Physics 505 **Fall 2005**

Homework Assignment #6 — Due Thursday, October 27

Textbook problems: Ch. 4: 4.2, 4.6 *a*) and *b*), 4.7 *a*) and *b*), 4.8

4.2 A point dipole with dipole moment $\vec{p}$ is located at the point $\vec{x}_0$. From the properties of the derivative of a Dirac delta function, show that for calculation of the potential Φ or the energy of a dipole in an external field, the dipole can be described by an effective charge density

$$\rho_{\text{eff}}(\vec{x}\,) = -\vec{p}\cdot\nabla\delta(\vec{x}-\vec{x}_0)$$

We first consider the potential

$$\begin{aligned}
\Phi(\vec{x}\,) &= \frac{1}{4\pi\epsilon_0}\int\rho(\vec{x}\,')\frac{1}{|\vec{x}-\vec{x}\,'|}\,d^3x' \\
&= -\frac{1}{4\pi\epsilon_0}\int[\vec{p}\cdot\vec{\nabla}_{x'}\delta^3(\vec{x}\,'-\vec{x}_0)]\frac{1}{|\vec{x}-\vec{x}\,'|}\,d^3x' \\
&= \frac{1}{4\pi\epsilon_0}\int\delta^3(\vec{x}\,'-\vec{x}_0)\vec{p}\cdot\vec{\nabla}_{x'}\left(\frac{1}{|\vec{x}-\vec{x}\,'|}\right)d^3x' \\
&= \frac{1}{4\pi\epsilon_0}\,\vec{p}\cdot\vec{\nabla}_{x'}\left(\frac{1}{|\vec{x}-\vec{x}\,'|}\right)\bigg|_{\vec{x}\,'=\vec{x}_0} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot(\vec{x}-\vec{x}_0)}{|\vec{x}-\vec{x}_0|^3}
\end{aligned}$$

This is the expected potential for a dipole.

Similarly, we can work out the energy of the dipole in an external field

$$\begin{aligned}
W &= \int\rho(\vec{x}\,)\Phi(\vec{x}\,)\,d^3x \\
&= -\int[\vec{p}\cdot\vec{\nabla}\delta^3(\vec{x}-\vec{x}_0)]\Phi(\vec{x}\,)\,d^3x \\
&= \int\delta^3(\vec{x}-\vec{x}_0)\vec{p}\cdot\vec{\nabla}\Phi(\vec{x}\,)\,d^3x \\
&= \vec{p}\cdot\vec{\nabla}\Phi(\vec{x}_0) = -\vec{p}\cdot\vec{E}(\vec{x}_0)
\end{aligned}$$

Essentially the derivative of a delta function serves to pick out derivatives of the function that it is multiplied against. Of course, a delta function itself is rather singular, and its derivatives are even more so. But as far as formal expressions (or distribution theory expressions) are concerned, these manipulations are in fact legitimate.

4.6 A nucleus with quadrupole moment Q finds itself in a cylindrically symmetric electric field with a gradient $(\partial E_z/\partial z)_0$ along the z axis at the position of the nucleus.

a) Show that the energy of quadrupole interaction is

$$W = -\frac{e}{4} Q \left(\frac{\partial E_z}{\partial z} \right)_0$$

The quadrupole interaction energy is

$$W = -\frac{1}{6} Q_{ij} \frac{\partial E_i}{\partial x_j}$$

To work out this expression, we first note that the cylindrically symmetric charge distribution for the nucleus gives rise to non-vanishing quadrupole moments

$$Q_{11} = Q_{22} = -\tfrac{1}{2} Q_{33}$$

(Note that this is obtained by demanding symmetry and tracelessness.) Since the nuclear quadrupole moment Q (without indices) is $(1/e)Q_{33}$ we end up with

$$Q_{11} = Q_{22} = -\tfrac{1}{2} eQ, \qquad Q_{33} = eQ$$

The energy is thus

$$W = -\frac{eQ}{6} \left(-\frac{1}{2} \frac{\partial E_x}{\partial x} - \frac{1}{2} \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) \tag{1}$$

