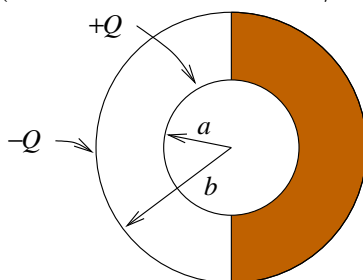## Homework Assignment #7 — Solutions

Textbook problems: Ch. 4: 4.10
Ch. 5: 5.3, 5.6, 5.7

4.10 Two concentric conducting spheres of inner and outer radii a and b, respectively, carry charges $\pm Q$. The empty space between the spheres is half-filled by a hemispherical shell of dielectric (of dielectric constant $\epsilon/\epsilon_0 0$, as shown in the figure.

a) Find the electric field everywhere between the spheres.

This is a somewhat curious problem. It should be obvious that without any dielectric the electric field between the spheres would be radial

$$\vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}$$

We cannot expect this to be unmodified by the dielectric. However, we note that the radial electric field is tangential to the interface between the dielectric and empty region. Thus the tangential matching condition $E_1^{\parallel} = E_2^{\parallel}$ is automatically satisfied. At the same time there is no perpendicular component to the interface, so there is nothing to worry about for the $D_1^{\perp} = D_2^{\perp}$ matching condition. This suggests that we guess a solution of the radial form

$$\vec{E} = A\frac{\hat{r}}{r^2}$$

where A is a constant to be determined. This guess is perhaps not completely obvious because one may have expected the field lines to bend into or out of the dielectric region. However, we could also recall that parallel fields do not get bent across the dielectric interface.

We may use the integral form of Gauss' law in a medium to determine the above constant A

$$\oint \vec{D}\cdot\hat{n}\,da = Q \qquad \Rightarrow \qquad \frac{\epsilon_0 A}{r^2}(2\pi r^2) + \frac{\epsilon A}{r^2}(2\pi r^2) = Q$$

or $A = Q/2\pi(\epsilon + \epsilon_0)$. Hence

$$\vec{E} = \frac{Q}{2\pi(\epsilon + \epsilon_0)} \frac{\hat{r}}{r^2}$$

Note that $\frac{1}{2}(\epsilon + \epsilon_0)$ may be viewed as the average permittivity in the volume between the spheres.

b) Calculate the surface-charge distribution on the inner sphere.

The surface-charge density is given by $\sigma = D^\perp\big|_{r=a}$ where either $D^\perp = \epsilon_0 E^\perp$ or $D^\perp = \epsilon E^\perp$ depending on region. This gives

$$\sigma = \begin{cases} \dfrac{\epsilon}{\epsilon + \epsilon_0} \dfrac{Q}{2\pi a^2}; & \text{dielectric side} \\[2ex] \dfrac{\epsilon_0}{\epsilon + \epsilon_0} \dfrac{Q}{2\pi a^2}; & \text{empty side} \end{cases} \tag{1}$$

Note that the total charge obtained by integrating σ over the surface of the inner sphere gives Q as expected.

c) Calculate the polarization-charge density induced on the surface of the dielectric at $r = a$.

The polarization charge density is given by

$$\rho_{\text{pol}} = -\nabla \cdot \vec{P}$$

where $\vec{P} = \epsilon_0 \chi_e \vec{E} = (\epsilon - \epsilon_0)\vec{E}$. Since the surface of the dielectric at $r = a$ is against the inner sphere, we can take the polarization to be zero inside the metal ('outside' the dielectric). Gauss' law in this case gives

$$\sigma_{\text{pol}} = -P^\perp\big|_{r=a} = -(\epsilon - \epsilon_0)E^\perp\big|_{r=a} = -\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} \frac{Q}{2\pi a^2}$$

Note that when this is combined with (1), the total (free and polarization) charge density is
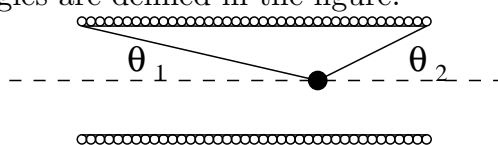
$$\sigma_{\text{tot}} = \sigma + \sigma_{\text{pol}} = \frac{\epsilon_0}{\epsilon + \epsilon_0} \frac{Q}{2\pi a^2}$$

on either half of the sphere. Since this is uniform, this is why the resulting electric field is radially symmetric.

