Physics 505 **Fall 2005**

Homework Assignment #5 — Solutions

Textbook problems: Ch. 3: 3.14, 3.26, 3.27
Ch. 4: 4.1

3.14 A line charge of length $2d$ with a total charge Q has a linear charge density varying as $(d^2 - z^2)$, where z is the distance from the midpoint. A grounded, conducting, spherical shell of inner radius $b > d$ is centered at the midpoint of the line charge.

a) Find the potential everywhere inside the spherical shell as an expansion in Legendre polynomials.

We first ought to specify the charge density $\rho(\vec{x}\,)$ corresponding to the line charge. By symmetry, we place the line charge along the z axis. In this case, it is specified by $\cos\theta = \pm 1$. As a slight subtlety, in order to get a *uniform* charge density in spherical coordinates, we need to divide out by r^2. Hence for charge density varying as $(d^2 - z^2)$ we end up with

$$\rho(\vec{x}\,) = \frac{\rho_0}{r^2}(d^2 - r^2)[\delta(\cos\theta - 1) + \delta(\cos\theta + 1)]$$

with the caveat that $r < d$. (This can be specified with a Heaviside step function $\Theta(d - r)$, but we will not bother with that.) The constant ρ_0 is specified by evaluating the total charge

$$\begin{aligned} Q = \int \rho(\vec{x}\,) d^2x &= \int \frac{\rho_0}{r^2}(d^2 - r^2)[\delta(\cos\theta - 1) + \delta(\cos\theta + 1)] r^2\, dr\, d\phi\, d(\cos\theta) \\ &= 2\pi \cdot 2 \cdot \rho_0 \int_0^d (d^2 - r^2)\, dr = \frac{8\pi}{3}\rho_0 d^3 \end{aligned}$$

Thus $\rho_0 = 3Q/(8\pi d^3)$.

Since the spherical shell is grounded, the potential inside the shell is given by

$$\Phi(\vec{x}\,) = \frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}\,')G(\vec{x}, \vec{x}\,')\, d^3x'$$

where

$$G(\vec{x}, \vec{x}\,') = \sum_{l,m} \frac{4\pi}{2l+1} r_<^l \left(\frac{1}{r_>^{l+1}} - \frac{r >^l}{b^{2l+1}} \right) Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi)$$

is the Dirichlet Green's function inside a sphere of radius b. Because of spherical symmetry, we see that only $m = 0$ terms will contribute in the integral. This indicates that the expression for $\Phi(\vec{x}\,)$ reduces to one with ordinary Legendre polynomials

$$\begin{aligned}
\Phi(\vec{x}\,) &= \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty} P_l(\cos\theta)\int \frac{\rho_0}{r'^2}(d^2-r'^2)[\delta(\cos\theta'-1)+\delta(\cos\theta'+1)] \\
&\qquad\qquad \times r_<^l\left(\frac{1}{r_>^{l+1}}-\frac{r_>^l}{b^{2l+1}}\right) P_l(\cos\theta')\, r'^2\, dr'\, d\phi'\, d(\cos\theta') \\
&= \frac{2\pi\rho_0}{4\pi\epsilon_0}\sum_{l=0}^{\infty} P_l(\cos\theta)[P_l(1)+P_l(-1)]\int_0^d (d^2-r'^2) r_<^l\left(\frac{1}{r_>^{l+1}}-\frac{r_>^l}{b^{2l+1}}\right) dr' \\
&= \frac{\rho_0}{\epsilon_0}\sum_{l\,\mathrm{even}} I_l(r) P_l(\cos\theta)
\end{aligned}$$

where

$$I_l(r) = \int_0^d (d^2-r'^2) r_<^l\left(\frac{1}{r_>^{l+1}}-\frac{r_>^l}{b^{2l+1}}\right) dr'$$

The reason only even values of l contribute is simply because the source is an even parity one. We are now left with evaluating the integral $I_l(r)$. There are two cases to consider.

