnetic induction (not including that of the body in question). The force is now expressed in terms of the effective charge densities ρ_M and σ_M. If the distribution of magnetization is now discontinuous, the surface can be at infinity and the force given by just the volume integral.

Given volume and surface current densities, we may write the force as

$$\vec{F} = \int_V \vec{J} \times \vec{B} \, d^3x + \int_S \vec{K} \times \vec{B} \, da$$

Using $\vec{J}_m = \vec{\nabla} \times M$ and $\vec{K}_m = \vec{M} \times \hat{n}$, we have

$$\begin{aligned}
\vec{F} &= \int_V (\vec{\nabla} \times \vec{M}) \times \vec{B}_e \, d^3x + \int_S (\vec{M} \times \hat{n}) \times \vec{B}_e \, da \\
&= -\int_V \vec{B}_e \times (\vec{\nabla} \times \vec{M}) \, d^3x + \int_S (\vec{M} \times \hat{n}) \times \vec{B}_e \, da
\end{aligned}$$

All that remains now is an exercise in vector calculus. We begin with the identity

$$\vec{\nabla}(\vec{a} \cdot \vec{b}) = (\vec{a} \cdot \vec{\nabla})\vec{b} + (\vec{b} \cdot \vec{\nabla})\vec{a} + \vec{a} \times (\vec{\nabla} \times \vec{b}) + \vec{b} \times (\vec{\nabla} \times \vec{a})$$

with $\vec{a} = \vec{M}$ and $\vec{b} = \vec{B}_e$ as well as the BAC–CAB rule on the surface term to write

$$\begin{aligned}
\vec{F} = &\int_V [-\vec{\nabla}(\vec{M} \cdot \vec{B}_e) + (\vec{M} \cdot \vec{\nabla})\vec{B}_e + (\vec{B}_e \cdot \vec{\nabla})\vec{M} + \vec{M} \times (\vec{\nabla} \times \vec{B}_e)] \, d^3x \\
&+ \int_S [(\vec{B}_e \cdot \vec{M})\hat{n} - (\vec{B}_e \cdot \hat{n})\vec{M}] \, da \qquad (1) \\
= &\int_V [(\vec{M} \cdot \vec{\nabla})\vec{B}_e + (\vec{B}_e \cdot \vec{\nabla})\vec{M}] \, d^3x - \int_S (\vec{B}_e \cdot \hat{n})\vec{M} \, da
\end{aligned}$$

To obtain the second line, we made use of the fact that we are in a source free region for the applied $\vec{B}_e$ (so that $\vec{\nabla} \times \vec{B}_e = 0$) and we also integrated the total divergence to cancel one of the surface terms. To proceed, we note that the volume terms may be integrated by parts. In particular

$$\int_V (\vec{a} \cdot \vec{\nabla}) b_i \, d^3x = \int_V [\nabla \cdot (\vec{a} b_i) - (\vec{\nabla} \cdot \vec{a}) b_i] \, d^3x = -\int_V (\vec{\nabla} \cdot \vec{a}) b_i \, d^3x + \int_S (\hat{n} \cdot \vec{a}) b_i \, da$$

or in a full vector notation

$$\int_V (\vec{a} \cdot \vec{\nabla}) \vec{b} \, d^3x = -\int_V (\vec{\nabla} \cdot \vec{a}) \vec{b} \, d^3x + \int_S (\hat{n} \cdot \vec{a}) \vec{b} \, da$$

Using this partial integration on (1) results in

$$\begin{aligned} \vec{F} &= -\int_V [(\vec{\nabla} \cdot \vec{M}) \vec{B}_e + (\vec{\nabla} \cdot \vec{B}_e) \vec{M}] \, d^3x + \int_S (\hat{n} \cdot \vec{M}) \vec{B}_e \, da \\ &= -\int_V (\vec{\nabla} \cdot \vec{M}) \vec{B}_e \, d^3x + \int_S (\hat{n} \cdot \vec{M}) \vec{B}_e \, da \end{aligned} \tag{2}$$

where we also used $\vec{\nabla} \cdot \vec{B}_e = 0$.

b) A sphere of radius R with uniform magnetization has its center at the origin of coordinates and its direction of magnetization making spherical angles θ_0, ϕ_0. If the external magnetic field is the same as in Problem 5.11, use the expression of part *a*) to evaluate the components of the force acting on the sphere.

