Physics 505 **Fall 2005**

Homework Assignment #8 — Solutions

Textbook problems: Ch. 5: 5.10, 5.14, 5.17, 5.19

5.10 A circular current loop of radius a carrying a current I lies in the x-y plane with its center at the origin.

a) Show that the only nonvanishing component of the vector potential is

$$A_\phi(\rho, z) = \frac{\mu_0 I a}{\pi} \int_0^\infty dk\, \cos kz\, I_1(k\rho_<)K_1(k\rho_>)$$

where $\rho_<$ ($\rho_>$) is the smaller (larger) of a and ρ.

The vector potential may be obtained by

$$\vec{A}(\vec{x}\,) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|}\, d^3x'$$

where (for a circular current loop)

$$\vec{J}(\vec{x}\,') = I\delta(z')\delta(\rho' - a)\hat{\phi}'$$

in cylindrical coordinates. Note that to obtain the cylindrical components of $\vec{A}(\vec{x}\,)$ we have to be careful to convert the basis vector $\hat{\phi}'$ at the point x' to components at x. (This is because the basic vectors depend on position.) A bit of geometry gives

$$\hat{\phi}' = \hat{\rho}\sin(\phi - \phi') + \hat{\phi}\cos(\phi - \phi')$$

[Or, alternatively, we may choose the point x to lie at $\phi = 0$, so that $\hat{\phi} = \hat{y}$ and $\hat{\rho} = \hat{x}$. Then it is straightforward to see that $\hat{\phi}' = \hat{y}\cos\phi' - \hat{x}\sin\phi' = \hat{\phi}\cos\phi' - \hat{\rho}\sin\phi'$. Using symmetry, we can see that only the $\hat{\phi}$ component of $\vec{A}$ is nonvanishing.]

The integral expression for the vector potential is then

$$\begin{aligned}\vec{A}(\vec{x}) &= \frac{\mu_0 I}{4\pi} \int \frac{\delta(z')\delta(\rho' - a)[\hat{\rho}\sin(\phi - \phi') + \hat{\phi}\cos(\phi - \phi')]}{|\vec{x} - \vec{x}\,'|}\, \rho' d\rho'\, d\phi'\, dz' \\ &= \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} \frac{\hat{\rho}\sin(\phi - \phi') + \hat{\phi}\cos(\phi - \phi')}{|\vec{x} - \vec{x}\,'|}\, d\phi' \end{aligned} \tag{1}$$

where the integrand in the second line is to be evaluated at $z' = 0$ and $\rho' = a$. We now use the cylindrical Green's function expressed as

$$\frac{1}{|\vec{x} - \vec{x}\,'|} = \frac{4}{\pi} \int_0^\infty dk\, \cos[k(z - z')]\Big[\tfrac{1}{2} I_0(k\rho_<)K_0(k\rho_>) + \sum_{m=1}^\infty \cos[m(\phi - \phi')] I_m(k\rho_<)K_m(k\rho_>)\Big]$$

Note that the integral over ϕ' picks out the $m = 1$ term in the sum. Furthermore, the $\hat{\rho}$ component drops out because $\sin(\phi - \phi')$ is orthogonal to $\cos(\phi - \phi')$, a result that could have been obtained by symmetry. We end up with

$$\begin{aligned} \vec{A}(\vec{x}\,) &= \frac{\mu_0 I a}{4\pi} \frac{4}{\pi} \pi \hat{\phi} \int_0^\infty dk\, \cos(kz) I_1(k\rho_<) K_1(k\rho_>) \\ &= \frac{\mu_0 I a}{\pi} \hat{\phi} \int_0^\infty dk\, \cos(kz) I_1(k\rho_<) K_1(k\rho_>) \end{aligned}$$

b) Show that an alternative expression for A_ϕ is

$$A_\phi(\rho, z) = \frac{\mu_0 I a}{2} \int_0^\infty dk\, e^{-k|z|} J_1(ka) J_1(k\rho)$$

