Physics 505 **Fall 2005**

Homework Assignment #3 — Solutions

Textbook problems: Ch. 2: 2.12, 2.13, 2.15
Ch. 3: 3.1

2.12 Starting with the series solution (2.71) for the two-dimensional potential problem with the potential specified on the surface of a cylinder of radius b, evaluate the coefficients formally, substitute them into the series, and sum it to obtain the potential *inside* the cylinder in the form of Poisson's integral:

$$\Phi(\rho,\phi)=\frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}\,d\phi'$$

What modification is necessary if the potential is desired in the region of space bounded by the cylinder and infinity?

The series solution (2.71) is given as

$$\Phi(\rho,\phi)=a_0+b_0\ln\rho+\sum_{n=1}^{\infty}[a_n\rho^n\sin(n\phi+\alpha_n)+b_n\rho^{-n}\sin(n\phi+\beta_n)]$$

Since we want an inside solution, we take $b_n=0$ so the potential is well behaved at $\rho=0$. With some rewriting of the series, we can then turn it into the equivalent form

$$\Phi(\rho,\phi)=\tfrac{1}{2}c_0+\sum_{n=1}^{\infty}\left(\frac{\rho}{b}\right)^n[c_n\cos(n\phi)+d_n\sin(n\phi)] \tag{1}$$

Breaking up the $\sin(n\phi+\alpha_n)$ terms into sines and cosines is convenient because we now end up with a standard Fourier series. The Fourier coefficients are

$$c_n=\frac{1}{\pi}\int_0^{2\pi}\Phi(b,\phi')\cos(n\phi')\,d\phi'$$
$$d_n=\frac{1}{\pi}\int_0^{2\pi}\Phi(b,\phi')\sin(n\phi')\,d\phi'$$

Substituting this back into (1) yields

$$\begin{aligned}\Phi(b,\phi)&=\frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')\\&\qquad\times\left[1+2\sum_{n=1}^{\infty}\left(\frac{\rho}{b}\right)^n\big(\cos(n\phi)\cos(n\phi')+\sin(n\phi)\sin(n\phi')\big)\right]d\phi'\\&=\frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')\left[1+2\sum_{n=1}^{\infty}\left(\frac{\rho}{b}\right)^n\cos n(\phi'-\phi)\right]d\phi'\end{aligned} \tag{2}$$

The series in the square brackets can be summed as a geometric series

$$
\begin{aligned}
1 + 2\sum_{n=1}^{\infty}\left(\frac{\rho}{b}\right)^n \cos n(\phi' - \phi)& \\
&= \Re\left[1 + 2\sum_{n=1}^{\infty}\left(\frac{\rho}{b}\right)^n e^{in(\phi'-\phi)}\right] = \Re\left[-1 + 2\sum_{n=0}^{\infty}\left(\frac{\rho}{b}e^{i(\phi'-\phi)}\right)^n\right] \\
&= \Re\left[-1 + \frac{2}{1 - \frac{\rho}{b}e^{i(\phi'-\phi)}}\right] = \Re\frac{1 + \frac{\rho}{b}e^{i(\phi'-\phi)}}{1 - \frac{\rho}{b}e^{i(\phi'-\phi)}} \\
&= \Re\frac{\left(1 + \frac{\rho}{b}e^{i(\phi'-\phi)}\right)\left(1 - \frac{\rho}{b}e^{-i(\phi'-\phi)}\right)}{\left(1 - \frac{\rho}{b}e^{i(\phi'-\phi)}\right)\left(1 - \frac{\rho}{b}e^{-i(\phi'-\phi)}\right)} \\
&= \Re\frac{1 - (\rho/b)^2 + 2i(\rho/b)\sin(\phi'-\phi)}{1 + (\rho/b)^2 - 2(\rho/b)\cos(\phi'-\phi)} = \frac{b^2 - \rho^2}{b^2 + \rho^2 - 2\rho b\cos(\phi'-\phi)}
\end{aligned}
$$

Inserting this into (2) finally yields the result

$$
\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')\frac{b^2-\rho^2}{b^2+\rho^2-2\rho b\cos(\phi'-\phi)}\,d\phi' \tag{3}
$$

Note that this can be obtained more directly from the Cauchy integral formula of complex analysis (Jackson problem 2.21).