Since the magnetization is uniform (ie constant), the volume gradient term vanishes, and we are left with a surface integral. Explicitly, the magnetization vector may be written

$$\vec{M} = M_0(\sin\theta_0 \cos\phi_0, \sin\theta_0 \sin\phi_0, \cos\theta_0)$$

while the magnetic induction vector is

$$\vec{B}_e = B_0(1 + \beta y, 1 + \beta x, 0) = B_0(1 + \beta r \sin\theta \sin\phi, 1 + \beta r \sin\theta \cos\phi, 0)$$

We have used spherical coordinates where the normal vector is

$$\hat{n} = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$$

Then

$$\begin{aligned} \vec{F} = \int_S (\vec{M} \cdot \hat{n}) \vec{B}_e \, da = R^2 M_0 B_0 \int d\Omega \, [\cos\theta \cos\theta_0 + \sin\theta \sin\theta_0 \cos(\phi - \phi_0)] \\ \times (1 + \beta R \sin\theta \sin\phi, 1 + \beta R \sin\theta \cos\phi, 0) \end{aligned}$$

It is straightforward to perform the ϕ integral. The result is

$$\begin{aligned}\vec{F} &= 2\pi R^2 M_0 B_0 \int_{-1}^{1} d\cos\theta\,(\cos\theta\cos\theta_0 + \tfrac{1}{2}\beta R\sin^2\theta\sin\theta_0\sin\phi_0, \\ &\qquad\qquad\qquad\qquad \cos\theta\cos\theta_0 + \tfrac{1}{2}\beta R\sin^2\theta\sin\theta_0\cos\phi_0, 0) \\ &= \tfrac{2\pi}{3} M_0 B_0 R^3 \beta(\sin\theta_0\sin\phi_0, \sin\theta_0\cos\phi_0, 0)\end{aligned}$$

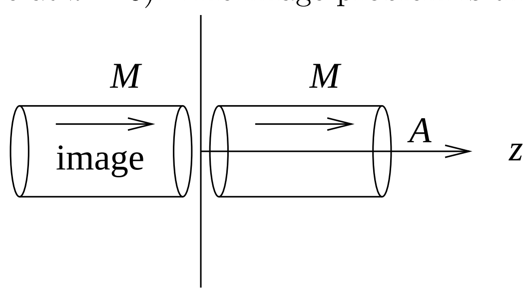
Note that the uniform (β independent) component of the magnetic field does not contribute to the force, as expected.

5.22 Show that in general a long, straight bar of uniform cross-sectional area A with uniform lenghwise magnetization M, when placed with its flat end against an infinitely permeable flat surface, adheres with a force given approximately by

$$F \simeq \frac{\mu_0}{2} A M^2$$

Relate your discussion to the electrostatic considerations in Section 1.11.

This problem is best solved by considering an image magnet. The infinite permeability of the flat surface ensures that the magnetic field must be perpendicular to the surface. As a result, this is similar to the electrostatic case of electric field lines being perpendicular to the surface of a perfect conductor. For magnetostatics, this means that we may use a magnetic scalar potential Φ_M (since there are no free currents) subject to the condition $\Phi_M = 0$ at $z = 0$ (taking the surface to lie in the x-y plane at $z = 0$). The image problem is then set up as follows



Fortunately we may make use of some of our previous results. For the magnetic induction of the image magnet, we may take the result of Homework #8, Problem 5.19

$$B_z(z) = -\frac{\mu_0 M}{2}\left[\frac{z}{\sqrt{a^2+z^2}} - \frac{z+L}{\sqrt{a^2+(z+L)^2}}\right]$$

Where we have shifted to coordinates so that the image magnet lies between $z = -L$ and $z = 0$. Note that, strictly speaking, this is the magnetic induction on the axis of the magnet. However, for a long straight bar, we may approximate the magnetic induction to be roughly uniform across the face of the magnet. Since we assume the magnetization to be uniform, we may compute the force from the surface term of (2)

$$\begin{aligned}
\vec{F} &= \int_S (\hat{n}\cdot\vec{M})\vec{B}_e\,da = \hat{z}M\int[-B_z(0)+B_z(L)]\,da \\
&\approx \hat{z}MA[B_z(L)-B_z(0)] = \hat{z}\frac{\mu_0 M^2 A}{2}\left[\frac{2L}{\sqrt{a^2+4L^2}} - \frac{L}{\sqrt{a^2+L^2}} - \frac{L}{\sqrt{a^2+L^2}}\right] \\
&\approx -\hat{z}\frac{\mu_0 M^2 A}{2}
\end{aligned}$$

where in the last line we used $L \gg a$ (a condition that we needed anyway to ensure that B_z is nearly uniform on the endcaps).