Case 1: $r < d$. This is the more involved computation, as the integral has to be divided into two segments

$$\begin{aligned}
I_l(r) &= \left(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\right)\int_0^r (d^2-r'^2) r'^l\, dr' + r^l\int_r^d (d^2-r'^2)\left(\frac{1}{r'^{l+1}}-\frac{r'^l}{b^{2l+1}}\right) dr' \\
&= \frac{1}{r^{l+1}}\left(1-\left(\frac{r}{b}\right)^{2l+1}\right)\left(\frac{d^2}{l+1}-\frac{r^2}{l+3}\right) r^{l+1} \\
&\quad + r^l\left[-\frac{1}{r'^l}\left(\frac{d^2}{l}-\frac{r'^2}{l-2}\right)-\frac{1}{r'^l}\left(\frac{r'}{b}\right)^{2l+1}\left(\frac{d^2}{l+1}-\frac{r'^2}{l+3}\right)\right]_r^d \\
&= d^2\left(\frac{2l+1}{l(l+1)}+\frac{2}{l(l-2)}\left(\frac{r}{d}\right)^l-\frac{2}{(l+1)(l+3)}\left(\frac{r}{d}\right)^l\left(\frac{d}{b}\right)^{l+1}\right) \\
&\quad - r^2\frac{2l+1}{(l-2)(l+3)}
\end{aligned} \tag{1}$$

Note that for either $l = 0$ or $l = 2$ we end up with a log

$$\begin{aligned}
I_0(r) &= d^2\left(\frac{1}{2}-\frac{2}{3}\left(\frac{d}{b}\right)-\ln\frac{r}{d}\right)+\frac{1}{6}r^2 \\
I_2(r) &= \frac{5}{6}d^2 - r^2\left(\frac{7}{10}+\frac{2}{15}\left(\frac{d}{b}\right)^2-\ln\frac{r}{d}\right)
\end{aligned} \tag{2}$$

Case 2: $r > d$. In this case, since $r' < d < r$ there is only one integral. This is in fact the first term of (1) with limits extended from 0 to d

$$I_l(r) = \left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)\int_0^d (d^2 - r'^2)r'^l dr' = \frac{2d^2}{(l+1)(l+3)}\left(\frac{d}{r}\right)^{l+1}\left(1 - \left(\frac{r}{b}\right)^{2l+1}\right) \tag{3}$$

The potential everywhere inside the sphere is thus given by

$$\Phi(\vec{x}\,) = \frac{3Q}{8\pi\epsilon_0 d^3}\sum_{l\text{ even}} I_l(r)P_l(\cos\theta)$$

where $I_l(r)$ is given by either (1), (2) or (3) as appropriate.

b) Calculate the surface-charge density induced on the shell.

For the surface-charge density, we need to know $\Phi(\vec{x}\,)$ near $r = b$. This falls into Case 2 above, which gives

$$\Phi(\vec{x}\,) = \frac{3Q}{4\pi\epsilon_0 d}\sum_{l\text{ even}}\frac{P_l(\cos\theta)}{(l+1)(l+3)}\left(\frac{d}{r}\right)^{l+1}\left(1 - \left(\frac{r}{b}\right)^{2l+1}\right) \tag{4}$$

The surface charge density is then given by

$$\sigma = \epsilon_0 E_\perp = \epsilon_0\frac{\partial\Phi}{\partial r}\bigg|_{r=b} = -\frac{3Q}{4\pi b^2}\sum_{l\text{ even}}\frac{(2l+1)P_l(\cos\theta)}{(l+1)(l+3)}\left(\frac{d}{b}\right)^l = -\frac{Q}{4\pi b^2} + \cdots \tag{5}$$

Only the $l = 0$ term contributes to the total charge on the shell. Thus integrating σ over the entire area $(4\pi b^2)$ demonstrates that there is a total charge of $-Q$ on the shell.

c) Discuss your answers to parts *a*) and *b*) in the limit that $d \ll b$.