5.3 A right-circular solenoid of finite length L and radius a has N turns per unit length and carries a current I. Show that the magnetic induction on the cylinder axis in the limit $NL \to \infty$ is

$$B_z = \frac{\mu_0 N I}{2}(\cos\theta_1 + \cos\theta_2)$$

where the angles are defined in the figure.

We start by computing the magnetic field on axis for a single loop of wire carrying a current I. This may be done by an elementary application of the Biot-Savart law.
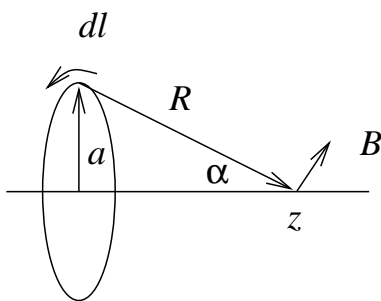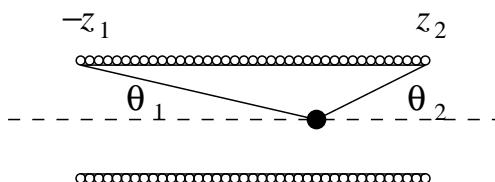

By symmetry, only the z component contributes

$$B_z = \frac{\mu_0 I}{4\pi} \int \frac{[d\vec{\ell} \times \vec{R}]_z}{R^3} = \frac{\mu_0 I}{4\pi} \int \frac{d\ell\, R \sin \alpha}{R^3} = \frac{\mu_0 I}{4\pi} 2\pi a \frac{a}{R^3} = \frac{\mu_0 I a^2}{2R^3}$$

Substituting in $R^2 = a^2 + z^2$ yields

$$B_z = \frac{\mu_0 I a^2}{2(a^2 + z^2)^{3/2}} \tag{2}$$

We now use linear superposition to obtain the field of the solenoid. Defining z_1 and z_2 as follows

(where $z_1 + z_2 = L$) we have

$$B_z = \frac{\mu_0 I a^2}{2} \int_{-z_1}^{z_2} \frac{N\, dz}{(a^2 + z^2)^{3/2}}$$

A simple trig substitution $z = a \tan \alpha$ converts this integral to

$$B_z = \frac{\mu_0 N I}{2} \int_{-\tan^{-1}(z_1/a)}^{\tan^{-1}(z_2/a)} \cos \alpha\, d\alpha = \frac{\mu_0 N I}{2} \sin \alpha \Big|_{-\tan^{-1}(z_1/a)}^{\tan^{-1}(z_2/a)}$$

A bit of geometry then demonstrates that this is equivalent to

$$B_z = \frac{\mu_0 N I}{2} (\cos \theta_1 + \cos \theta_2)$$

5.6 A cylindrical conductor of radius a has a hole of radius b bored parallel to, and centered a distance d from, the cylinder axis $(d + b < a)$. The current density is uniform throughout the remaining metal of the cylinder and is parallel to the axis. Use Ampère's law and principle of linear superposition to find the magnitude and the direction of the magnetic-flux density in the hole.

Ampère's law in integral form states

$$\oint_V \vec{B} \cdot d\vec{\ell} = \mu_0 i_{\text{enc}}$$

For a cylindrically symmetric geometry this gives simply

$$B = \frac{\mu_0 i_{\text{enc}}}{2\pi r} = \frac{\mu_0 (j\pi r^2)}{2\pi r} = \frac{\mu_0 j r}{2}$$

where we have assumed a uniform current density j. The direction of the magnetic induction is given by the right hand rule. For a conductor oriented along the z axis (so that the current is flowing in the $+\hat{z}$ direction), we may write

$$\vec{B} = \frac{\mu_0 j r}{2} \hat{z} \times \hat{r} = \frac{\mu_0 j}{2} \hat{z} \times \vec{r}$$

where $\vec{r}$ is the vector from the center of the conductor to the position where we are measuring the field. We now use linear superposition to start with a solid cylindrical conductor and then subtract the 'missing' current from the hole

$$\vec{B} = \frac{\mu_0 j}{2} \hat{z} \times \vec{x} - \frac{\mu_0 j}{2} \hat{z} \times (\vec{x} - \vec{d}) = \frac{\mu_0 j}{2} \hat{z} \times \vec{d}$$

Here $\vec{d}$ is the vector displacement of the hole from the center of the cylinder. This somewhat remarkable result demonstrates that the magnetic induction is uniform in the hole, and is in a direction given by the right hand rule.