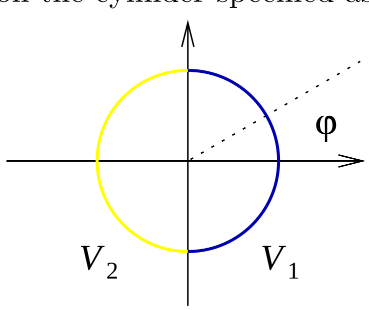
For the exterior solution, we may simply take $\rho/b \to b/\rho$ in (1). This corresponds to making the replacement $\rho \leftrightarrow b$ in the fraction in the integrand of (3). Since this only affects the numerator, the simple result is that we change the sign of $\Phi(\rho, \phi)$.

2.13 *a*) Two halves of a long hollow conducting cylinder of inner radius b are separated by small lengthwise gaps on each side, and are kept at different potentials V_1 and V_2. Show that the potential inside is given by

$$
\Phi(\rho,\phi) = \frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi}\tan^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right)
$$

where ϕ is measured from a plane perpendicular to the plane through the gap.

For the potential on the cylinder specified as



we may write the potential as an average piece on the cylinder plus a deviation from the average

$$V(\phi) = \tfrac{1}{2}(V_1 + V_2) + \tfrac{1}{2}(V_1 - V_2)\,\mathrm{sgn}\,(\cos\phi)$$

By linear superposition, the potential $\Phi(\rho,\phi)$ is a sum of the constant (average) term plus the term proportional to $V_1 - V_2$. Using (3), we have

$$\Phi(\rho,\phi) = \frac{V_1+V_2}{2} + \frac{V_1-V_2}{4\pi}\int_0^{2\pi} \mathrm{sgn}\,(\cos\phi')\frac{b^2-\rho^2}{b^2+\rho^2-2\rho b\cos(\phi'-\phi)}\,d\phi' \quad (4)$$

We may break the integral up into two pieces

$$\int_0^{2\pi} \mathrm{sgn}\,(\cos\phi')\,d\phi' = \int_{-\pi/2}^{\pi/2} d\phi' - \int_{\pi/2}^{3\pi/2} d\phi'$$

The second integral can be brought into the domain $[-\pi/2, \pi/2]$ by the shift $\phi' \to \phi' + \pi$. This has the effect of flipping the sign of $\cos(\phi'-\phi)$ in the denominator of (4). Hence

$$\begin{aligned}
\Phi(\rho,\phi) &= \frac{V_1+V_2}{2} \\
&\quad + \frac{V_1-V_2}{4\pi}\int_{-\pi/2}^{\pi/2}\left[\frac{b^2-\rho^2}{b^2+\rho^2-2\rho b\cos(\phi'-\phi)} - \frac{b^2-\rho^2}{b^2+\rho^2+2\rho b\cos(\phi'-\phi)}\right]d\phi' \\
&= \frac{V_1+V_2}{2} + \frac{(V_1-V_2)b\rho(b^2-\rho^2)}{\pi}\int_{-\pi/2}^{\pi/2}\frac{\cos(\phi'-\phi)}{(b^2+\rho^2)^2 - 4b^2\rho^2\cos^2(\phi'-\phi)}\,d\phi'
\end{aligned}$$

This can be integrated by making the substitution

$$u = \sin(\phi'-\phi) \qquad du = \cos(\phi'-\phi)\,d\phi'$$

The result is

$$\Phi(\rho,\phi) = \frac{V_1+V_2}{2} + \frac{(V_1-V_2)b\rho(b^2-\rho^2)}{\pi}\int_{-\cos\phi}^{\cos\phi}\frac{du}{(b^2-\rho^2)^2+4b^2\rho^2u^2}$$

This can be converted into an arctan integral with the substitution

$$x = \frac{2b\rho}{b^2-\rho^2}u \qquad x_0 = \frac{2b\rho}{b^2-\rho^2}\cos\phi$$

giving

$$\begin{aligned}
\Phi(\rho,\phi) &= \frac{V_1+V_2}{2} + \frac{V_1-V_2}{2\pi}\int_{-x_0}^{x_0}\frac{dx}{1+x^2} = \frac{V_1+V_2}{2} + \frac{V_1-V_2}{2\pi}\tan^{-1}x_0\Big|_{-x_0}^{x_0} \\
&= \frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi}\tan^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right)
\end{aligned}$$

b) Calculate the surface-charge density on each half of the cylinder.