To obtain the alternative expression, we use the alternate form of the Greens' function

$$\frac{1}{|\vec{x} - \vec{x}\,'|} = 2 \int_0^\infty dk\, e^{-k(z_> - z_<)} \Big[\tfrac{1}{2} J_0(k\rho) J_0(k\rho') + \sum_{m=1}^\infty \cos[m(\phi - \phi')] J_m(k\rho) J_m(k\rho') \Big]$$

Since, for $z' = 0$, we have $z_> - z_< = |z|$, it is clear that when we stick this into (1) we end up with

$$\vec{A}(\vec{x}\,) = \frac{\mu_0 I a}{2} \hat{\phi} \int_0^\infty dk\, e^{-k|z|} J_1(k\rho) J_1(ka)$$

c) Write down integral expressions for the components of magnetic induction, using the expressions of parts *a*) and *b*). Evaluate explicitly the components of $\vec{B}$ on the z axis by performing the necessary integrations.

Since $\vec{B} = \vec{\nabla} \times \vec{A}$ and the only non-vanishing component of $\vec{A}$ is A_ϕ, we end up with

$$B_\rho = -\partial_z A_\phi, \qquad B_z = \frac{1}{\rho} \partial_\rho (\rho A_\phi)$$

The z derivative is straightforward. For the ρ derivative, on the other hand, we may use the Bessel equation identity

$$\frac{d}{dz} X_1(z) + \frac{1}{z} X_1(z) = X_0(z)$$

where X_m denotes either J_m, N_m, I_m or K_m. This gives, in particular

$$\frac{1}{\rho} \partial_\rho [\rho X_1(k\rho)] = k X_0(k\rho)$$

Hence, for the expression of a) we find

$$B_\rho = \frac{\mu_0 I a}{\pi} \int_0^\infty dk\, k \sin(kz) I_1(k\rho_<) K_1(k\rho_>)$$

and

$$B_z = \frac{\mu_0 I a}{\pi} \int_0^\infty dk\, k \cos(kz) \left\{ \begin{array}{c} I_0(k\rho)K_1(ka) \\ I_1(ka)K_0(k\rho) \end{array} \right\}$$

where the top line holds for $\rho < a$, while the bottom line holds for $\rho > a$.

Similarly, the vector potential of b) yields the magnetic induction

$$B_\rho = -\frac{\mu_0 I a}{2} \operatorname{sgn(z)} \int_0^\infty dk\, k e^{-k|z|} J_1(k\rho) J_1(ka)$$

and

$$B_z = \frac{\mu_0 I a}{2} \int_0^\infty dk\, k e^{-k|z|} J_0(k\rho) J_1(ka)$$

The z axis corresponds to $\rho = 0$. In this case, it is easy to see that $B_\rho = 0$ (a result demanded by symmetry) follows from the result that either $J_1(0) = 0$ or $I_1(0) = 0$. For the B_z component, we take the representation of part b). Noting that $J_0(0) = 1$, we end up with

$$\begin{aligned} B_z(\rho = 0) &= \frac{\mu_0 I a}{2} \int_0^\infty dk\, k e^{-k|z|} J_1(ka) \\ &= \frac{\mu_0 I a}{2} \frac{a}{(z^2 + a^2)^{3/2}} \\ &= \frac{\mu_0 I a^2}{2(z^2 + a^2)^{3/2}} \end{aligned}$$

which agrees with the elementary result for a current loop on axis. [This integral was performed by noting that it is a Laplace transform $\mathcal{L}\{tJ_1(at)\}$, which in turn is the derivative $-d/ds$ of the transform $\mathcal{L}\{J_1(at)\}$. The Laplace transform of a Bessel function can be looked up, with the result $\mathcal{L}\{J_n(at)\} = a^{-n}(\sqrt{s^2 + a^2} - s)^n/\sqrt{s^2 + a^2}$.]

5.14 A long, hollow, right circular cylinder of inner (outer) radius a (b), and of relative permeability μ_r, is placed in a region of initially uniform magnetic-flux density $\vec{B}_0$ at right angles to the field. Find the flux density at all points in space, and sketch the logarithm of the ratio of the magnitudes of $\vec{B}$ on the cylinder axis to $\vec{B}_0$ as a function of $\log_{10} \mu_r$ for $a^2/b^2 = 0.5, 0.1$. Neglect end effects.