5.26 A two-wire transmission line consists of a pair of nonpermeable parallel wires of radii a and b separated by a distance $d > a+b$. A current flows down one wire and back the other. It is uniformly distributed over the cross section of each wire. Show that the self-inductance per unit length is

$$L = \frac{\mu_0}{4\pi}\left[1 + 2\ln\left(\frac{d^2}{ab}\right)\right]$$

This problem appears straightforward, but can be quite tedious if not approached with care. Considering both wires as a single circuit, we could take the definition

$$L = \frac{\mu_0}{4\pi I^2}\int\frac{\vec{J}(\vec{x}\,)\cdot\vec{J}(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\,d^3x\,d^3x' \tag{3}$$

where both $\vec{x}$ and $\vec{x}\,'$ are individually taken over wires a and b. However, working out this integral requires care both because of the geometry and because the total inductance is actually infinite (it is only the inductance per unit length that is finite). [For uniform anti-parallel current densities, the numerator is almost trivial, but the denominator causes difficulty.]

Instead, using

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\,d^3x' \tag{4}$$

we may rewrite the inductance expression as

$$L = \frac{1}{I^2}\int\vec{J}(\vec{x}\,)\cdot\vec{A}(\vec{x}\,)\,d^3x \tag{5}$$

which is exactly the same as we would have gotten from energy considerations $W = \frac{1}{2}LI^2$ and $W = \frac{1}{2}\int\vec{J}\cdot\vec{A}\,d^3x$. To proceed, we need to find an expression for the vector potential $\vec{A}$. Of course, we would like to avoid using the integral (4). as that would be just as bad as using (3). What we can do instead is to consider each wire one at a time (because of linear superposition). Then for a single wire (say the one of radius a), Ampère's law gives the simple result

$$\vec{B} = \frac{\mu_0 I}{2\pi a}\frac{\rho_<}{\rho_>}\hat{\phi} \tag{6}$$

Based on (4), and the fact that the current density $\vec{J}$ is only along the $\hat{z}$ direction, we seek a vector potential A_z that gives rise to this magnetic induction. From the cross product $\vec{B} = \vec{\nabla} \times \vec{A}$ we see that

$$B_\phi(\rho) = -\frac{\partial}{\partial\rho}A_z(\rho) \qquad \Rightarrow \qquad A_z = -\int B_\phi(\rho)\,d\rho$$

Integrating the magnetic induction, we obtain

$$\vec{A} = -\frac{\mu_0 I}{4\pi}\hat{z}\begin{cases}(\rho/a)^2, & \rho < a\\ 1+\ln[(\rho/a)^2], & \rho > a\end{cases}$$

where the constants were chosen to make $\vec{A}$ continuous at the surface of the wire. [It may appear that an overall constant shift of $\vec{A}$ would modify the inductance according to (5). However, once we assume the vector current density $\vec{J}$ integrates to zero over the entire volume of the inductor, as must happen for a closed circuit, then this constant drops out.] Note, in addition, that this choice of vector potential trivially satisfies the Coulomb gauge condition $\vec{\nabla}\cdot\vec{A} = 0$. Combining both wires (with currents in opposite directions), we arrive at

$$\vec{A} = -\frac{\mu_0 I}{4\pi}\hat{z}\left[\left\{\begin{array}{ll}(\rho/a)^2, & \rho < a\\ 1+\ln[(\rho/a)^2], & \rho > a\end{array}\right\} - \left\{\begin{array}{ll}(\rho'/b)^2, & \rho' < b\\ 1+\ln[(\rho'/b)^2], & \rho' > b\end{array}\right\}\right]$$

where ρ and ρ' are the perpendicular distances to the centers of wires a and b, respectively. Returning to (5), we see that the inductance integral will have two contributions, one from the current flowing in wire a and the other from the current in wire b. We may break this up into two terms, and only calculate the wire a contribution. The wire b contribution may then be obtained by the interchange $a \leftrightarrow b$. In wire a, the current density is $\vec{J}_a = (I/\pi a^2)\hat{z}$. Furthermore, since the wires are separated, we have

$$\vec{A}_a = -\frac{\mu_0 I}{4\pi}\hat{z}\left((\rho/a)^2 - 1 - \ln[(\rho'/b)^2]\right)$$