The surface charge density is given by

$$\begin{aligned}
\sigma = -\epsilon_0 \frac{d\Phi}{d\rho}\bigg|_{\rho=b} &= -\epsilon_0 \frac{V_1 - V_2}{\pi} \partial\rho \tan^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right)\bigg|_{\rho=b} \\
&= -\epsilon_0 \frac{V_1 - V_2}{\pi} \frac{1}{1+\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right)^2} \frac{2b(b^2+\rho^2)}{(b^2-\rho^2)^2}\cos\phi\bigg|_{\rho=b} \\
&= -\epsilon_0 \frac{V_1 - V_2}{\pi} \frac{2b(b^2+\rho^2)\cos\phi}{(b^2-\rho^2)^2 + 4b^2\rho^2\cos^2\phi}\bigg|_{\rho=b} \\
&= -\epsilon_0 \frac{V_1 - V_2}{\pi b \cos\phi}
\end{aligned}$$

Note that we would have ran into trouble had we made the substitution $\rho = b$ at too early a stage. Furthermore, this expression is valid for either half of the cylinder. However the surface charge density blows up at the boundaries between halves (where $\cos\phi$ vanishes).

2.15 *a*) Show that the Green function $G(x, y; x', y')$ appropriate for Dirichlet boundary conditions for a square two-dimensional region, $0 \le x \le 1$, $0 \le y \le 1$, has an expansion

$$G(x, y; x', y') = 2\sum_{n=1}^{\infty} g_n(y, y') \sin(n\pi x) \sin(n\pi x')$$

where $g_n(y, y')$ satisfies

$$\left(\frac{\partial^2}{\partial y'^2} - n^2\pi^2\right) g_n(y, y') = -4\pi\delta(y' - y) \quad \text{and} \quad g_n(y, 0) = g_n(y, 1) = 0$$

We start by recalling the the Green's function is defined by

$$(\partial_{x'}^2 + \partial_{y'}^2) G(x, y; x', y') = -4\pi\delta(x' - x)\delta(y' - y) \tag{5}$$

Although this is symmetric in x' and y', the problem suggests that we begin by expanding in x' (and also x). This of course breaks the symmetry in the *expanded* form of the Green's function. Nevertheless $G(x, y; x', y')$ is unique for the given boundary conditions; it just may admit different expansions, and we are free to choose whatever expansion is the most convenient.

Given the boundary condition that G vanishes for $x' = 0$ and $x' = 1$, this suggests an expansion in a Fourier sine series

$$G(x, y; x', y') = \sum_{n=1}^{\infty} f_n(x, y; y') \sin(n\pi x')$$

Substituting this into (5) then gives

$$\sum_{n=1}^{\infty}(\partial_{y'}^2 - n^2\pi^2) f_n(x,y;y') \sin(n\pi x') = -4\pi\delta(x'-x)\delta(y'-y) \tag{6}$$

However this is not particularly useful (yet), since the $\delta(x'-x)$ on the right hand side does not match with the Fourier sine series on the left. We can get around this by invoking the completeness relation for the sine series

$$\sum_{n=1}^{\infty} \sin(n\pi x)\sin(n\pi x') = \tfrac{1}{2}\delta(x-x')$$

Replacing the delta function by the sum, we end up by rewriting (6) as

$$\sum_{n=1}^{\infty}(\partial_{y'}^2 - n^2\pi^2) f_n(x,y;y') \sin(n\pi x') = -8\pi\delta(y'-y)\sum_{n=1}^{\infty}\sin(n\pi x)\sin(n\pi x') \tag{7}$$

