Physics 505 **Fall 2005**

Homework Assignment #1 — Solutions

Textbook problems: Ch. 1: 1.4, 1.5, 1.10, 1.14

1.4 Each of three charged spheres of radius a, one conducting, one having a uniform charge density within its volume, and one having a spherically symmetric charge density that varies radially as r^n ($n > -3$), has a total charge Q. Use Gauss' theorem to obtain the electric fields both inside and outside each sphere. Sketch the behavior of the fields as a function of radius for the first two spheres, and for the third with $n = -2, +2$.

Because of spherical symmetry, this may be solved by a straightforward application of Gauss' law. In all cases, the electric field (as a function of r) is given by

$$\vec{E} = k\frac{q_{\rm enc}}{r^2}\hat{r} \qquad k = \frac{1}{4\pi\epsilon_0}$$

i) For the conducting sphere, the charge Q resides on the surface of the sphere (the electric field vanishes inside). Hence

$$\vec{E} = \begin{cases} 0 & r < a \\ k\frac{Q}{r^2}\hat{r} & r > a \end{cases}$$

ii) For the sphere with uniform charge density, we note that the charge enclosed inside a radius $r < a$ must be proportional to the volume $\frac{4}{3}\pi r^3$. Hence $q_{\rm enc} = Q(r/a)^3$ and we are left with

$$\vec{E} = \begin{cases} k\frac{Q}{a^3}\vec{r} & r < a \\ k\frac{Q}{r^2}\hat{r} & r > a \end{cases}$$

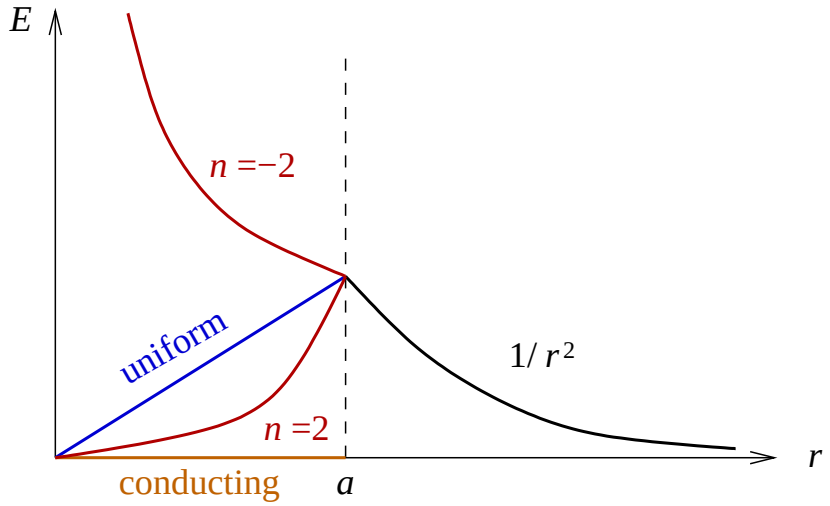
Note that the electric field is linearly proportional to $\vec{r}$ inside the sphere.

iii) For the sphere with varying charge density $\rho \sim r^n$ the charge enclosed is now proportional to r^{n+3}. Hence $q_{\rm enc} = Q(r/a)^{n+3}$ and the electric field becomes

$$\vec{E} = \begin{cases} k\frac{Q}{a^3}\left(\frac{r}{a}\right)^n\vec{r} & r < a \\ k\frac{Q}{r^2}\hat{r} & r > a \end{cases}$$

This reduces to the previous case for $n = 0$. Note that the expression for $q_{\rm enc}$ breaks down for $n < -3$. Furthermore, for $n = -3$, $q_{\rm enc} = Q$ is constant independent of radius r, signifying the charge is concentrated at $r = 0$. This accounts for the point-charge like behavior when $n = -3$. Furthermore, note that in all three cases, the field outside the sphere is identical.