If desired, we note that the total current carried by the wire is $I = j(\pi a^2 - \pi b^2)$, so we may express the magnetic induction in terms of I as

$$\vec{B} = \frac{\mu_0 I}{2\pi (a^2 - b^2)} \hat{z} \times \vec{d}$$

5.7 A compact circular coil of radius a, carrying a current I (perhaps N turns, each with current I/N), lies in the x-y plane with its center at the origin.

a) By elementary means [Eq. (5.4)] find the magnetic induction at any point on the z axis

By appropriate integration of $\vec{J}(\vec{x}')/|\vec{x} - \vec{x}'|$ we could find the magnetic induction anywhere in space. However we have already computed the magnetic induction when restricted to the z axis. The result is given by (2)

$$B_z = \frac{\mu_0 I a^2}{2(a^2 + z^2)^{3/2}}$$

b) An identical coil with the same magnitude and sense of the current is located on the same axis, parallel to, and a distance b above, the first coil. With the coordinate origin relocated at the point midway between the centers of the two coils, determine the magnetic induction on the axis near the origin as an expansion in powers of z, up to z^4 inclusive:

$$B_z = \left(\frac{\mu_0 I a^2}{d^3}\right)\left[1 + \frac{3(b^2 - a^2)z^2}{2d^4} + \frac{15(b^4 - 6b^2 a^2 + 2a^4)z^4}{16d^8} + \cdots\right]$$

where $d^2 = a^2 + b^2/4$.

By shifting the origin around, it should be obvious that the magnetic induction is given exactly by

$$B_z = \frac{\mu_0 I a^2}{2}\left((a^2 + (z - \tfrac{1}{2}b)^2)^{-3/2} + (a^2 + (z + \tfrac{1}{2}b)^2)^{-3/2}\right) \tag{3}$$

All we must do now is to Taylor expand the terms to order z^4. Noting that we are seeking an expansion in powers of z/d^2, we may write

$$\begin{aligned} B_z &= \frac{\mu_0 I a^2}{2}\left((d^2 - bz + z^2)^{-3/2} + (d^2 + bz + z^2)^{-3/2}\right) \\ &= \frac{\mu_0 I a^2}{2d^3}\left((1 - b\zeta + d^2\zeta^2)^{-3/2} + (1 + b\zeta + d^2\zeta^2)^{-3/2}\right) \\ &= \frac{\mu_0 I a^2}{2d^3}\left((1 - b\zeta + (a^2 + \tfrac{1}{4}b^2)\zeta^2)^{-3/2} + (1 + b\zeta + (a^2 + \tfrac{1}{4}b^2)\zeta^2)^{-3/2}\right) \end{aligned} \tag{4}$$

where we have introduced $\zeta = z/d^2$. Expanding this in powers of ζ yields

$$B_z = \frac{\mu_0 I a^2}{2d^3}\left[1 + \tfrac{3}{2}(b^2 - a^2)\zeta^2 + \tfrac{15}{16}(b^4 - 6b^2 a^2 + 2a^4)\zeta^4 + \cdots\right]$$

which is the desired result.

c) Show that, off-axis near the origin, the axial and radial components, correct to second order in the coordinates, take the form

$$B_z = \sigma_0 + \sigma_2\left(z^2 - \frac{\rho^2}{2}\right); \qquad B_\rho = -\sigma_2 z\rho$$

In principle, we may compute the vector potential or magnetic induction off-axis through the Biot-Savart law. However, near the axis, it is perhaps more convenient to perform a series expansion of the magnetic induction subject to the source-free constraints $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{\nabla} \times \vec{B} = 0$. By symmetry, we start with

$$B_z(0, z) = b(z), \qquad B_\rho(0, z) = 0$$

where $b(z)$ is the known on-axis solution of part b). We now develop a Taylor expansion in ρ

$$B_z(\rho, z) = b(z) + \rho b_1(z) + \rho^2 b_2(z) + \cdots, \qquad B_\rho(\rho, z) = \rho c_1(z) + \rho^2 c_2(z) + \cdots$$