In this limit, the line charge shrinks to a point compared with the sphere. Thus we assume $d \ll r$ as well as $d \ll b$ when examining the resulting limit. By rewriting (4), we have

$$\Phi(\vec{x}\,) = \frac{3Q}{4\pi\epsilon_0 b}\sum_{l\text{ even}}\frac{P_l(\cos\theta)}{(l+1)(l+3)}\left(\frac{b}{r}\left(\frac{d}{r}\right)^l - \left(\frac{r}{b}\right)^l\left(\frac{d}{b}\right)^l\right) \tag{6}$$

Because of the d/r and d/b factors, only the $l = 0$ term is important in this limit. The result is

$$\Phi(\vec{x}\,) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{b}\right)$$

which is the potential of a point charge surrounded by a grounded conducting sphere. Similarly, taking the limit $d/b \to 0$ in (5) yields

$$\sigma = -\frac{Q}{4\pi b^2} \tag{7}$$

which is the expected uniform induced charge.

Note that if we did not assume $r \gg d$ (but still take $d/b \to 0$) only the second term in (6) would disappear for $l > 0$. This more general limit gives a multipole expansion

$$\Phi(\vec{x}\,) = \frac{Q}{4\pi\epsilon_0}\left(-\frac{1}{b} + \sum_{l\,\text{even}} \frac{3P_l(\cos\theta)}{(l+1)(l+3)}\frac{d^l}{r^{l+1}}\right)$$

while the induced charge on the sphere is still uniform, and is given by (7).

3.26 Consider the Green function appropriate for Neumann boundary conditions for the volume V between the concentric spherical surfaces defined by $r = a$ and $r = b$, $a < b$. To be able to use (1.46) for the potential, impose the simple constraint (1.45). Use an expansion in spherical harmonics of the form

$$G(\vec{x}, \vec{x}\,') = \sum_{l=0}^{\infty} g_l(r, r')P_l(\cos\gamma)$$

where $g_l(r, r') = r_<^l/r_>^{l+1} + f_l(r, r')$.

a) Show that for $l > 0$, the radial Green function has the symmetric form

$$g_l(r, r') = \frac{r_<^l}{r_>^{l+1}} + \frac{1}{(b^{2l+1} - a^{2l+1})}\left[\frac{l+1}{l}(rr')^l + \frac{l}{l+1}\frac{(ab)^{2l+1}}{(rr')^{l+1}} + a^{2l+1}\left(\frac{r^l}{r'^{l+1}} + \frac{r'^l}{r^{l+1}}\right)\right]$$

There are several approaches to this problem. However, we first consider the Neumann boundary condition (1.45)

$$\left.\frac{\partial G(\vec{x}\,\vec{x}\,')}{\partial n'}\right|_{\text{bndy}} = -\frac{4\pi}{S}$$

For this problem with two boundaries, the surface area S must be the area of both boundaries (ie it is the *total* area surrounding the volume). Hence $S = 4\pi(a^2+b^2)$, and in particular this is uniform (constant) in the angles. As a result, this will only contribute to the $l = 0$ term in the expansion of the Green's function. More precisely, we could write

$$\left.\frac{\partial G(\vec{x}\,\vec{x}\,')}{\partial n'}\right|_{\text{bndy}} = \sum_l \left.\frac{\partial g_l(r, r')}{\partial n'}P_l(\cos\gamma)\right|_{\text{bndy}} = -\frac{1}{a^2+b^2}P_0(\cos\gamma)$$

Since the Legendre polynomials are orthogonal, this implies that

$$\left.\frac{\partial g_l(r, r')}{\partial n'}\right|_{\text{bndy}} = -\frac{1}{a^2+b^2}\delta_{l,0}$$

Noting that the outward normal is either in the $-\hat{r}'$ or the $\hat{r}'$ direction for the sphere at a or b, respectively, we end up with two boundary condition equations

$$\left.\frac{\partial g_l(r,r')}{\partial r'}\right|_a = \frac{1}{a^2+b^2}\delta_{l,0} \qquad \left.\frac{\partial g_l(r,r')}{\partial r'}\right|_b = -\frac{1}{a^2+b^2}\delta_{l,0} \tag{8}$$