Thus

$$L_a = \frac{\mu_0}{4\pi^2 a^2}\int_{\rho<a}\left(1-(\rho/a)^2 + \ln[(\rho'/b)^2]\right)\,\rho d\rho\,d\phi$$

The first two terms in the integrand are trivial to integrate. However, the log term takes some care, as it is based on the distance to the second center, ρ' (which may be obtained from the law of cosines, $\rho'^2 = \rho^2 + d^2 - 2\rho d \cos\phi$). Integrating the first two terms gives

$$L_a = \frac{\mu_0}{4\pi^2 a^2}\left(\frac{\pi}{2}a^2 + \int_0^a \rho d\rho \int_0^{2\pi} \ln[(\rho'/b)^2]\, d\phi\right)$$

To proceed, we note that the vector potential is harmonic in a source-free region, $\nabla^2 \vec{A} = 0$. In particular, since this problem is independent of z, this implies that $\vec{A}$ due to wire b is harmonic in two dimensions inside of wire a (where the integral is performed). We then recall that the average of any harmonic function around a circle is equal to the function at the center of the circle. Hence

$$\int_0^{2\pi} \ln[(\rho'/b)^2]\, d\phi = 2\pi \ln[(d/b)^2]$$

since $\rho' = d$ at the center of wire a. [Equivalently, we see that $\ln(\rho'/b)^2$ is the real part of an analytic function $\ln[(z/b-1)^2]$, which explicitly demonstrates that it is harmonic. Showing that the average value of an analytic function around a circle is equal to the value at the center is then a simple application of Cauchy's integral formula.] In any case, this allows us to integrate the log term exactly. The result is simply

$$L_a = \frac{\mu_0}{4\pi^2 a^2}\left(\frac{\pi}{2}a^2 + \pi a^2 \ln[(d/b)^2]\right) = \frac{\mu_0}{4\pi}\left(\frac{1}{2} + \ln\frac{d^2}{b^2}\right)$$

Combining L_a and a similar expression for L_b finally gives

$$L = \frac{\mu_0}{4\pi}\left(1 + \ln\frac{d^4}{a^2 b^2}\right) \tag{7}$$

which we stress is an *exact* result.

Incidentally, a more direct use of energetics would yield an inductance expression of the form

$$L = \frac{1}{\mu_0 I^2}\int |\vec{B}(\vec{x}\,)|^2\, d^3x$$

However, any attempt to integrate the magnetic energy in this fashion turns out to be extremely tedious. This is because the integral is over two centers (ie with distances ρ and ρ'), and hence one must give up cylindrical symmetry. Of course, the magnetic induction is straightforwardly given by applying (6) to both wires

$$\vec{B} = \frac{\mu_0 I}{2\pi}\left(\frac{\rho_<}{a\rho_>}\hat{\phi} - \frac{\rho'_<}{b\rho'_>}\hat{\phi}'\right)$$

so that

$$|\vec{B}\,|^2 = \left(\frac{\mu_0 I}{2\pi}\right)\left(\frac{1}{a^2}\left(\frac{\rho_<}{\rho_>}\right)^2 + \frac{1}{b^2}\left(\frac{\rho'_<}{\rho'_>}\right)^2 - \frac{1}{ab}\frac{\rho_<\rho'_<}{\rho_>\rho'_>}\frac{\rho^2+\rho'^2-d^2}{\rho\rho'}\right)$$

where we have used some geometry to evaluate

$$\hat{\phi} \cdot \hat{\phi}' = \frac{\rho^2 + \rho'^2 - d^2}{2\rho\rho'}$$

The volume (actually area, since we work per unit length in the z direction) integral can be performed by a suitable change of variables

$$\int d^2x = 2 \int J \, d\rho \, d\rho'$$

where the Jacobian is

$$J = \left| \frac{\partial(x, y)}{\partial(\rho, \rho')} \right| = \frac{2\rho\rho'}{\sqrt{\Delta}}$$

