Physics 505 **Fall 2005**

Homework Assignment #4 — Solutions

Textbook problems: Ch. 3: 3.4, 3.6, 3.9, 3.10

3.4 The surface of a hollow conducting sphere of inner radius a is divided into an *even number* of equal segments by a set of planes; their common line of intersection is the z axis and they are distributed uniformly in the angle ϕ. (The segments are like the skin on wedges of an apple, or the earth's surface between successive meridians of longitude.) The segments are kept at fixed potentials $\pm V$, alternately.

a) Set up a series representation for the potential inside the sphere for the general case of $2n$ segments, and carry the calculation of the coefficients in the series far enough to determine exactly which coefficients are different from zero. For the nonvanishing terms, exhibit the coefficients as an integral over $\cos\theta$.

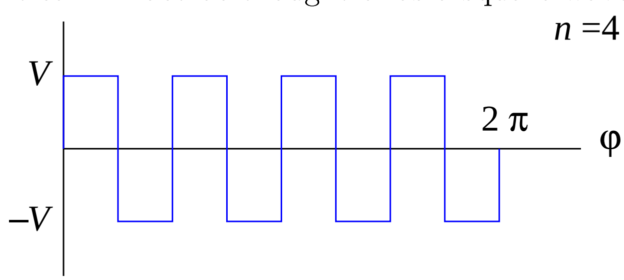
The general spherical harmonic expansion for the potential inside a sphere of radius a is

$$\Phi(r,\theta,\phi) = \sum_{l,m} \alpha_{lm} \left(\frac{r}{a}\right)^l Y_{lm}(\theta,\phi)$$

where

$$\alpha_{lm} = \int V(\theta,\phi) Y^*_{lm}(\theta,\phi) d\Omega$$

In this problem, $V(\theta,\phi) = \pm V$ is independent of θ, but depends on the azimuthal angle ϕ. It can in fact be thought of as a square wave in ϕ



This has a familiar Fourier expansion

$$V(\phi) = \frac{4V}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin[(2k+1)n\phi]$$

This is already enough to demonstrate that the m values in the spherical harmonic expansion can only take on the values $\pm(2k+1)n$. In terms of associated Legendre polynomials, the expansion coefficients are

$$\begin{aligned}
\alpha_{lm} &= \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\int_0^{2\pm} V(\phi)e^{-im\phi}\,d\phi \int_{-1}^{1} P_l^m(x)\,dx \\
&= \frac{4V}{\pi}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\sum_{k=0}^{\infty}\frac{1}{2k+1}\int_0^{2\pi}\sin[(2k+1)n\phi]e^{-im\phi}\,d\phi \\
&\qquad\qquad \times \int_{-1}^{1} P_l^m(x)\,dx \\
&= -4iV\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\sum_{k=0}^{\infty}\frac{\delta_{m,(2k+1)n}-\delta_{m,-(2k+1)n}}{2k+1}\int_{-1}^{1} P_l^m(x)\,dx
\end{aligned}$$

Using $P_l^{-m}(x) = (-)^m[(l-m)!/(l+m)!]P_l^m(x)$, we may write the non-vanishing coefficients as

$$\begin{aligned}
\alpha_{l,-(2k+1)n} &= (-)^{n+1}\alpha_{l,(2k+1)n} \\
&= -\frac{4iV}{2k+1}\sqrt{\frac{2l+1}{4\pi}\frac{(l-(2k+1)n)!}{(l+(2k+1)n)!}}\int_{-1}^{1} P_l^{(2k+1)n}(x)\,dx
\end{aligned} \tag{1}$$

for $k = 0, 1, 2, \ldots$. Since $l \geq (2k+1)n$, we see that the first non-vanishing term enters at order $l = n$. Making note of the parity of associated Legendre polynomials, $P - l^m(-x) = (-)^{l+m}P_l^m(x)$, we see that the non-vanishing coefficients are given by the sequence