For a long cylinder (neglecting end effects) we may think of this as a two-dimensional problem. Since there are no current sources, we use a magnetic scalar potential Φ_M which must be harmonic in two dimensions. Since $\vec{H} = -\vec{\nabla}\Phi_M$, we orient the uniform magnetic field H_0 along the $+x$ axis and write

$$\Phi_M(\rho,\phi) = \begin{cases} (-H_0\rho + \sum \frac{\alpha}{\rho})\cos\phi, & \rho > b \\ (\beta\rho + \frac{\gamma}{\rho})\cos\phi, & a < \rho < b \\ \delta\rho\cos\phi, & \rho < a \end{cases} \tag{2}$$

Of course, the general harmonic expansion would be of the form $(A_m\rho^m + B_m\rho^{-m})\cos m\phi + (C_m\rho^m + D_m\rho^{-m})\sin m\phi$. However here we have already used the shortcut that all matching conditions for $m \neq 1$ lead to homogeneous equations admitting only a trivial (zero) solution.

The magnetostatic boundary conditions demand that H_ϕ and B_ρ are continuous at both $\rho = a$ and $\rho = b$. The magnetic field (and magnetic induction) components are

$$H_\phi = -\frac{1}{\rho}\partial_\phi\Phi_M = \begin{cases} (-H_0 + \frac{\alpha}{\rho^2})\sin\phi, & \rho > b \\ (\beta + \frac{\gamma}{\rho^2})\sin\phi, & a < \rho < b \\ \delta\sin\phi, & \rho, a \end{cases}$$

and

$$B_\rho = \mu\partial_\rho\Phi_M = \begin{cases} \mu_0(-H_0 - \frac{\alpha}{\rho^2})\cos\phi, & \rho > b \\ \mu(\beta - \frac{\gamma}{\rho^2})\cos\phi, & a < \rho < b \\ \mu_0\delta\cos\phi, & \rho < a \end{cases}$$

The resulting matching conditions at a and b are

$$\begin{aligned} -H_0 + \frac{\alpha}{b^2} &= \beta + \frac{\gamma}{b^2}, & -H_0 - \frac{\alpha}{b^2} &= \mu_r\left(\beta - \frac{\gamma}{b^2}\right) \\ \beta + \frac{\gamma}{a^2} &= \delta, & \beta - \frac{\gamma}{a^2} &= \frac{1}{\mu_r}\delta \end{aligned}$$

where $\mu_r = \mu/\mu_0$. These equations may be solved to yield

$$\begin{aligned} \alpha &= \Delta^{-1}(\mu_r - \mu_r^{-1})(b^2 - a^2)H_0 \\ \beta &= -2\Delta^{-1}(1 + \mu_r^{-1})H_0 \\ \gamma &= -2\Delta^{-1}(1 - \mu_r^{-1})a^2H_0 \\ \delta &= -4\Delta^{-1}H_0 \end{aligned}$$

where

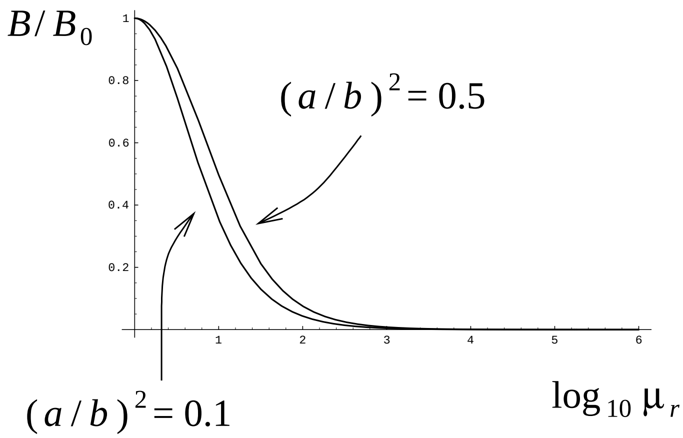
$$\Delta = (1+\mu_r)(1+\mu_r^{-1}) + (1-\mu_r)(1-\mu_r^{-1})\left(\frac{a}{b}\right)^2 = \frac{1}{\mu_r}\left[(\mu_r+1)^2 - (\mu_r-1)^2\left(\frac{a}{b}\right)^2\right]$$

The magnetic scalar potential is then given by (2) with the above values of the coefficients. We see that the magnetic induction for $\rho < a$ is uniform, pointed along the same direction as $\vec{B}_0$. The other two regions contain a dipole field in addition a uniform component.