Now that we have written down the boundary conditions for $g_l(r,r')$, we proceed to obtain its explicit form. The suggestion of the problem is to write

$$g_l(r,r') = \frac{r_<^l}{r_>^{l+1}} + f_l(r,r')$$

Since

$$\frac{1}{|\vec{x}-\vec{x}\,'|} = \sum_l \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)$$

we see that the first term in $g_l(r,r')$ is designed to give the singular source delta function. The remaining term

$$F(\vec{x},\vec{x}\,') = \sum_l f_l(r,r') P_l(\cos\gamma)$$

then solves the homogeneous equation $\nabla^2_{x'} F(\vec{x},\vec{x}\,') = 0$. But we know how to solve Laplace's equation in spherical coordinates, and the result is that the radial function must be of the form

$$f_l(r,r') = A_l r'^l + B_l \frac{1}{r'^{l+1}}$$

Note that we are taking the Green's function equation to act on the $\vec{x}\,'$ variable, where $\vec{x}$ may be thought of as a parameter (constant) giving the location of the delta function source. We thus have

$$g_l(r,r') = \frac{r_<^l}{r_>^{l+1}} + A_l r'^l + B_l \frac{1}{r'^{l+1}} \tag{9}$$

All that remains is to use the boundary conditions (8) to solve for A_l and B_l. For the inside sphere (at a), we have

$$l\frac{a^{l-1}}{r^{l+1}} + lA_l a^{l-1} - (l+1)B_l \frac{1}{a^{l+2}} = \frac{\delta_{l,0}}{a^2+b^2} \tag{10}$$

while for the outside sphere we have

$$-(l+1)\frac{r^l}{b^{l+2}} + lA_l b^{l-1} - (l+1)B_l \frac{1}{b^{l+2}} = -\frac{\delta_{l,0}}{a^2+b^2} \tag{11}$$

For $l \neq 0$ we rewrite these equations as

$$\begin{pmatrix} la^{2l+1} & -(l+1) \\ lb^{2l+1} & -(l+1) \end{pmatrix} \begin{pmatrix} A_l \\ B_l \end{pmatrix} = \begin{pmatrix} -la^{2l+1}/r^{l+1} \\ (l+1)r^l \end{pmatrix}$$

which may be solved to give

$$\begin{aligned} \begin{pmatrix} A_l \\ B_l \end{pmatrix} &= \frac{1}{l(l+1)(b^{2l+1}-a^{2l+1})} \begin{pmatrix} -(l+1) & (l+1) \\ -lb^{2l+1} & la^{2l+1} \end{pmatrix} \begin{pmatrix} -la^{2l+1}/r^{l+1} \\ (l+1)r^l \end{pmatrix} \\ &= \frac{r^l}{b^{2l+1}-a^{2l+1}} \begin{pmatrix} (a/r)^{2l+1} + (l+1)/l \\ a^{2l+1} + l/(l+1)(ab/r)^{2l+1} \end{pmatrix} \end{aligned}$$

Inserting this into (9) yields

$$\begin{aligned} g_l(r,r') &= \frac{r_<^l}{r_>^{l+1}} + \frac{r^l}{b^{2l+1}-a^{2l+1}} \left[\left(\left(\frac{a}{r}\right)^{2l+1} + \frac{l+1}{l} \right) r'^l \right. \\ &\qquad \left. + \left(a^{2l+1} + \frac{l}{l+1} \left(\frac{ab}{r}\right)^{2l+1} \right) \frac{1}{r'^{l+1}} \right] \\ &= \frac{r_<^l}{r_>^{l+1}} \\ &\quad + \frac{1}{b^{2l+1}-a^{2l+1}} \left[\frac{l+1}{l}(rr')^l + \frac{l}{l+1}\frac{(ab)^{2l+1}}{(rr')^{l+1}} + a^{2l+1}\left(\frac{r'^l}{r^{l+1}} + \frac{r^l}{r'^{l+1}} \right) \right] \\ &= \frac{1}{b^{2l+1}-a^{2l+1}} \left[\frac{l+1}{l}(r_<r_>)^l + \frac{l}{l+1}\frac{(ab)^{2l+1}}{(r_<r_>)^{l+1}} \right. \\ &\qquad \left. + b^{2l+1}\frac{r_<^l}{r_>^{l+1}} + a^{2l+1}\frac{r_>^l}{r_<^{2l+1}} \right] \\ &= \frac{l+1}{l(b^{2l+1}-a^{2l+1})} \left(r_<^l + \frac{l}{l+1}\frac{a^{2l+1}}{r_<^{l+1}} \right) \left(r_>^l + \frac{l}{l+1}\frac{b^{2l+1}}{r_>^{l+1}} \right) \end{aligned} \tag{10}$$