$$\begin{array}{llll}
\alpha_{n,n}, & \alpha_{n+2,n}, & \alpha_{n+4,n}, & \alpha_{n+6,n}, \ldots \\
\alpha_{3n,3n}, & \alpha_{3n+2,3n}, & \alpha_{3n+4,3n}, \ldots & \\
\alpha_{5n,5n}, & \alpha_{5n+2,5n}, & \alpha_{5n+4,5n}, \ldots & \\
\vdots & & &
\end{array}$$

b) For the special case of $n = 1$ (two hemispheres) determine explicitly the potential up to an including all terms with $l = 3$. By a coordinate transformation verify that this reduces to result (3.36) of Section 3.3.

For $n = 1$, explicit computation shows that

$$\int_{-1}^{1} P_1^1(x)\,dx = -\frac{\pi}{2}, \qquad \int_{-1}^{1} P_3^1(x)\,dx = -\frac{3\pi}{16}, \qquad \int_{-1}^{1} P_3^3(x)\,dx = -\frac{45\pi}{8}$$

Inserting this in to (1) yields

$$\begin{aligned}
&\alpha_{1,-1} = \alpha_{1,1} = iV\sqrt{\frac{3\pi}{2}} \\
&\alpha_{3,-1} = \alpha_{3,1} = iV\sqrt{\frac{21\pi}{256}}, \qquad \alpha_{3,-3} = \alpha_{3,3} = iV\sqrt{\frac{35\pi}{256}}
\end{aligned}$$

Hence

$$
\begin{aligned}
\Phi &= iV\Bigg[\left(\frac{r}{a}\right)\sqrt{\frac{3\pi}{2}}(Y_{1,1}+Y_{1,-1}) \\
&\qquad\qquad + \left(\frac{r}{a}\right)^3\left(\sqrt{\frac{21\pi}{256}}(Y_{3,1}+Y_{3,-1})+\sqrt{\frac{35\pi}{256}}(Y_{3,3}+Y_{3,-3})\right)+\cdots\Bigg] \\
&= -2V\Im\left[\left(\frac{r}{a}\right)\sqrt{\frac{3\pi}{2}}Y_{1,1}+\left(\frac{r}{a}\right)^3\left(\sqrt{\frac{21\pi}{256}}Y_{3,1}+\sqrt{\frac{35\pi}{256}}Y_{3,3}\right)+\cdots\right] \\
&= 2V\Im\Bigg[\left(\frac{r}{a}\right)\frac{3}{4}\sin\theta\, e^{i\phi} \\
&\qquad\qquad + \left(\frac{r}{a}\right)^3\left(\frac{21}{128}\sin\theta(5\cos^2\theta-1)e^{i\phi}+\frac{35}{128}\sin^3\theta\, e^{3i\phi}\right)+\cdots\Bigg] \\
&= V\Bigg[\left(\frac{r}{a}\right)\frac{3}{2}\sin\theta\sin\phi \\
&\qquad\qquad \left(\frac{r}{a}\right)^3\frac{7}{128}\big(3\sin\theta(5\cos^2\theta-1)\sin\phi+5\sin^3\theta\sin^3\phi\big)+\cdots\Bigg]
\end{aligned}
\tag{2}
$$

To relate this to the previous result, we note that the way we have set up the wedges corresponds to taking the 'top' of the $+V$ hemisphere to point along the $\hat{y}$ axis. This may be rotated to the $\hat{z}'$ axis by a 90° rotation along the $\hat{x}$ axis. Explicitly, we take

$$
\hat{y}=\hat{z}', \qquad \hat{z}=-\hat{y}', \qquad \hat{x}=\hat{x}'
$$

or

$$
\sin\theta\sin\phi=\cos\theta', \qquad \cos\theta=-\sin\theta'\sin\phi', \qquad \sin\theta\cos\phi=\sin\theta'\cos\phi'
$$