Since $\vec{H} = -\vec{\nabla}\Phi_M = -\delta\hat{x}$ for $\rho < a$, the ratio of $\vec{B}$ on axis ($\rho = 0$) to $\vec{B}_0$ is given by

$$\frac{B}{B_0} = 4\Delta^{-1} = \frac{4}{(1+\mu_r)(1+\mu_r^{-1}) + (1-\mu_r)(1-\mu_r^{-1})(a/b)^2}$$

This may be plotted as follows



5.17 A current distribution $\vec{J}(\vec{x}\,)$ exists in a medium of unit relative permeability adjacent to a semi-infinite slab of material having relative permeability μ_r and filling the half-space, $z < 0$.

a) Show that for $z > 0$ the magnetic induction can be calculated by replacing the medium of permeability μ_r by an image current distribution, $\vec{J}^*$, with components,

$$\left(\frac{\mu_r - 1}{\mu_r + 1}\right) J_x(x, y, -z), \qquad \left(\frac{\mu_r - 1}{\mu_r + 1}\right) J_y(x, y, -z), \qquad -\left(\frac{\mu_r - 1}{\mu_r + 1}\right) J_z(x, y, -z)$$

We will end up solving parts *a*) and *b*) simultaneously. We start, however, by defining the reflection (Parity) operator $P : z \to -z$ so that

$$P : (x, y, z) \to (x, y, -z)$$

On the right ($z > 0$), we assume the magnetic induction is generated by both the original current $\vec{J}$ (contained entirely on the right) and an image current $\vec{J}^*$ (contained entirely on the left). Thus

$$\vec{B}_R(\vec{x}\,) = \frac{\mu_0}{4\pi}\int \frac{(\vec{J}(\vec{x}\,') + \vec{J}^*(\vec{x}\,')) \times (\vec{x} - \vec{x}\,')}{|\vec{x} - \vec{x}\,'|^3}\, d^3x'$$

By changing variables $z' \to -z'$ in the $\vec{J}^*$ term, we may restrict this volume integral to $z' > 0$

$$\vec{B}_R(\vec{x}\,) = \frac{\mu_0}{4\pi}\int_{z'>0}\left(\frac{\vec{J}(\vec{x}\,') \times (\vec{x} - \vec{x}\,')}{|\vec{x} - \vec{x}\,'|^3} + \frac{\vec{J}^*(P\vec{x}\,') \times (\vec{x} - P\vec{x}\,')}{|\vec{x} - P\vec{x}\,'|^3}\right) d^3x' \qquad (3)$$

On the left ($z < 0$), we assume the magnetic induction is generated by a current of the same form as the original $\vec{J}$, but with possibly modified strength (because of the change of permeability). Given a modified current $\lambda\vec{J}$ and permeability μ, we write

$$\vec{B}_L(\vec{x}\,) = \frac{\mu\lambda}{4\pi}\int_{z'>0} \frac{\vec{J}(\vec{x}\,') \times (\vec{x} - \vec{x}\,')}{|\vec{x} - \vec{x}\,'|^3} d^3x' \tag{4}$$