We now make use of the fact that the (external) electric field is divergence free

$$0 = \vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

and cylindrically symmetric (so that $\partial E_x / \partial x = \partial E_y / \partial y$) to write

$$\frac{\partial E_x}{\partial x} = \frac{\partial E_y}{\partial y} = -\frac{1}{2} \frac{\partial E_z}{\partial z}$$

Making this substitution in (1) allows us to write W entirely in terms of $\partial E_z / \partial z$

$$W = -\frac{eQ}{4} \frac{\partial E_z}{\partial z}$$

b) If it is known that $Q = 2 \times 10^{-28}\,\mathrm{m}^2$ and that W/h is 10 MHz, where h is Planck's constant, calculate $(\partial E_z / \partial z)_0$ in units of $e/4\pi\epsilon_0 a_0^3$, where $a_0 = 4\pi\epsilon_0 \hbar^2 / me^2 = 0.529 \times 10^{-10}\,\mathrm{m}$ is the Bohr radius in hydrogen.

Note that

$$\frac{\partial E_z}{\partial z} = -\frac{4}{e} \frac{W}{Q} = -\frac{4h}{ea_0^2} \frac{(W/h)}{(Q/a_0^2)}$$

In units of $e/4\pi\epsilon_0 a_0^3$, we have

$$\frac{\partial E_z}{\partial z}\bigg/\left(\frac{e}{4\pi\epsilon_0 a_0^3}\right) = -\frac{16\pi h\epsilon_0 a_0}{e^2}\frac{(W/h)}{(Q/a_0^2)} = -\frac{8\pi}{\alpha}\frac{a_0}{c}\frac{(W/h)}{(Q/a_0^2)}$$

where $\alpha = e^2/4\pi\epsilon_0\hbar c = 1/137.036\ldots$ is the fine structure constant. Putting in numbers yields

$$\left|\frac{\partial E_z}{\partial z}\right|\bigg/\left(\frac{e}{4\pi\epsilon_0 a_0^3}\right) \approx 0.085$$

4.7 A localized distribution of charge has a charge density

$$\rho(\vec{r}) = \frac{1}{64\pi}r^2 e^{-r}\sin^2\theta$$

a) Make a multipole expansion of the potential due to this charge density and determine all the nonvanishing multipole moments. Write down the potential at large distances as a finite expansion in Legendre polynomials.

This charge distribution is azimuthally symmetric. As a result, only $m = 0$ moments will be nonvanishing. Furthermore, noting that

$$\sin^2\theta = 1 - \cos^2\theta = \tfrac{2}{3}[P_0(\cos\theta) - P_2(\cos\theta)]$$

we may write down the moments

$$\begin{aligned}
q_{l0} &= \int r^l Y_{l0}^*(\theta,\phi)\rho(r,\theta)\, r^2\, dr\, d\phi\, d(\cos\theta) \\
&= 2\pi\sqrt{\frac{2l+1}{4\pi}}\int r^l P_l(\cos\theta)\rho(r,\theta)\, r^2\, dr\, d(\cos\theta) \\
&= \frac{2\pi}{64\pi}\frac{2}{3}\sqrt{\frac{2l+1}{4\pi}}\int_0^\infty r^{l+4}e^{-r}\, dr\int_{-1}^{1} P_l(\cos\theta)[P_0(\cos\theta) - P_2(\cos\theta)]\, d(\cos\theta) \\
&= \frac{1}{48}\sqrt{\frac{2l+1}{4\pi}}\Gamma(l+5)[2\delta_{l,0} - \tfrac{2}{5}\delta_{l,2}]
\end{aligned}$$