Noting that $\sin 3\phi=-\sin^3\phi+3\sin\phi\cos^2\phi$, the last line of (2) transforms into

$$
\begin{aligned}
\Phi &= V\Bigg[\left(\frac{r}{a}\right)\frac{3}{2}\cos\theta'+\left(\frac{r}{a}\right)^3\frac{7}{128}\big(3\cos\theta'(5\sin^2\theta'\sin^2\phi'-1) \\
&\qquad\qquad + 5(-\cos^3\theta'+3\cos\theta'\sin^2\theta'\sin^2\phi')\big)+\cdots\Bigg] \\
&= V\left[\frac{3}{2}\left(\frac{r}{a}\right)\cos\theta'-\frac{7}{8}\left(\frac{r}{a}\right)^3\tfrac{1}{2}(5\cos^3\theta'-3\cos\theta')+\cdots\right] \\
&= V\left[\frac{3}{2}\left(\frac{r}{a}\right)P_1(\cos\theta')-\frac{7}{8}\left(\frac{r}{a}\right)^3P_3(\cos\theta')+\cdots\right]
\end{aligned}
$$

which reproduces the result (3.36).

3.6 Two point charges q and $-q$ are located on the z axis at $z = +a$ and $z = -a$, respectively.

a) Find the electrostatic potential as an expansion in spherical harmonics and powers of r for both $r > a$ and $r < a$.

The potential is clearly

$$\Phi = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\vec{x}-\vec{a}\,|} - \frac{1}{|\vec{x}+\vec{a}\,|}\right)$$

where $\vec{a} = a\hat{z}$ points from the origin to the positive charge. Using the spherical harmonic expansion

$$\frac{1}{|\vec{x}-\vec{x}\,'|} = 4\pi\sum_{l,m}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\hat{x}')Y_{lm}(\hat{x})$$

as well as $\vec{a} = a\hat{z}$, we obtain

$$\Phi = \frac{q}{\epsilon_0}\sum_{l,m}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}[Y_{lm}^*(0,\phi') - Y_{lm}^*(\pi,\phi')]Y_{lm}(\theta,\phi) \tag{3}$$

Noting that $Y_{lm}(0,\phi) \sim P_l^m(1)$ and that $P_l^m(1) = \delta_{m,0}$ we see that only terms with $m = 0$ contribute. This is also obvious from symmetry. Since

$$Y_{l0}(0,\phi) = (-)^l Y_{l0}(\pi,\phi) = \sqrt{\frac{2l+1}{4\pi}}$$

the potential (3) becomes

$$\begin{aligned}\Phi &= \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}[1-(-)^l]\frac{r_<^l}{r_>^{l+1}}\sqrt{\frac{4\pi}{2l+1}}Y_{l0}(\theta,\phi)\\ &= \frac{q}{2\pi\epsilon_0}\sum_{l\ \text{odd}}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)\end{aligned} \tag{4}$$

b) Keeping the product $qa = p/2$ constant, take the limit of $a \to 0$ and find the potential for $r \neq 0$. This is by definition a dipole along the z axis and its potential.

Since we will take $a \to 0$, we have $r_< = a$ and $r_> = r$. This yields an expansion of (4)

$$\Phi = \frac{qa}{2\pi\epsilon_0 r^2}\sum_{k=0}^{\infty}\left(\frac{a}{r}\right)^{2k}P_{2k+1}(\cos\theta)$$

Setting $qa = p/2$ and taking $a \to 0$, only the $k = 0$ term survives in the sum. The result is

$$\Phi = \frac{p}{4\pi\epsilon_0} \frac{1}{r^2} P_1(\cos\theta) = \frac{p}{4\pi\epsilon_0} \frac{\cos\theta}{r^2} \tag{5}$$

which is the potential due to a dipole.

c) Suppose now that the dipole of part *b*) is surrounded by a *grounded* spherical shell of radius b concentric with the origin. By linear superposition find the potential everywhere inside the shell.