Our aim is now to match the left and right magnetic field and magnetic induction. More precisely, at $z = 0$, both H_x and H_y (the parallel components) must be continuous, and B_z (the perpendicular component) must also be continuous. To perform this matching, we first note that the norms $|\vec{x} - \vec{x}\,'|$ and $|\vec{x} - P\vec{x}\,'|$ are identical at $z = 0$. (The are both equal to $\sqrt{(x - x')^2 + (y - y')^2 + z'^2}$.) Thus all denominators are the same, and we deduce that the numerators of (3) and (4) must be matched as appropriate. For B_z, we have

$$(J_x + J_x^*)(y - y') - (J_y + J_y^*)(x - x') = \mu_r\lambda(J_x(y - y') - J_y(x - x'))$$

where any component of $\vec{J}^*$ is understood to have argument $P\vec{x}$. For H_x and H_y matching, we find

$$\begin{aligned} -(J_y - J_y^*)z' - (J_z + J_z^*)(x - x') &= \lambda(-J_y z' - J_z(x - x')) \\ (J_z + J_z^*)(x - x') + (J_x - J_x^*)z' &= \lambda(J_z(x - x') + J_x z') \end{aligned}$$

Since these equations hold for all values of (x, y), they separate into

$$\begin{aligned} \lambda J_y &= J_y - J_y^* & \lambda J_z &= J_z + J_z^* \\ \lambda J_z &= J_z + J_z^* & \lambda J_x &= J_x - J_x^* \\ \mu_r\lambda J_x &= J_x + J_x^* & \mu_r\lambda J_y &= J_y + J_y^* \end{aligned}$$

These equations may be solved to yield

$$J_x^* = (1 - \lambda)J_x, \qquad J_y^* = (1 - \lambda)J_y, \qquad J_z = -(1 - \lambda)J_z$$

provided $\mu_r\lambda - 1 = 1 - \lambda$, or $\lambda = 2/(\mu_r + 1)$. This may be given in a more concise form using the reflection operator

$$\vec{J}^*(\vec{x}\,) = (1 - \lambda)P\vec{J}(P\vec{x}\,) = \frac{\mu_r - 1}{\mu_r + 1}P\vec{J}(P\vec{x}\,)$$

b) Show that for $z < 0$ the magnetic induction appears to be due to a current distribution $[2\mu_r/(\mu_r + 1)]\vec{J}$ in a medium of unit relative permeability.

From the expression (4) for $\vec{B}_L$, the magnetic induction appears to be due to a current $\lambda\vec{J} = [2/(\mu_r + 1)]\vec{J}$ in a medium of permeability μ. This is equivalent to having a current distribution $[2\mu_r/(\mu_r+1)]\vec{J}$ in a medium of *unit* relative permeability.

5.19 A magnetically "hard" material is in the shape of a right circular cylinder of length L and radius a. The cylinder has a permanent magnetization M_0, uniform throughout its volume and parallel to its axis.

a) Determine the magnetic field $\vec{H}$ and magnetic induction $\vec{B}$ at all points on the axis of the cylinder, both inside and outside.

We use a magnetic scalar potential and the expression

$$\Phi_M = -\frac{1}{4\pi}\int_V \frac{\vec{\nabla}\cdot\vec{M}(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\,d^3x' + \frac{1}{4\pi}\oint_S \frac{\hat{n}'\cdot\vec{M}(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\,da'$$

Orienting the cylinder along the z axis, we take a uniform magnetization $\vec{M} = M_0\hat{z}$. In this case the volume integral drops out, and the surface integral only picks up contributions on the endcaps. Thus

$$\Phi_M = \frac{M_0}{4\pi}\left[\int_{\text{top}} \frac{1}{|\vec{x}-\vec{x}\,'|}\,da' - \int_{\text{bottom}} \frac{1}{|\vec{x}-\vec{x}\,'|}\,da'\right]$$

where 'top' and 'bottom' denote $z = \pm L/2$, and the integrals are restricted to $\rho < a$. On axis ($\rho = 0$) we have simply

$$\begin{aligned}
\Phi_M(z) &= \frac{M_0}{4\pi}\int\left(\frac{1}{\sqrt{\rho^2+(z-L/2)^2}} - \frac{1}{\sqrt{\rho^2+(z+L/2)^2}}\right)\rho\,d\rho\,d\phi \\
&= \frac{M_0}{4}\int_0^{a^2}\left(\frac{1}{\sqrt{\rho^2+(z-L/2)^2}} - \frac{1}{\sqrt{\rho^2+(z+L/2)^2}}\right)d\rho^2 \\
&= \frac{M_0}{2}\left[\sqrt{a^2+(z-L/2)^2} - \sqrt{a^2+(z+L/2)^2} - |z-L/2| + |z+L/2|\right]
\end{aligned}$$