As a result, we read off the only nonvanishing multipole moments

$$q_{00} = \sqrt{\frac{1}{4\pi}}, \qquad q_{20} = -6\sqrt{\frac{5}{4\pi}}$$

The multipole expansion then yields the large distance potential

$$\begin{aligned}
\Phi &= \frac{1}{4\pi\epsilon_0}\sum_{l,m}\frac{4\pi}{2l+1}q_{lm}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}} \\
&= \frac{1}{4\pi\epsilon_0}\sum_l\sqrt{\frac{4\pi}{2l+1}}q_{l0}\frac{P_l(\cos\theta)}{r^{l+1}} \qquad (2) \\
&= \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r} - \frac{6}{r^3}P_2(\cos\theta)\right]
\end{aligned}$$

b) Determine the potential explicitly at any point in space, and show that near the origin, correct to r^2 inclusive,

$$\Phi(\vec{r}) \simeq \frac{1}{4\pi\epsilon_0}\left[\frac{1}{4} - \frac{r^2}{120}P_2(\cos\theta)\right]$$

We may use a Green's function to obtain the potential at any point in space. In general (since there are no boundaries, except at infinity)

$$G(\vec{x},\vec{x}\,') = \frac{1}{|\vec{x}-\vec{x}\,'|} = \sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

However, for azimuthal symmetry, it is sufficient to focus on the $m = 0$ terms in the expansion

$$G(\vec{x},\vec{x}\,') = \sum_l \frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)P_l(\cos\theta') + (m \neq 0)$$

Then

$$\begin{aligned}
\Phi(\vec{x}\,) &= \frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}\,')G(\vec{x},\vec{x}\,')\,d^3x' \\
&= \frac{1}{4\pi\epsilon_0}\frac{2\pi}{64\pi}\frac{2}{3}\int_0^\infty r'^4 e^{-r'}\frac{r_<^l}{r_>^{l+1}}\,dr' \\
&\quad\times\int_{-1}^1[P_0(\cos\theta') - P_2(\cos\theta')]P_l(\cos\theta')P_l(\cos\theta)\,d(\cos\theta') \\
&= \frac{1}{4\pi\epsilon_0}\frac{1}{48}\left[\frac{1}{r^{l+1}}\int_0^r r'^{l+4}e^{-r'}\,dr' + r^l\int_r^\infty r'^{3-l}e^{-r'}\,dr'\right][2\delta_{l,0} - \tfrac{2}{5}\delta_{l,2}P_2(\cos\theta)]
\end{aligned}$$

Instead of writing this out in terms of incomplete Gamma functions, it is better just to integrate for $l = 0$ and $l = 2$. The result is

$$\begin{aligned}
\Phi = \frac{1}{4\pi\epsilon_0}\frac{1}{24}\bigg[&\frac{1}{r}\left(24 - e^{-r}(24 + 18r + 6r^2 + r^3)\right) \\
&- \frac{1}{r^3}P_2(\cos\theta)\left(144 - e^{-r}(144 + 144r + 72r^2 + 24r^3 + 6r^4 + r^5)\right)\bigg]
\end{aligned}$$

Note that as $r \to \infty$ the e^{-r} factors are exponentially small. As a result, we simply reproduce (2) in this limit. On the other hand, as $r \to 0$ a Taylor expansion yields

$$\Phi = \frac{1}{4\pi\epsilon_0}\left[\left(\frac{1}{4} + \cdots\right) - \left(\frac{r^2}{120} + \cdots\right)P_2(\cos\theta)\right]$$

Obtaining the correct $l = 2$ term involves the cancellation of the first five terms in the Taylor expansion. The leading terms in the final expression have the 'correct' powers of $r^l P_l(\cos\theta)$.

4.8 A very long, right circular, cylindrical shell of dielectric constant ϵ/ϵ_0 and inner and outer radii a and b, respectively, is placed in a previously uniform electric field E_0 with its axis perpendicular to the field. The medium inside and outside the cylinder has a dielectric constant of unity.

a) Determine the potential and electric field in the three regions, neglecting end effects.