To account for the spherical shell, we add to (5) a solution to the (homogeneous) Laplace's equation. For an inside solution, we have

$$\Phi = \frac{p}{4\pi\epsilon_0} \left[\frac{1}{r^2} P_1(\cos\theta) + \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \right]$$

The boundary condition $\Phi(r = b) = 0$ corresponds to having

$$\sum_{l=0}^{\infty} A_l b^{l+2} P_l(\cos\theta) = -P_1(\cos\theta)$$

Since the Legendre polynomials form an orthonormal set, the only term that can show up on the left hand side is the $l = 1$ term. We then take $A_1 = -1/b^{l+2}$, and the resulting solution is

$$\Phi = \frac{p}{4\pi\epsilon_0} \left(\frac{1}{r^2} - \frac{r}{b^3} \right) \cos\theta$$

3.9 A hollow right circular cylinder of radius b has its axis coincident with the z axis and its ends at $z = 0$ and $z = L$. The potential on the end faces is zero, while the potential on the cylindrical surface is given as $V(\phi, z)$. Using the appropriate separation of variables in cylindrical coordinates, find a series solution for the potential anywhere inside the cylinder.

The general solution obtained by separation of variables has the form

$$\Phi(\rho, \phi, z) = \sum \left\{ J_m(k\rho) \text{ or } N_m(k\rho) \right\} \left\{ e^{\pm im\phi} \right\} \left\{ e^{\pm kz} \right\}$$

However, since the potential vanishes on the endcaps, it is natural to take $k \to ik$ so that the z function obeying boundary conditions is $\sin(n\pi z/L)$. The result is to use the modified Bessel functions $I_m(k\rho)$ and $K_m(k\rho)$ instead. However, for the solution to be regular at $\rho = 0$ we discard the $K_\nu(k\rho)$ functions, which blow up at vanishing argument. The resulting series expression for the potential is

$$\Phi(\rho, \phi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} I_m \left(\frac{n\pi}{L} \rho \right) \left(a_{mn} \sin m\phi + b_{mn} \cos m\phi \right) \sin \left(\frac{n\pi}{L} z \right) \tag{6}$$

In order to satisfy the boundary conditions on the cylindrical surface, we need to have

$$V(\phi,z)=\sum_{m,n} I_m\left(\frac{n\pi b}{L}\right)(a_{mn}\sin m\phi+b_{mn}\cos m\phi)\sin\left(\frac{n\pi}{L}z\right)$$

This is a double Fourier series in ϕ and z. As a result, the Fourier coefficients are

$$\begin{Bmatrix} a_{mn} \\ b_{mn} \end{Bmatrix} I_m\left(\frac{n\pi b}{L}\right)=\frac{1}{\pi}\int_0^{2\pi} d\phi\,\frac{2}{L}\int_0^L dz\,V(\phi,z)\begin{Bmatrix} \sin m\phi \\ \cos m\phi \end{Bmatrix}\sin\left(\frac{n\pi}{L}z\right)$$

with the caveat that $b_{0,n}$ must be divided by two. This can be rewritten as

$$\begin{Bmatrix} a_{mn} \\ b_{mn} \end{Bmatrix}=\frac{2}{\pi L I_m(n\pi b/L)}\int_0^{2\pi} d\phi\int_0^L dz\,V(\phi,z)\begin{Bmatrix} \sin m\phi \\ \cos m\phi \end{Bmatrix}\sin\left(\frac{n\pi}{L}z\right) \qquad (7)$$

(where $b_{0,n}$ has to be divided by two).

3.10 For the cylinder in Problem 3.9 the cylindrical surface is made of two equal half-cylinders, one at potential V and the other at potential $-V$, so that

$$V(\phi,z)=\begin{cases} V & \text{for } -\pi/2<\phi<\pi/2 \\ -V & \text{for } \pi/2<\phi<3\pi/2 \end{cases}$$

a) Find the potential inside the cylinder.