Matching left and right sides of the Fourier sine series indicates that the x behavior of $f_n(x,y;y')$ must be given by $\sin(n\pi x)$. Putting in a factor of two for convenience

$$f_n(x,y;y') = 2g_n(y,y')\sin(n\pi x)$$

finally motivates the expansion

$$G(x,y;x',y') = 2\sum_{n=1}^{\infty} g_n(y,y')\sin(n\pi x)\sin(n\pi x')$$

When this is inserted into (7), we match the x and x' behavior perfectly, and we are left with an equation in y'

$$(\partial_{y'}^2 - n^2\pi^2) g_n(y,y') = -4\pi\delta(y'-y) \tag{8}$$

The boundary conditions are that G vanishes at $y'=0$ and $y'=1$. Hence we must also demand $g_n(y,0) = g_n(y,1) = 0$.

b) Taking for $g_n(y,y')$ appropriate linear combinations of $\sinh(n\pi y')$ and $\cosh(n\pi y')$ in the two regions, $y' < y$ and $y' > y$, in accord with the boundary conditions and the discontinuity in slope required by the source delta function, show that the explicit form of G is

$$G(x,y;x',y') = 8\sum_{n=1}^{\infty}\frac{1}{n\sinh(n\pi)}\sin(n\pi x)\sin(n\pi x')\sinh(n\pi y_<)\sinh[n\pi(1-y_>)]$$

where $y_<(y_>)$ is the smaller (larger) of y and y'.

To find the Green's function for (8), we begin with the solution to the homogeneous equation $(\partial_{y'}^2 - n^2\pi^2)g_n(y,y') = 0$. This clearly has exponential solutions $e^{\pm n\pi y'}$, or equivalently $\sinh(n\pi y')$ and $\cosh(n\pi y')$. As a result, we can write the Green's function as

$$g_n(y,y') = \begin{cases} g_< \equiv a_< \sinh(n\pi y') + b_< \cosh(n\pi y') & y' < y \\ g_> \equiv a_> \sinh(n\pi y') + b_> \cosh(n\pi y') & y' > y \end{cases}$$

We wish to solve for the four constants $a_<, b_<, a_>, b_>$ given the boundary conditions $g_n(y,0) = 0$, $g_n(y,1) = 0$ and the continuity and jump conditions

$$g_> = g_< \qquad \partial_{y'} g_> = \partial_{y'} g_< - 4\pi \qquad \text{when} \quad y' = y$$

We start with the boundary conditions. For $g_<$ to vanish at $y' = 0$ we must take the sinh solution, while for $g_>$ to vanish at $y' = 1$ we end up with $a_> \sinh(n\pi) + b_> \cosh(n\pi) = 0$ or $b_> = -a_> \tanh(n\pi)$. Thus

$$g_n(y,y') = \begin{cases} a_< \sinh(n\pi y') & y' < y \\ a_>[\sinh(n\pi y') - \tanh(n\pi)\cosh(n\pi y')] & y' > y \end{cases} \tag{9}$$

The continuity and jump conditions yield the system of equations

$$\begin{pmatrix} \sinh(n\pi y) & -\sinh(n\pi y) + \tanh(n\pi)\cosh(n\pi y) \\ \cosh(n\pi y) & -\cosh(n\pi y) + \tanh(n\pi)\sinh(n\pi y) \end{pmatrix} \begin{pmatrix} a_< \\ a_> \end{pmatrix} = \begin{pmatrix} 0 \\ 4/n \end{pmatrix}$$

which is solved by

$$\begin{aligned} \begin{pmatrix} a_< \\ a_> \end{pmatrix} &= -\frac{4}{n\tanh(n\pi)} \begin{pmatrix} \sinh(n\pi y) - \tanh(n\pi)\cosh(n\pi y) \\ \sinh(n\pi y) \end{pmatrix} \\ &= -\frac{4}{n\sinh(n\pi)} \begin{pmatrix} \cosh(n\pi)\sinh(n\pi y) - \sinh(n\pi)\cosh(n\pi y) \\ \cosh(n\pi)\sinh(n\pi y) \end{pmatrix} \end{aligned}$$