which is valid for $l \neq 0$. Note that in the last few lines we have been able to rewrite the Green's function in terms of a product of $u(r_<)$ and $v(r_>)$ where u and v satisfies Neumann boundary conditions at $r = a$ and $r = b$, respectively. This is related to another possible method of solving this problem. Using the Legendre identity

$$\sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l(\cos\gamma) = \delta(\phi - \phi')\delta(\cos\theta - \cos\theta')$$

the Green's function equation may be reduced to the one-dimensional problem

$$\left[\frac{d}{dr'} r'^2 \frac{d}{dr'} - l(l+1) \right] g_l(r,r') = -(2l+1)\delta(r-r')$$

Using the general method for the Sturm-Liouville problem, the Green's function is given by

$$g_l(r,r') = -\frac{2l+1}{A_l} u_l(r_<) v_l(r_>) \tag{12}$$

where $u(r')$ and $v(r')$ solve the homogeneous equation and the constant A_l is fixed by the Wronskian, $W(u,v) = A_l/r'^2$. For $l \neq 0$ the boundary conditions (8) are homogeneous

$$u'(r')\big|_{r'=a} = 0 \qquad v'(r')\big|_{r'=b} = 0$$

It is easy to see that these are satisfied by

$$u(r') = r'^l + \frac{l}{l+1}\frac{a^{2l+1}}{r'^{l+1}} \qquad v(r') = r'^l + \frac{l}{l+1}\frac{b^{2l+1}}{r'^{l+1}}$$

Computing the Wronskian gives

$$\begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \frac{l(2l+1)(a^{2l+1} - b^{2l+1})}{(l+1)r'^2}$$

which allows us to identify

$$A_l = -(2l+1)\frac{l}{l+1}(b^{2l+1} - a^{2l+1})$$

This gives the result of the last line of (10).

b) Show that for $l = 0$

$$g_0(r,r') = \frac{1}{r_>} - \left(\frac{a^2}{a^2+b^2}\right)\frac{1}{r'} + f(r)$$

where $f(r)$ is arbitrary. Show explicitly in (1.46) that answers for the potential $\Phi(\vec{x}\,)$ are independent of $f(r)$.

The $l = 0$ case involves a non-homogeneous boundary condition. Hence the result of (12) will not work. Of course, we can still work out the one-dimensional delta function problem with matching and jump conditions at $r' = r$. However it is more direct to return to (10) and (11) and to simply solve those conditions for $l = 0$. Both (10) and (11) result in

$$B_0 = -\frac{a^2}{a^2+b^2}$$

while leaving A_0 completely undetermined. Finally, since r is thought of as a parameter, this indicates that $A_0 = f(r)$ can be an arbitrary function of r. The $l = 0$ Green's function is given by (9)

$$g_0(r,r') = \frac{1}{r_>} - \frac{a^2}{a^2+b^2}\frac{1}{r'} + f(r) \tag{13}$$

Incidentally, we note that without the inhomogeneous Neumann boundary condition term $-4\pi/S$ there will be no solution to the system (10) and (11) for $l=0$ (unless b is taken to ∞). This demonstrates the inconsistency of simply setting $\partial G/\partial n' = 0$ for the Neumann Green's function.