To obtain the potential, we want to find the coefficients a_{mn} and b_{mn} of the expansion (6) and (7). Noting first that $V(\phi,z)=V(\phi)=\pm V$ is an even function of ϕ, we see that all the a_{mn} coefficients vanish. We are left with

$$\begin{aligned} b_{mn}&=\frac{2V}{\pi L I_m(n\pi b/L)}\left[\int_{-\pi/2}^{\pi/2}-\int_{\pi/2}^{3\pi/2}\right]d\phi\,\cos m\phi\int_0^L dz\,\sin\frac{n\pi z}{L}\\ &=\frac{2V}{\pi^2 I_m(n\pi b/L)}\frac{4\sin(m\pi/2)}{m}\frac{1-(-)^n}{n} \qquad (m\neq 0)\end{aligned}$$

This is non-vanishing only when both m and n are odd. Introducing $m=2k+1$ and $n=2l+1$, we have

$$b_{2k+1,2l+1}=\frac{16V}{\pi^2 I_{2k+1}((2l+1)\pi b/L)}\frac{(-)^k}{(2k+1)(2l+1)}$$

Inserting this into (6) yields

$$\Phi=\frac{16V}{\pi^2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-)^k}{(2k+1)(2l+1)}\frac{I_{2k+1}(\frac{(2l+1)\pi\rho}{L})}{I_{2k+1}(\frac{(2l+1)\pi b}{L})}\cos(2k+1)\phi\sin\frac{(2l+1)\pi z}{L} \qquad (8)$$

b) Assuming $L \gg b$, consider the potential at $z = L/2$ as a function of ρ and ϕ and compare it with two-dimensional Problem 2.13.

For $L \gg b$ both ρ/L and b/L are much less than one. This allows us to use a small argument expansion of the modified Bessel function

$$I_\nu(x) \approx \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu$$

In addition, for $z = L/2$ we have

$$\sin\frac{(2l+1)\pi z}{L} = \sin(l+\tfrac{1}{2})\pi = (-)^l$$

Hence in this limit (8) becomes

$$\begin{aligned}\Phi &= \frac{16V}{\pi^2}\sum_{k,l}\frac{(-)^k}{2k+1}\frac{(-)^l}{2l+1}\left(\frac{\rho}{b}\right)^{2k+1}\cos(2k+1)\phi \\ &= \frac{16V}{\pi^2}\left[\sum_l \frac{(-)^l}{2l+1}\right]\Re\left[\sum_k \frac{(-)^k}{2k+1}\left(\frac{\rho}{b}e^{i\phi}\right)^{2k+1}\right]\end{aligned}$$

Noting the Taylor series expansion for arctan

$$\tan^{-1} z = \sum_n \frac{(-)^n}{2n+1} z^{2n+1}$$

we arrive at

$$\Phi = \frac{16V}{\pi^2}\tan^{-1}(1)\Re\tan^{-1}\left(\frac{\rho}{b}e^{i\phi}\right) = \frac{4V}{\pi}\Re\tan^{-1}\left(\frac{\rho}{b}e^{i\phi}\right)$$

To calculate $\Re\tan^{-1} z$ we recall that

$$\tan^{-1} a + \tan^{-1} b = \tan^{-1}\frac{a+b}{1-ab}$$

Hence

$$\Re\tan^{-1} z = \tfrac{1}{2}(\tan^{-1} z + \tan^{-1} z^*) = \tfrac{1}{2}\tan^{-1}\frac{z+z^*}{1-|z|^2}$$

For $z = (\rho/b)e^{i\phi}$ we find

$$\Phi = \frac{2V}{\pi}\tan^{-1}\frac{2(\rho/b)\cos\phi}{1-(\rho/b)^2} = \frac{2V}{\pi}\tan^{-1}\frac{2b\rho\cos\phi}{b^2-\rho^2}$$

which reproduces the answer to Problem 2.13 (where $V_1 = -V_2 = V$).