Since the cylinder is very long, we treat this as a two-dimensional problem. In this case, the potential admits a general expansion

$$\Phi = \sum_m [\alpha_m \rho^m + \beta_m \rho^{-m}] \cos(m\phi - \delta_m)$$

(where the $m = 0$ term should actually be $\alpha_0 + \beta_0 \ln\rho$). Furthermore, by orienting the electric field along the $+x$ direction, we may use the $\phi \leftrightarrow -\phi$ symmetry of this problem to eliminate the phases δ_m. As a result, we are able to write the potential as an expansion in each of the three regions

$$\Phi = \begin{cases} \Phi_1 = A_m \rho^{-m} \cos(m\phi) - E_0 \rho \cos\phi, & \rho > b \\ \Phi_2 = (B_m \rho^m + C_m \rho^{-m}) \cos(m\phi), & a < \rho < b \\ \Phi_3 = D_m \rho^m \cos(m\phi), & \rho < a \end{cases}$$

For each value of m, there are four unknowns, A_m, B_m, C_m and D_m. On the other hand, there are also four matching conditions ($D^\perp$ and $E^\parallel$ both at a and at b). Note, however, that when $m \neq 1$ these matching conditions yield homogeneous equations which only admit the trivial solution

$$A_m = B_m = C_m = D_m = 0 \qquad m \neq 1$$

Thus we may write

$$\Phi = \begin{cases} \Phi_1 = (A\rho^{-1} - E_0 \rho) \cos\phi, & \rho > b \\ \Phi_2 = (B\rho + C\rho^{-1}) \cos\phi, & a < \rho < b \\ \Phi_3 = D\rho \cos\phi, & \rho < a \end{cases} \tag{3}$$

We may obtain the electric field by taking a gradient

$$E_\rho = -\frac{\partial\Phi}{\partial\rho} = \begin{cases} E_\rho^1 = (A\rho^{-2} + E_0) \cos\phi, & \rho > b \\ E_\rho^2 = (-B + C\rho^{-2}) \cos\phi, & a < \rho < b \\ E_\rho^3 = -D \cos\phi, & \rho < a \end{cases}$$
$$E_\phi = -\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi} = \begin{cases} E_\phi^1 = (A\rho^{-2} - E_0) \sin\phi, & \rho > b \\ E_\phi^2 = (B + C\rho^{-2}) \sin\phi, & a < \rho < b \\ E_\phi^3 = D \sin\phi, & \rho < a \end{cases} \tag{4}$$

The matching at $\rho = a$ is

$$\epsilon_0 E^3_\rho = \epsilon E^2_\rho, \qquad E^3_\phi = E^2_\phi$$

or

$$(\epsilon_0/\epsilon)D - B + Ca^{-2} = 0, \qquad D - B - Ca^{-2} = 0$$

This may be solved for C and D in terms of B

$$C = \frac{1-\epsilon_0/\epsilon}{1+\epsilon_0/\epsilon} Ba^2, \qquad D = \frac{2}{1+\epsilon_0/\epsilon} B \tag{5}$$

Similarly, the matching at $\rho = b$ is

$$\epsilon E^2_\rho = \epsilon_0 E^3_\rho, \qquad E^2_\phi = E^3_\phi$$

or

$$(\epsilon_0/\epsilon)Ab^{-2} + B - Cb^{-2} = -(\epsilon_0/\epsilon)E_0, \qquad Ab^{-2} - B - Cb^{-2} = E_0$$

Eliminating C using (5) gives rise to the simultaneous equations

$$\begin{pmatrix} b^{-2} & -1 - \frac{1-\epsilon_0/\epsilon}{1+\epsilon_0/\epsilon}\left(\frac{a}{b}\right)^2 \\ (\epsilon_0/\epsilon)b^{-2} & 1 - \frac{1-\epsilon_0/\epsilon}{1+\epsilon_0/\epsilon}\left(\frac{a}{b}\right)^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = E_0 \begin{pmatrix} 1 \\ -\epsilon_0/\epsilon \end{pmatrix}$$