The factor of two in the integral is a result of the two-fold degeneracy of specifying a point (x, y) based on (ρ, ρ'). The quantity

$$\Delta = -d^4 - \rho^4 - \rho'^4 + 2(d^2\rho^2 + d^2\rho'^2 + \rho^2\rho'^2)$$

is actually non-negative in the integration region

$$\rho + \rho' > d, \qquad |\rho - \rho'| < d$$

(obtained by satisfying the triangle inequality). The expression for the inductance then has the form

$$L = \frac{\mu_0}{2\pi^2} \int J \, d\rho \, d\rho' \left(\frac{1}{a^2} \left(\frac{\rho_<}{\rho_>} \right)^2 + \frac{1}{b^2} \left(\frac{\rho'_<}{\rho'_>} \right)^2 - \frac{1}{ab} \frac{\rho_< \rho'_<}{\rho_> \rho'_>} \frac{\rho^2 + \rho'^2 - d^2}{\rho\rho'} \right)$$

This integration can be broken up into three regions, $\rho < a$ (inside the first wire), $\rho' < b$ (inside the second wire) and $\{\rho > a, \rho' > b\}$ (outside both wires). Specifically, these three integrals are

$$\begin{aligned}
I_1 &= \frac{\mu_0}{\pi^2} \int_0^a \rho d\rho \int_{d-\rho}^{d+\rho} \rho' d\rho' \frac{1}{\sqrt{\Delta}} \left(\frac{\rho^2 - a^2}{a^4} + \frac{d^2 + a^2 - \rho^2}{a^2 \rho'^2} \right) \\
I_2 &= \frac{\mu_0}{\pi^2} \int \frac{d\rho}{\rho} \int \frac{d\rho'}{\rho'} \frac{d^2}{\sqrt{\Delta}} \\
I_3 &= \frac{\mu_0}{\pi^2} \int_0^b \rho' d\rho' \int_{d-\rho'}^{d+\rho'} \rho d\rho \frac{1}{\sqrt{\Delta}} \left(\frac{\rho'^2 - b^2}{b^4} + \frac{d^2 + b^2 - \rho'^2}{b^2 \rho^2} \right)
\end{aligned}$$

(where the integration region is left implicit in I_2). The integrals I_1 and I_3 are straightforward, and may be evaluated by first using

$$\int \frac{1}{\sqrt{(a-x)(b+x)}} dx = \tan^{-1} \left(\frac{(b+x) - (a-x)}{2\sqrt{(a-x)(b+x)}} \right)$$

The result is

$$I_1 = \frac{\mu_0}{4\pi}\left(\frac{1}{2} + \ln\frac{d^2}{d^2 - a^2}\right) \qquad I_3 = \frac{\mu_0}{4\pi}\left(\frac{1}{2} + \ln\frac{d^2}{d^2 - b^2}\right)$$

The I_2 integral is somewhat more involved, as the integration region is somewhat awkward to specify. One way to handle it is to start with the complete region (which must be regulated by taking, eg, an ϵ cutoff, $\rho > \epsilon$ and $\rho' > \epsilon$) and then subtracting away the $\epsilon < \rho < a$ and $\epsilon < \rho' < b$ contributions. The result is

$$I_2 = \frac{\mu_0}{4\pi}\left(\ln\frac{d^4}{\epsilon^4} - \ln\frac{d^2a^2}{\epsilon^2(d^2 - a^2)} - \ln\frac{d^2b^2}{\epsilon^2(d^2 - b^2)}\right) = \frac{\mu_0}{4\pi}\ln\frac{(d^2 - a^2)(d^2 - b^2)}{a^2b^2}$$

Adding all three contributions together gives

$$L = I_1 + I_2 + I_3 = \frac{\mu_0}{4\pi}\left(1 + \ln\frac{d^4}{a^2b^2}\right)$$

which is the same result as (7).