Note that, by setting $f(r) = -a^2/[(a^2+b^2)r]$ we obtain a symmetrical Green's function

$$g_0(r,r') = \frac{1}{r_>} - \frac{a^2}{a^2+b^2}\left(\frac{1}{r'}+\frac{1}{r}\right)$$

On the other hand, the choice of $f(r)$ is unphysical. This arises because, for the Neumann Green's function, the $f(r)$ contribution to the potential is given by

$$\begin{aligned}\Phi(\vec{x}\,) &= \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}\,')f(r)\,d^3x' + \frac{1}{4\pi}\oint_S \frac{\partial\Phi(\vec{x}\,')}{\partial n'}f(r)\,da' \\ &= \frac{f(r)}{4\pi\epsilon_0}\left(\int_V \rho(\vec{x}\,')\,d^3x' - \epsilon_0\oint_S \vec{E}(\vec{x}\,')\cdot d\hat{a}'\right) \\ &= \frac{f(r)}{4\pi\epsilon_0}\left(q_{\text{enc}} - \epsilon_0\oint_S \vec{E}(\vec{x}\,')\cdot d\hat{a}'\right) = 0\end{aligned}$$

by Gauss' law. It is important not to mix up r and r' in this derivation.

3.27 Apply the Neumann Green function of Problem 3.26 to the situation in which the normal electric field is $E_r = -E_0\cos\theta$ at the outer surface ($r=b$) and is $E_r = 0$ on the inner surface ($r=a$).

a) Show that the electrostatic potential inside the volume V is

$$\Phi(\vec{x}\,) = E_0\frac{r\cos\theta}{1-p^3}\left(1+\frac{a^3}{2r^3}\right)$$

where $p = a/b$. Find the components of the electric field

$$E_r(r,\theta) = -E_0\frac{\cos\theta}{1-p^3}\left(1-\frac{a^3}{r^3}\right), \qquad E_\theta(r,\theta) = E_0\frac{\sin\theta}{1-p^3}\left(1+\frac{a^3}{2r^3}\right)$$

Since there is no charge between the spheres, the solution to be boundary value problem is given by

$$\begin{aligned}\Phi(\vec{x}\,) &= \frac{1}{4\pi}\oint_S \frac{\partial\Phi(\vec{x}\,')}{\partial n'}G(\vec{x},\vec{x}\,')\,da' \\ &= -\frac{1}{4\pi}\int_{r'=b} E_r(\Omega')G(\vec{x},\vec{x}\,')b^2\,d\Omega' \\ &= \frac{E_0 b^2}{4\pi}\int_{r'=b} G(\vec{x},\vec{x}\,')P_1(\cos\theta')\,d\Omega'\end{aligned}$$

By writing $P_l(\cos\gamma)$ in terms of spherical harmonics, and by using orthogonality of the Y_{lm}, we see that $\Phi(\vec{x}\,')$ has only a $l = 1$ component. Inserting $l = 1$ into (10), and making note that only $Y_{10} = \sqrt{3/4\pi}\cos\theta$ is important because of symmetry, we find

$$\begin{aligned}\Phi(\vec{x}\,) &= \frac{E_0 b^2}{4\pi}\int_{r'=b}[g_1(r,r')\cos\theta\cos\theta']\cos\theta' d\Omega' \\ &= \frac{E_0 b^2\cos\theta}{3}g_1(r,b) = \frac{E_0 b^2\cos\theta}{3}\frac{2}{b^3-a^3}\left(r+\frac{a^3}{2r^2}\right)\frac{3b}{2} \\ &= \frac{E_0 r\cos\theta}{1-(a/b)^3}\left(1+\frac{a^3}{2r^3}\right)\end{aligned} \tag{14}$$

This is the potential for a constant electric field combined with an electric dipole. Defining $p = a/b$, the components of the electric field are

$$E_r = -\frac{\partial\Phi}{\partial r} = -\frac{E_0\cos\theta}{1-p^3}\left(1-\frac{a^3}{r^3}\right), \qquad E_\theta = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = \frac{E_0\sin\theta}{1-p^3}\left(1+\frac{a^3}{2r^3}\right)$$

b) Calculate the Cartesian or cylindrical components of the field, E_z and E_ρ, and make a sketch or computer plot of the lines of electric force for a typical case of $p = 0.5$.