The magnitude of the electric field looks roughly as follows



1.5 The time-averaged potential of a neutral hydrogen atom is given by

$$\Phi = \frac{q}{4\pi\epsilon_0}\frac{e^{-\alpha r}}{r}\left(1+\frac{\alpha r}{2}\right)$$

where q is the magnitude of the electronic charge, and $\alpha^{-1} = a_0/2$, a_0 being the Bohr radius. Find the distribution of charge (both continuous and discrete) that will give this potential and interpret your result physically.

We may obtain the charge distribution by computing $\rho = -\epsilon_0\nabla^2\Phi$. However, since Φ blows up as $r \to 0$, we must be a bit careful. We first consider $r > 0$

$$\begin{aligned}\rho = -\epsilon_0\nabla^2\Phi &= -\frac{q}{4\pi}\frac{1}{r^2}\partial_r r^2 \partial_r e^{-\alpha r}\left(\frac{1}{r}+\frac{\alpha}{2}\right)\\ &= -\frac{q}{4\pi}\frac{1}{r^2}\partial_r e^{-\alpha r}\left(1+\alpha r+\frac{\alpha^2 r^2}{2}\right) = -\frac{q\alpha^3}{8\pi}e^{-\alpha r}\end{aligned}$$

For $r \approx 0$, on the other hand, we may expand

$$\Phi = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r}-\frac{\alpha}{2}+\cdots\right) \approx \frac{q}{4\pi\epsilon_0 r}$$

This is the potential of a point charge q at the origin. Hence the complete charge distribution can be written as

$$\rho = q\delta^3(r) - \frac{q\alpha^3}{8\pi}e^{-\alpha r}$$

The first term corresponds to the proton charge, and the second to the negatively charged electron cloud in the $1s$ orbital around the proton.

We can additionally verify that the hydrogen atom is indeed neutral

$$Q = \int \rho \, d^3x = q - \frac{q\alpha^3}{8\pi} \int_0^\infty e^{-\alpha r} 4\pi r^2 \, dr = q - \frac{q}{2}\Gamma(3) = 0$$

1.10 Prove the *mean value theorem*: For charge-free space the value of the electrostatic potential at any point is equal to the average of the potential over the surface of *any* sphere centered on that point.

There are many variations on the proof. However, they all tend to involve Green's theorem. For example, one could begin by making the substitution $\phi = \Phi(\vec{x}\,)$ and $\psi = 1/|\vec{x} - \vec{x}\,'|$ into Green's theorem. Here we take a shortcut and realize that we want to connect the value of the potential Φ inside the sphere to the value on the boundary. This is a Dirichlet problem, and we go directly to the Dirichlet Green's function solution

$$\Phi(\vec{x}\,) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\vec{x}\,') G_D(\vec{x}, \vec{x}\,') \, d^3x' - \frac{1}{4\pi} \oint_S \Phi(\vec{x}\,') \frac{\partial G_D(\vec{x}, \vec{x}\,')}{\partial n'} \, da' \qquad (1)$$

where $G_D(\vec{x}, \vec{x}\,')$ vanishes for $\vec{x}\,'$ on the surface S. Taking S to be a perfect sphere of radius R (not necessarily large) around the point $\vec{x}$, it is easy to see that

$$G_D(\vec{x}, \vec{x}\,') = \frac{1}{|\vec{x} - \vec{x}\,'|} - \frac{1}{R}$$

satisfies the appropriate requirements. Furthermore, since we are working in charge-free space, $\rho = 0$, and (1) reduces to

$$\begin{aligned} \Phi(\vec{x}\,) &= -\frac{1}{4\pi} \oint_S \Phi(\vec{x}\,') \frac{\partial G_D(\vec{x}, \vec{x}\,')}{\partial n'} \, da' \\ &= -\frac{1}{4\pi} \int_R \Phi(\vec{x}\,') \left(-\frac{1}{R^2} \right) da' = \frac{1}{4\pi R^2} \int_R \Phi(\vec{x}\,') \, da' \end{aligned}$$

Since $4\pi R^2$ is just the area of the sphere, this proves that the value of $\Phi(\vec{x}\,)$ is equal to the average of $\Phi(\vec{x}\,')$ over any sphere centered on $\vec{x}$.