Inserting this into (9) gives

$$\begin{aligned} g_n(y,y') = &\frac{4}{n\sinh(n\pi)} \\ &\times \begin{cases} \sinh(n\pi y')[\sinh(n\pi)\cosh(n\pi y) - \cosh(n\pi)\sinh(n\pi y)] & y' < y \\ \sinh(n\pi y)[\sinh(n\pi)\cosh(n\pi y') - \cosh(n\pi)\sinh(n\pi y')] & y' > y \end{cases} \end{aligned}$$

This is simplified by noting

$$\sinh[n\pi(1-y)] = \sinh(n\pi)\cosh(n\pi y) - \cosh(n\pi)\sinh(n\pi y)$$

and by using the definition $y_< = \min(y,y')$ and $y_> = \max(y,y')$. The result is

$$g_n(y,y') = \frac{4}{n\sinh(n\pi)} \sinh(n\pi y_<)\sinh[n\pi(1-y_>)]$$

which yields

$$G(x,y;x',y') = \sum_n \frac{8}{n\sinh(n\pi)} \sin(n\pi x)\sin(n\pi x')\sinh(n\pi y_<)\sinh[n\pi(1-y_>)]$$

3.1 Two concentric spheres have radii a, b ($b > a$) and each is divided into two hemispheres by the same horizontal plane. The upper hemisphere of the inner sphere and the lower hemisphere of the outer sphere are maintained at potential V. The other hemispheres are at zero potential.

Determine the potential in the region $a \le r \le b$ as a series in Legendre polynomials. Include terms at least up to $l = 4$. Check your solution against known results in the limiting cases $b \to \infty$, and $a \to 0$.

The general expansion in Legendre polynomials is of the form

$$\Phi(r,\theta) = \sum_\ell [A_\ell r^\ell + B_\ell r^{-\ell-1}]P_\ell(\cos\theta) \tag{10}$$

Since we are working in the region between spheres, neither A_ℓ nor B_ℓ can be assumed to vanish. To solve for both A_ℓ and B_ℓ we will need to consider boundary conditions at $r = a$ and $r = b$

$$\begin{aligned} \Phi(a,\theta) &= \sum_\ell [A_\ell a^\ell + B_\ell a^{-\ell-1}]P_\ell(\cos\theta) = V \quad \text{for} \quad \cos\theta \ge 0 \\ \Phi(b,\theta) &= \sum_\ell [A_\ell b^\ell + B_\ell b^{-\ell-1}]P_\ell(\cos\theta) = V \quad \text{for} \quad \cos\theta \le 0 \end{aligned}$$

Using orthogonality of the Legendre polynomials, we may write

$$\begin{aligned} A_\ell a^\ell + B_\ell a^{-\ell-1} &= \frac{2\ell+1}{2} V \int_0^1 P_\ell(x)\,dx \\ A_\ell b^\ell + B_\ell b^{-\ell-1} &= \frac{2\ell+1}{2} V \int_{-1}^0 P_\ell(x)\,dx = \frac{2\ell+1}{2} V(-)^\ell \int_0^1 P_\ell(x)\,dx \end{aligned}$$

where in the last expression we used the fact that $P_\ell(-x) = (-)^\ell P_\ell(x)$. Since the integral is only over half of the standard interval, it does not yield a particularly simple result. For now, we define

$$N_\ell = \int_0^1 P_\ell(x)\,dx \tag{11}$$

As a result, we have the system of equations

$$\begin{pmatrix} a^\ell & a^{-\ell-1} \\ b^\ell & b^{-\ell-1} \end{pmatrix} \begin{pmatrix} A_\ell \\ B_\ell \end{pmatrix} = \frac{2\ell+1}{2} V N_\ell \begin{pmatrix} 1 \\ (-)^\ell \end{pmatrix}$$

which may be solved to give

$$\begin{pmatrix} A_\ell \\ B_\ell \end{pmatrix} = \frac{2\ell+1}{2} V N_\ell \frac{1}{b^{2\ell+1} - a^{2\ell+1}} \begin{pmatrix} (-)^\ell b^{\ell+1} - a^{\ell+1} \\ (ab)^{\ell+1}(b^\ell + (-)^{\ell+1} a^\ell) \end{pmatrix}$$