Rewriting (14) as

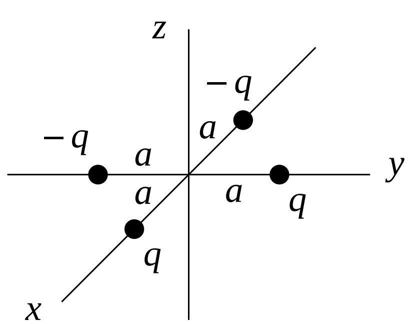
$$\Phi(\vec{x}\,) = \frac{E_0}{1-p^3}\left(z+\frac{a^3 z}{2r^3}\right)$$

we obtain

$$\begin{aligned}E_z &= -\frac{\partial\Phi}{\partial z} = -\frac{E_0}{1-p^3}\left(1+\frac{a^3(1-3\hat{z})}{2r^3}\right) \\ E_\rho &= -\frac{\partial\Phi}{\partial\rho} = -\frac{E_0}{1-p^3}\left(-\frac{3a^3\hat{z}\hat{\rho}}{2r^3}\right)\end{aligned}$$

4.1 Calculate the multipole moments q_{lm} of the charge distributions shown as parts *a*) and *b*). Try to obtain results for the nonvanishing moments valid for all l, but in each case find the first *two* sets of nonvanishing moments at the very least.

a)



The multipole moments are given by

$$q_{lm} = \int r^l Y^*_{lm}(\theta,\phi)\rho(\vec{x}\,)\,d^3x = qa^l[Y^*_{lm}(\tfrac{\pi}{2},0)+Y^*_{lm}(\tfrac{\pi}{2},\tfrac{\pi}{2})-Y^*_{lm}(\tfrac{\pi}{2},\pi)-Y^*_{lm}(\tfrac{\pi}{2},\tfrac{3\pi}{2})]$$

This is given in terms of associated Legendre polynomials by

$$q_{lm} = qa^l\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(0)[1+(-i)^m-(-1)^m-(i)^m]$$

The moments vanish unless m is odd. Writing $m = 2k+1$ gives

$$\begin{aligned} q_{l,2k+1} &= 2qa^l[1-i(-1)^k]\sqrt{\frac{2l+1}{4\pi}\frac{(l-(2k+1))!}{(l+(2k+1))!}}P_l^{2k+1}(0) \\ &= 2qa^l[1-i(-1)^k]Y_{l,2k+1}(\tfrac{\pi}{2},0) \end{aligned}$$

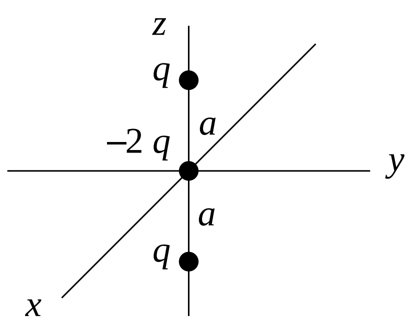
Note by parity this vanishes unless l is odd. Hence only the odd l and m moments are present. The lowest non-trivial ones are

$$q_{1,1} = -q^*_{1,-1} = -2qa(1-i)\sqrt{\frac{3}{8\pi}}$$

and

$$q_{3,3} = -q^*_{3,-3} = -2qa^3(1+i)\frac{1}{4}\sqrt{\frac{35}{4\pi}} \qquad q_{3,1} = -q^*_{3,-1} = 2qa^3(1-i)\frac{1}{4}\sqrt{\frac{21}{4\pi}}$$

b)



In this case, we have

$$q_{lm} = qa^l[Y^*_{lm}(0,0)+Y^*_{lm}(\pi,0)]$$

for $l > 0$ and $q_{00} = 0$. By azimuthal symmetry, only the $m = 0$ moments are non-vanishing. Hence

$$q_{l0} = qa^l\sqrt{\frac{2l+1}{4\pi}}[P_l(1)+P_l(-1)] = qa^l[1+(-1)^l]\sqrt{\frac{2l+1}{4\pi}} \qquad l > 0$$

We end up with even multipoles

$$q_{l0} = qa^l\sqrt{\frac{2l+1}{\pi}} \qquad l = 2, 4, 6, \ldots$$

Explicitly

$$q_{20} = qa^2\sqrt{\frac{5}{\pi}} \qquad q_{40} = 2qa^4\sqrt{\frac{9}{\pi}}$$

c) For the charge distribution of the second set *b*) write down the multipole expansion for the potential. Keeping only the lowest-order term in the expansion, plot the potential in the x-y plane as a function of distance from the origin for distances greater than a.