This mean value theorem also shows up in complex analysis, where the Cauchy Integral Formula may be used to prove that the value of an analytic function $f(z)$ at a point z is equal to its mean value on any circle centered at z. Of course, analytic functions are two-dimensional versions of harmonic functions (solutions to Laplace's equation) and yield automatic solutions to two-dimensional electrostatic problems. This is why conformal mapping is a powerful technique for solving two-dimensional electrostatic boundary value problems. The mean value theorem also holds for dimensions higher than two or three.

1.14 Consider the electrostatic Green functions of Section 1.10 for Dirichlet and Neumann boundary conditions on the surface S bounding the volume V. Apply Green's theorem (1.35) with integration variable $\vec{y}$ and $\phi = G(\vec{x},\vec{y})$, $\psi = G(\vec{x}\,',\vec{y})$, with $\nabla_y^2 G(\vec{z},\vec{y}) = -4\pi\delta(\vec{y}-\vec{z})$. Find an expression for the difference $[G(\vec{x},\vec{x}\,') - G(\vec{x}\,',\vec{x})]$ in terms of an integral over the boundary surface S.

Using ϕ and ψ as indicated in Green's theorem, we have

$$\int_V \left(G(\vec{x},\vec{y})\nabla_y^2 G(\vec{x}\,',\vec{y}) - G(\vec{x}\,',\vec{y})\nabla_y^2 G(\vec{x},\vec{y})\right) d^3y = \oint_S \left(G(\vec{x},\vec{y})\frac{\partial G(\vec{x}\,',\vec{y})}{\partial n_y} - G(\vec{x}\,',\vec{y})\frac{\partial G(\vec{x},\vec{y})}{\partial n_y}\right) da_y$$

Since $\nabla_y^2 G(\vec{x},\vec{y}) = -4\pi\delta(\vec{x}-\vec{y})$, the left hand side integrates to $-4\pi[G(\vec{x},\vec{x}\,') - G(\vec{x}\,',\vec{x})]$. Dividing both sides by -4π finally gives

$$G(\vec{x},\vec{x}\,') - G(\vec{x}\,',\vec{x}) = \frac{1}{4\pi}\oint_S \left(G(\vec{x}\,',\vec{y})\frac{\partial G(\vec{x},\vec{y})}{\partial n_y} - G(\vec{x},\vec{y})\frac{\partial G(\vec{x}\,',\vec{y})}{\partial n_y}\right) da_y \quad (2)$$

a) For Dirichlet boundary conditions on the potential and the associated boundary condition on the Green function, show that $G_D(\vec{x},\vec{x}\,')$ must be symmetric in $\vec{x}$ and $\vec{x}\,'$.

For the Dirichlet Green's function, $G_D(\vec{x},\vec{y}) = 0$ for $\vec{y}$ on the boundary S. This means that the right hand side of (2) vanishes. Then we automatically find

$$G_D(\vec{x},\vec{x}\,') = G_D(\vec{x}\,',\vec{x})$$

b) For Neumann boundary conditions, use the boundary condition (1.45) for $G_N(\vec{x},\vec{x}\,')$ to show that $G_N(\vec{x},\vec{x}\,')$ is not symmetric in general, but that $G_N(\vec{x},\vec{x}\,') - F(\vec{x})$ is symmetric in $\vec{x}$ and $\vec{x}\,'$, where