Inserting this into (10) gives

$$\begin{aligned} \Phi(r,\theta) = \tfrac{1}{2} V \sum_\ell \frac{(2\ell+1)N_\ell}{1-\left(\frac{a}{b}\right)^{2\ell+1}} \Bigg[(-)^\ell \left(1 + (-)^{\ell+1}\left(\frac{a}{b}\right)^{\ell+1}\right)\left(\frac{r}{b}\right)^\ell \\ + \left(1 + (-)^{\ell+1}\left(\frac{a}{b}\right)^{\ell}\right)\left(\frac{a}{r}\right)^{\ell+1}\Bigg] P_\ell(\cos\theta) \end{aligned} \tag{12}$$

We now examine the integral (11). First note that for even ℓ we may actually extend the region of integration

$$N_{2j} = \int_0^1 P_{2j}(x)\,dx = \tfrac{1}{2}\int_{-1}^1 P_{2j}(x)\,dx = \tfrac{1}{2}\int_{-1}^1 P_0(x)P_{2j}(x)\,dx = \delta_{j,0}$$

This demonstrates that the only contribution from even ℓ is for $\ell = 0$, corresponding to the average potential. Using this fact, the potential (12) reduces to

$$\begin{aligned} \Phi(r,\theta) = \frac{V}{2} + \frac{V}{2}\sum_{j=1}^\infty \frac{(4j-1)N_{2j-1}}{1-\left(\frac{a}{b}\right)^{4j-1}} \Bigg[-\left(1 + \left(\frac{a}{b}\right)^{2j}\right)\left(\frac{r}{b}\right)^{2j-1} \\ + \left(1 + \left(\frac{a}{b}\right)^{2j-1}\right)\left(\frac{a}{r}\right)^{2j}\Bigg] P_{2j-1}(\cos\theta) \end{aligned}$$

Physically, once the average $V/2$ is removed, the remaining potential is odd under the flip $z \to -z$ or $\cos\theta \to -\cos\theta$. This is why only odd Legendre polynomials may contribute.

At this stage, we may simply perform elementary integrations to obtain the first few terms N_1, N_3, etc. However, we may derive a fairly simple expression for N_ℓ by integrating the generating function

$$(1-2xt+t^2)^{-1/2} = \sum_{\ell=0}^\infty P_\ell(x)t^\ell$$

from $x = 0$ to 1. In other words

$$\sum_{\ell=0}^\infty N_\ell t^\ell = \int_0^1 (1-2xt+t^2)^{-1/2} dx = t^{-1}(-1+t+\sqrt{1+t^2})$$

The square root yields a binomial expansion

$$(1+t^2)^{1/2} = 1+\tfrac{1}{2}t^2+\tfrac{1}{2}(-\tfrac{1}{2})\tfrac{1}{2!}t^4+\tfrac{1}{2}(-\tfrac{1}{2})(-\tfrac{3}{2})\tfrac{1}{3!}t^6+\cdots = 1+\sum_{j=1}^{\infty}(-)^j\frac{\Gamma(j-\frac{1}{2})}{\Gamma(-\frac{1}{2})j!}t^{2j}$$

As a result

$$\sum_{\ell=0}^{\infty} N_\ell t^\ell = 1+\sum_{j=1}^{\infty}(-)^{j+1}\frac{\Gamma(j-\frac{1}{2})}{2\sqrt{\pi}j!}t^{2j-1}$$