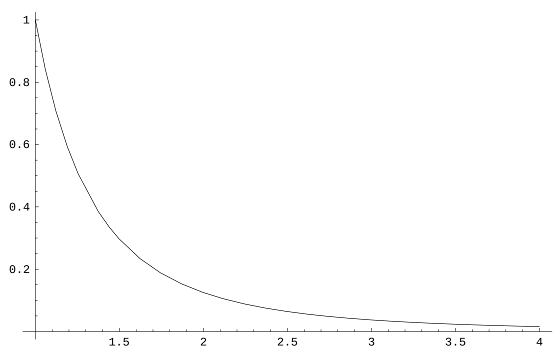
The expansion of the potential is

$$\begin{aligned}\Phi(\vec{x}\,) &= \frac{1}{4\pi\epsilon_0}\sum_{l,m}\frac{4\pi}{2l+1}q_{lm}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}} = \frac{1}{\epsilon_0}\sum_{l=2,4,\ldots}\frac{qa^l}{2l+1}\sqrt{\frac{2l+1}{\pi}}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}} \\ &= \frac{q}{2\pi\epsilon_0}\sum_{l=2,4,\ldots}\frac{a^l}{r^{l+1}}P_l(\cos\theta) = \frac{q}{4\pi\epsilon_0}\frac{a^2}{r^3}(3\cos^2\theta - 1) + \cdots\end{aligned}$$

In the x-y plane we have $\cos\theta = 0$, so the lowest order term is

$$\Phi = -\frac{q}{4\pi\epsilon_0 a}\left(\frac{a}{r}\right)^3 + \cdots$$

We all know what $1/r^3$ looks like when plotted, but here it is

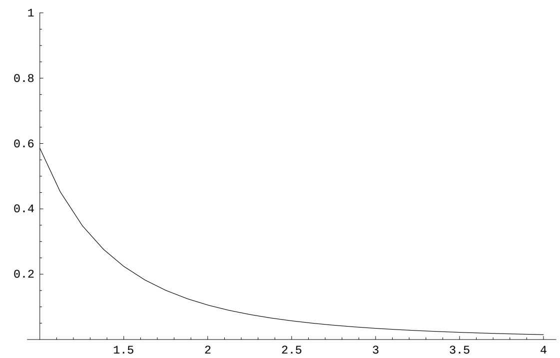


d) Calculate directly from Coulomb's law the exact potential for *b*) in the x-y plane. Plot it as a function of distance and compare with the result found in part *c*).

For three charges, the potential is simply the sum of three terms, one for each charge. In the x-y plane, if r is the distance from the origin we have

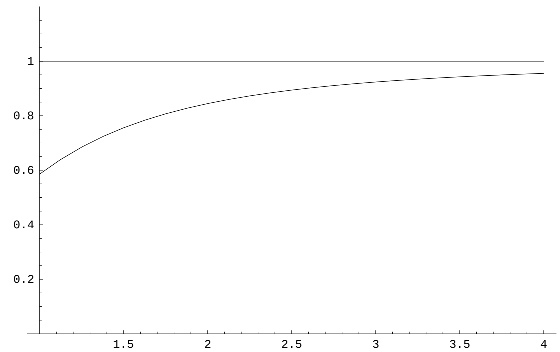
$$\begin{aligned}\Phi &= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{r^2+a^2}} - \frac{1}{r} + \frac{1}{\sqrt{r^2+a^2}}\right) = -\frac{q}{2\pi\epsilon_0 r}\left(1 - \frac{1}{\sqrt{1+(a/r)^2}}\right) \\ &= -\frac{q}{4\pi\epsilon_0 a}2\left(\frac{1}{(r/a)} - \frac{1}{\sqrt{1+(r/a)^2}}\right)\end{aligned}$$

The exact potential looks like



Divide out the asymptotic form in parts c) and d) to see the behavior at large distances more clearly.

If we divide out by $1/r^3$, the approximate and exact potentials are



where the straight line is the approximation of c) and the sloped line is the exact result. The approximation improves as $r \gg a$.