$$F(\vec{x}) = \frac{1}{S}\oint_S G_N(\vec{x},\vec{y})\, da_y$$

We use the Neumann boundary condition

$$\frac{\partial G_N(\vec{x},\vec{y})}{\partial n_y} = -\frac{4\pi}{S}$$

for $\vec{y}$ on the boundary S. This means the right hand side of (2) becomes

$$RHS = \frac{1}{S}\oint_S \left(G_N(\vec{x},\vec{y}) - G_N(\vec{x}\,',\vec{y})\right) da_y = F(\vec{x}) - F(\vec{x}\,')$$

where we used the definition of $F(\vec{x})$ given in the problem. This yields

$$G_N^{\rm new}(\vec{x},\vec{x}\,') \equiv G_N(\vec{x},\vec{x}\,') - F(\vec{x}) = G_N(\vec{x}\,',\vec{x}) - F(\vec{x}\,') \quad (3)$$

which demonstrates that $G_N^{\rm new}(\vec{x},\vec{x}\,')$ is symmetric in $\vec{x}$ and $\vec{x}'$.

c) Show that the addition of $F(\vec{x}\,)$ to the Green function does not affect the potential $\Phi(\vec{x}\,)$. See problem 3.26 for an example of the Neumann Green function.

What we need to do is to show that the Neumann Green's function solution

$$\Phi(\vec{x}\,) = \langle\Phi\rangle_S + \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}\,')G_N(\vec{x},\vec{x}\,')\,d^3x' + \frac{1}{4\pi}\oint_S \frac{\partial\Phi(\vec{x}\,')}{\partial n'}G_N(\vec{x},\vec{x}\,')\,da' \quad (4)$$

is unchanged when we replace G_N by $G_N^{\rm new}$. If we let $\Phi^{\rm new}$ denote the computation using $G_N^{\rm new}$, then

$$\begin{aligned}
\Phi^{\rm new}(\vec{x}\,) &= \langle\Phi\rangle_S + \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}\,')G_N^{\rm new}(\vec{x},\vec{x}\,')\,d^3x' + \frac{1}{4\pi}\oint_S \frac{\partial\Phi(\vec{x}\,')}{\partial n'}G_N^{\rm new}(\vec{x},\vec{x}\,')\,da' \\
&= \langle\Phi\rangle_S + \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}\,')G_N(\vec{x},\vec{x}\,')\,d^3x' + \frac{1}{4\pi}\oint_S \frac{\partial\Phi(\vec{x}\,')}{\partial n'}G_N(\vec{x},\vec{x}\,')\,da' \\
&\quad - \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}\,')F(\vec{x}\,)\,d^3x' - \frac{1}{4\pi}\oint_S \frac{\partial\Phi(\vec{x}\,')}{\partial n'}F(\vec{x}\,)\,da' \\
&= \Phi(\vec{x}\,) - \frac{F(\vec{x}\,)}{4\pi\epsilon_0}\left(\int_V \rho(\vec{x}\,')\,d^3x' + \epsilon_0\oint_S \frac{\partial\Phi(\vec{x}\,')}{\partial n'}\,da'\right) \\
&= \Phi(\vec{x}\,) - \frac{F(\vec{x}\,)}{4\pi\epsilon_0}\left(q_{\rm enc} - \epsilon_0\oint_S \vec{E}(\vec{x}\,')\cdot\hat{n}'\,da'\right)
\end{aligned}$$

where we used the fact that $\hat{n}\cdot\vec{E} = -\hat{n}\cdot\vec{\nabla}\Phi = -\partial\Phi/\partial n$. A simple application of Gauss' law then demonstrates that $\Phi^{\rm new} = \Phi$. Hence we have shown that the addition of $F(\vec{x}\,)$ leaves the solution unchanged. This demonstrates that we can always make G_N symmetric by appropriate modification with F.

Note that from (3) we could instead have defined the symmetric combination $G_N^{\rm bad}(\vec{x},\vec{x}\,') = G_N(\vec{x},\vec{x}\,') + F(\vec{x}\,')$. However this is a bad thing to do, as substitution of $G_N^{\rm bad}$ into (4) will generate an incorrect solution for Φ.