On axis, the field can only point in the z direction. It is given by

$$\begin{aligned}
H_z = -\partial_z\Phi_M = -\frac{M_0}{2}\Bigg[&\frac{z-L/2}{\sqrt{a^2+(z-L/2)^2}} - \frac{z+L/2}{\sqrt{a^2+(z+L/2)^2}} \\
&- \operatorname{sgn}(z-L/2) + \operatorname{sgn}(z+L/2)\Bigg]
\end{aligned}$$

Note that the last two terms cancel when $|z| > L/2$, but add up to 2 inside the magnet. Thus we may write

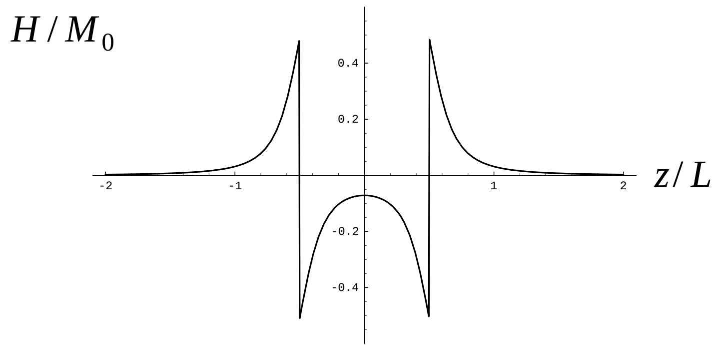
$$H_z = -\frac{M_0}{2}\left[\frac{z-L/2}{\sqrt{a^2+(z-L/2)^2}} - \frac{z+L/2}{\sqrt{a^2+(z+L/2)^2}} + 2\,\Theta(L/2-|z|)\right]$$

where $\Theta(\xi)$ denotes the unit step function, $\Theta = 1$ for $\xi > 0$ (and 0 otherwise). The magnetic induction is obtained by rewriting the relation $\vec{H} = \vec{B}/\mu_0 - \vec{M}$ as $\vec{B} = \mu_0(\vec{H} + \vec{M}\,)$. Since the magnetization is only nonzero inside the magnet [ie $M_z = M_0\,\Theta(L/2 - |z|)$], the addition $\vec{H} + \vec{M}$ simply removes the step function term. We find

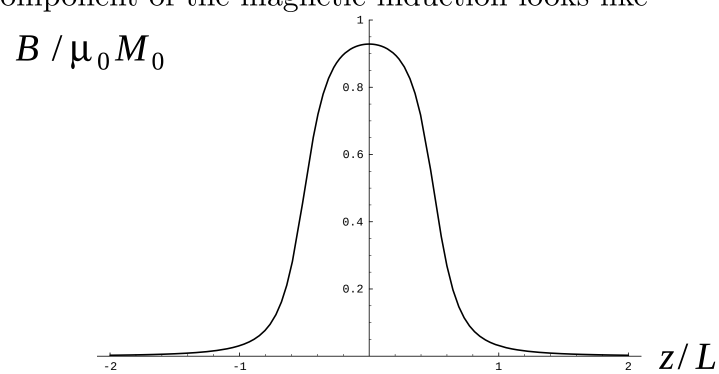
$$B_z = \mu_0(H_z + M_z) = -\frac{\mu_0 M_0}{2}\left[\frac{z - L/2}{\sqrt{a^2 + (z - L/2)^2}} - \frac{z + L/2}{\sqrt{a^2 + (z + L/2)^2}}\right]$$

b) Plot the ratios $\vec{B}/\mu_0 M_0$ and $\vec{H}/M_0$ on the axis as functions of z for $L/a = 5$.

The z component of the magnetic field looks like



while the z component of the magnetic induction looks like



Note that B_z is continuous, while H_z jumps at the ends of the magnet. This jump may be thought of as arising from effective magnetic surface charge.