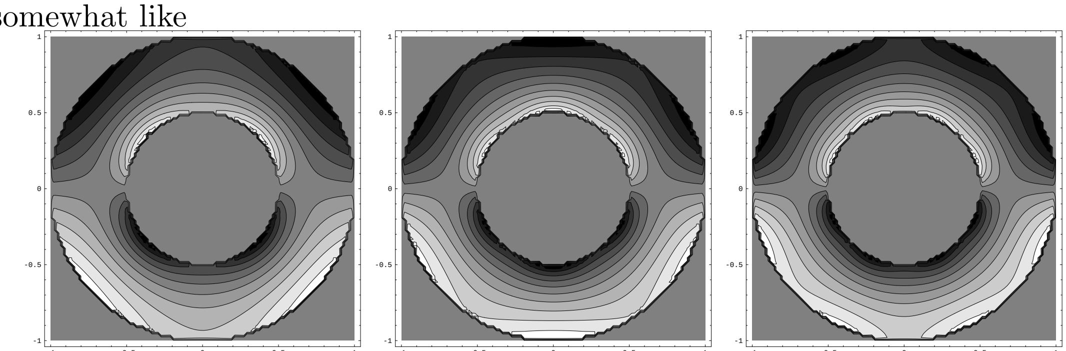
where we used the fact that $\Gamma(-\frac{1}{2}) = -2\Gamma(\frac{1}{2}) = -2\sqrt{\pi}$. Matching powers of t demonstrates that all even N_ℓ terms vanish except $N_0 = 1$ and that

$$N_{2j-1} = (-)^{j+1}\frac{\Gamma(j-\frac{1}{2})}{2\sqrt{\pi}j!}$$

The final result for the potential is thus

$$\begin{aligned}\Phi(r,\theta) &= \frac{V}{2} + V\sum_{j=1}^{\infty}\frac{(-)^{j+1}(4j-1)\Gamma(j-\frac{1}{2})}{4\sqrt{\pi}j!\left(1-\left(\frac{a}{b}\right)^{4j-1}\right)}\Bigg[-\left(1+\left(\frac{a}{b}\right)^{2j}\right)\left(\frac{r}{b}\right)^{2j-1} \\ &\qquad + \left(1+\left(\frac{a}{b}\right)^{2j-1}\right)\left(\frac{a}{r}\right)^{2j}\Bigg]P_{2j-1}(\cos\theta) \\ &= \frac{V}{2} \\ &\quad + V\Bigg[\frac{3}{4}\left(1-\left(\frac{a}{b}\right)^3\right)^{-1}\left(-\left(1+\left(\frac{a}{b}\right)^2\right)\left(\frac{r}{b}\right)+\left(1+\left(\frac{a}{b}\right)\right)\left(\frac{a}{r}\right)^2\right)P_1(\cos\theta) \\ &\quad - \frac{7}{16}\left(1-\left(\frac{a}{b}\right)^7\right)^{-1}\left(-\left(1+\left(\frac{a}{b}\right)^4\right)\left(\frac{r}{b}\right)^3+\left(1+\left(\frac{a}{b}\right)^3\right)\left(\frac{a}{r}\right)^4\right)P_3(\cos\theta) \\ &\qquad + \cdots\Bigg]\end{aligned}$$

Taking a constant ϕ slice of the region between the spheres, the potential looks somewhat like



up to P_3 | up to P_5 | up to P_7

We note that including the higher Legendre modes improves the potential near the surfaces of the spheres. This is very much like summing the first few terms of a Fourier series. On the other hand, the potential midway between the spheres is well estimated by just the first term or two in the series. This is because both r/b and a/r are small in this region, and the series rapidly converges (assuming $a \ll b$, that is).

In the limit when $b \to \infty$ we may remove (a/b) and (r/b) terms. Removing the latter corresponds to having only inverse powers of r surviving, which is the expected case for an exterior solution. The result is

$$\Phi(r,\theta) \to \frac{V}{2} + \frac{V}{2}\left[\frac{3}{2}\left(\frac{a}{r}\right)^2 P_1(\cos\theta) - \frac{7}{8}\left(\frac{a}{r}\right)^4 P_3(\cos\theta) + \cdots\right]$$

which agrees with the exterior solution for a sphere with oppositely charged hemispheres (except that here we have the average potential $V/2$ and that the potential difference between northern and southern hemispheres is only half as large).

Similarly, when $a \to 0$ we remove (a/b). But this time we get rid of the inverse powers (a/r) instead. The result is the interior solution

$$\Phi(r,\theta) \to \frac{V}{2} - \frac{V}{2}\left[\frac{3}{2}\left(\frac{r}{b}\right) P_1(\cos\theta) - \frac{7}{8}\left(\frac{r}{b}\right)^3 P_3(\cos\theta) + \cdots\right]$$

which is again a reasonable result (this time with the hemispheres oppositely charged from the previous case).