In cylindrical coordinates, we then have

$$0 = \vec{\nabla} \cdot \vec{B} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho B_\rho + \frac{\partial}{\partial z}B_z = 2c_1(z) + 3\rho c_2(z) + b'(z) + \rho b_1'(z) + \cdots$$

Since this vanishes for any value of ρ, we match powers to obtain

$$c_1(z) = -\tfrac{1}{2}b'(z), \qquad c_2(z) = -\tfrac{1}{3}b_1'(z)$$

We now continue with the curl equation. The only non-trivial component is

$$0 = [\vec{\nabla} \times \vec{B}]_\phi = \frac{\partial}{\partial z}B_\rho - \frac{\partial}{\partial \rho}B_z = \rho c_1'(z) - b_1(z) - 2\rho b_2(z) + \cdots$$

This now gives

$$b_1(z) = 0, \qquad b_2(z) = \tfrac{1}{2}c_1'(z)$$

Combining this with the above gives us the solution

$$B_z(\rho, z) = b(z) - \tfrac{1}{4}\rho^2 b''(z) + \cdots, \qquad B_\rho(\rho, z) = -\tfrac{1}{2}\rho b'(z) + \cdots$$

valid up to and including $\mathcal{O}(\rho^2)$. Incidentally, we have basically almost solved problem 5.4 in this manner. The idea here is that if we know the behavior of the field along a symmetry axis (or in a sufficiently large region of space), the equations of motion (Maxwell's equations in this case) allow us to extend the solution away from the axis in a unique manner.

We now insert $b(z) = \sigma_0 + \sigma_2 z^2$ to obtain

$$B_z(\rho, z) = \sigma_0 + \sigma_2(z^2 - \tfrac{1}{2}\rho^2) + \cdots, \qquad B_\rho(\rho, z) = -\sigma_2\rho z + \cdots$$

d) For the two coils in part b) show that the magnetic induction on the z axis for large $|z|$ is given by the expansion in inverse odd powers of $|z|$ obtained from the small z expansion of part b) by the formal substitution $d \to |z|$.

For large $|z|$ we Taylor expand (3) in inverse powers of z

$$B_z = \frac{\mu_0 I a^2}{2|z|^3}\left((1 - bz^{-1} + (a^2 + \tfrac{1}{4}b^2)z^{-2})^{-3/2} + (1 + bz^{-1} + (a^2 + \tfrac{1}{4}b^2)z^{-2})^{-3/2}\right)$$

Comparing this with the last line of (4) shows that the Taylor series is formally equivalent under the substitution $\zeta \to z^{-1}$, which may be accomplished by taking $d \to |z|$.

e) If $b = a$, the two coils are known as a pair of Helmholtz coils. For this choice of geometry the second terms in the expansions of parts b) and d) are absent [$\sigma_2 = 0$ in part c)]. The field near the origin is then very uniform. What is the maximum permitted value of $|z|/a$ if the axial field is to be uniform to one part in 10^4, one part in 10^2?

For $b = a$ the axial field is of the form

$$B_z = \frac{\mu_0 I a^2}{2d^3} \left(1 - \frac{45}{16} \frac{a^4 z^4}{d^8} + \cdots \right)$$

$$= \frac{4\mu_0 I a^2}{5^{3/2} a^3} \left(1 - \frac{144}{125} \left(\frac{z}{a} \right)^4 + \cdots \right)$$

Taking the $(|z|/a)^4$ term as a small correction, the field non-uniformity is

$$\frac{\delta B}{B} \approx \frac{144}{125} \left(\frac{z}{a} \right)^4$$

For uniformity to one part in 10^4, we find $|z|/a < 0.097$, while for uniformity to one part in 10^2, we instead obtain $|z|/a < 0.305$. These numbers are actually pretty good because of the fourth power. For example, the first value indicates we can move $\approx \pm 10\%$ of the distance between the coils while maintaining field uniformity at the level of 0.01%. Helmholtz coils are very useful in the lab for canceling out the Earth's magnetic field.