This yields a solution

$$\begin{aligned} A &= E_0 \Delta^{-1}(1-\epsilon_0/\epsilon)\left(1 - \left(\frac{a}{b}\right)^2\right) b^2 \\ B &= -E_0\Delta^{-1}(2\epsilon_0/\epsilon) \end{aligned} \tag{6}$$

where

$$\Delta = (1+\epsilon_0/\epsilon)\left(1 - \left(\frac{1-\epsilon_0/\epsilon}{1+\epsilon_0/\epsilon}\frac{a}{b}\right)^2\right)$$

Substituting B into (5) then gives the remaining coefficients

$$\begin{aligned} C &= -E_0\Delta^{-1}\frac{(1-\epsilon_0/\epsilon)2\epsilon_0/\epsilon}{1+\epsilon_0/\epsilon}a^2 \\ D &= -E_0\Delta^{-1}\frac{4\epsilon_0/\epsilon}{1+\epsilon_0/\epsilon} \end{aligned} \tag{7}$$

The potential and electric field are obtained by substituting these coefficients into (3) and (4)

b) Sketch the lines of force for a typical case of $b \simeq 2a$.

c) Discuss the limiting forms of your solution appropriate for a solid dielectric cylinder in a uniform field, and a cylindrical cavity in a uniform dielectric.

A solid dielectric cylinder of radius b may be obtained by taking the limit $a \to 0$. In this case the expressions (6) and (7) simplify considerably. We give the potential

$$\Phi = \begin{cases} \Phi_1 = -E_0 x + E_0 \frac{1-\epsilon_0/\epsilon}{1+\epsilon_0/\epsilon} \frac{b^2 x}{\rho^2}, & \rho > b \\ \Phi_2 = -E_0 \frac{2\epsilon_0/\epsilon}{1+\epsilon_0/\epsilon} x, & \rho < b \end{cases} \tag{8}$$

where $x = \rho \cos \phi$. The potential Φ_3 is irrelevant in this case. Here we see that the potential Φ_2 inside the cylinder is uniform (but corresponds to a reduced electric field provided $\epsilon > \epsilon_0$). The potential outside is that of the original uniform electric field combined with a two-dimensional dipole.

For the opposite limit, we obtain a cylindrical cavity of radius a by taking the limit $b \to \infty$. In this case, we end up with

$$\Phi = \begin{cases} \Phi_2 = -E_0 \frac{2\epsilon_0/\epsilon}{1+\epsilon_0/\epsilon} x - E_0 \frac{(1-\epsilon_0/\epsilon) 2\epsilon_0/\epsilon}{(1+\epsilon_0/\epsilon)^2} \frac{a^2 x}{\rho^2}, & \rho > a \\ \Phi_3 = -E_0 \frac{4\epsilon_0/\epsilon}{(1+\epsilon_0/\epsilon)^2} x, & \rho < a \end{cases}$$

At first glance, this appears to be considerably different from (8). However, note that the physical electric field we measure as $\rho \to \infty$ is $\tilde{E}_0 = E_0(2\epsilon_0/\epsilon)/(1+\epsilon_0/\epsilon)$. In terms of $\tilde{E}_0$, we have

$$\Phi = \begin{cases} \Phi_2 = -\tilde{E}_0 x - \tilde{E}_0 \frac{1-\epsilon_0/\epsilon}{1+\epsilon_0/\epsilon} \frac{a^2 x}{\rho^2}, & \rho > a \\ \Phi_3 = -\tilde{E}_0 \frac{2}{1+\epsilon_0/\epsilon} x, & \rho < a \end{cases}$$

which may be rewritten as

$$\Phi = \begin{cases} \Phi_2 = -\tilde{E}_0 x + \tilde{E}_0 \frac{1-\epsilon/\epsilon_0}{1+\epsilon/\epsilon_0} \frac{a^2 x}{\rho^2}, & \rho > a \\ \Phi_3 = -\tilde{E}_0 \frac{2\epsilon/\epsilon_0}{1+\epsilon/\epsilon_0} x, & \rho < a \end{cases}$$

This agrees with (8) after the replacement $\epsilon \leftrightarrow \epsilon_0$ (and $a \to b$), as it must.