6.1 In three dimensions the solution to the wave equation (6.32) for a point source in space and time (a light flash at $t' = 0, \vec{x}\,' = 0$) is a spherical shell disturbance of radius $R = ct$, namely the Green function $G^{(+)}$ (6.44). It may be initially surprising that in one or two dimensions, the disturbance possesses a "wake", even though the source is a "point" in space and time. The solutions for fewer dimensions than three can be found by superposition in the superfluous dimension(s), to eliminate dependence on such variable(s). For example, a flashing line source of uniform amplitude is equivalent to a point source in two dimensions.

a) Starting with the retarded solution to the three-dimensional wave equation (6.47), show that the source $f(\vec{x}\,', t) = \delta(x')\delta(y')\delta(t')$, equivalent to a $t = 0$ point source at the origin in two spatial dimensions, produces a two-dimensional wave

$$\Psi(x, y, t) = \frac{2c\Theta(ct - \rho)}{\sqrt{c^2t^2 - \rho^2}}$$

where $\rho^2 + x^2 + y^2$ and $\Theta(\xi)$ is the unit step function [$\Theta(\xi) = 0$ (1) if $\xi <$ ($>$) 0.]

Using

$$\Psi(\vec{x}, t) = \int \frac{[f(\vec{x}\,', t')]_{\mathrm{ret}}}{|\vec{x} - \vec{x}\,'|}\, d^3x'$$

we find

$$\begin{aligned}\Psi(\vec{x}, t) &= \int \frac{\delta(x')\delta(y')\delta(t - R/c)}{R}\, dx'dy'dz' \\ &= \int_{-\infty}^{\infty} \frac{\delta(t - R/c)}{R}\, dz'\end{aligned}$$

where

$$R = |\vec{x} - \vec{x}\,'| = \sqrt{\rho^2 + (z - z')^2} \qquad \text{when } x' = y' = 0$$

By shifting $z' \to z' + z$, we end up with the integral

$$\Psi(\rho, t) = \int_{-\infty}^{\infty} \frac{\delta(t - \sqrt{\rho^2 + z'^2}/c)}{\sqrt{\rho^2 + z'^2}}\, dz' \tag{8}$$

Using

$$\delta(f(\zeta)) = \sum_i \frac{1}{|f'(\zeta)|} \delta(\zeta - \zeta_i) \tag{9}$$

where the sum is over the zeros of $f(\zeta)$, we see that

$$\delta(t - \sqrt{\rho^2 + z'^2}/c) = \sum_i \frac{c\sqrt{\rho^2 + z'^2}}{|z'|} \delta(z' - z'_i)$$

The zeros z'_i are given by

$$\rho^2 + z'^2 = c^2 t^2 \qquad \Rightarrow \qquad z' = \pm\sqrt{c^2 t^2 - \rho^2}$$

However it is clear that there are real zeros only if $c^2 t^2 \geq \rho^2$ or $\rho < ct$. Going back to (8), and noting there are two zeros (one for each sign of the square root), we end up with

$$\Psi(\rho, t) = \frac{2c\Theta(ct - \rho)}{\sqrt{c^2 t^2 - \rho^2}}$$

b) Show that a "sheet" source, equivalent to a point pulsed source at the origin in one space dimension, produces a one-dimensional wave proportional to

$$\psi(x, t) = 2\pi c \Theta(ct - |x|)$$

For the sheet source, we use $f(\vec{x}\,', t') = \delta(x')\delta(t')$ to write

$$\Psi(\vec{x}, t) = \int \frac{\delta(x')\delta(t - R/c)}{R}\, dx' dy' dz'$$

where $R = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$. By integrating x' and shifting $y' \to y' + y$ and $z' \to z' + z$ we end up with

$$\Psi(x, t) = \int \frac{\delta(t - \sqrt{x^2 + y'^2 + z'^2}/c)}{\sqrt{x^2 + y'^2 + z'^2}}\, dy' dz' = \int \frac{\delta(t - \sqrt{\rho'^2 + x^2}/c)}{\sqrt{\rho'^2 + x^2}}\, \rho' d\rho' d\phi'$$

where we have gone to polar coordinates in the y'-z' plane. The ϕ' integral is now trivial. Treating the delta function as in (9) results in

$$\Psi(x, t) = 2\pi \int_0^{\infty} \sum_i c\delta(\rho' - \rho'_i) d\rho'$$

where the zeros ρ_i' corespond to

$$\rho'^2 + x^2 = c^2t^2 \qquad \Rightarrow \qquad \rho' = \pm\sqrt{c^2t^2 - x^2}$$

Since ρ' is non-negative, only the positive zero contributes, and we end up with

$$\Psi(x,t) = 2\pi c\Theta(ct - |z|)$$

where the step function enforces the condition for a real zero to exist.